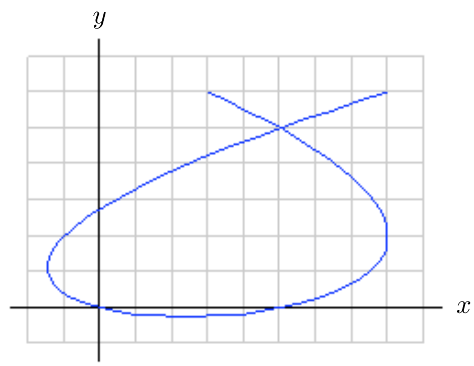
MATH 148 Calculus 2, Solutions to the Exercises for Chapter 4

1: Consider the parametric curve $(x,y) = \left(t^3+2t^2-4t,\, t^2+t\right)$ with $-3 \le t \le 2$.

(a) Sketch the curve showing all of the horizontal and vertical points.

Solution: We have $x' = 3t^2+4t-4 = (3t-2)(t+2)$ so $x'=0 \iff x = -2, \frac{2}{3}$, and we have $y' = 2t+1$ so $y'=0 \iff t = -\frac{1}{2}$. Thus the curve is horizontal when $t=-\frac{1}{2}$ and vertical when $t=-2,\frac{2}{3}$. We make a table of values and plot the curve.

t	x	y
-3	3	6
-2	8	2
-1	5	0
$-\frac{1}{2}$	$\frac{19}{8}$	$-\frac{1}{4}$
0	0	0
$\frac{2}{3}$	$-\frac{44}{27}$	$\frac{10}{9}$
1	-1	2
2	8	6



(b) Find the equation of the tangent line at the point where $t=1$.

Solution: When $t=1$ we have $(x,y) = (t^3+2t^2-4t, t^2+t) = (-1,2)$ and $(x',y') = (3t^2+4t-4, 2t+1) = (3,3)$ so $\dfrac{dy}{dx} = \dfrac{y'}{x'} = 1$. Thus the equation of the tangent line is $y-2 = 1(x+1)$ or equivalently $y = x+3$.

(c) Find $\dfrac{d^2y}{dx^2}$ at the point where $t=1$.

Solution: We have $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)} = \dfrac{\frac{d}{dt}\left(\frac{2t+1}{3t^2+4t-4}\right)}{3t^2+4t-4} = \dfrac{(2)(3t^2+4t-4)-(2t+1)(6t+4)}{(3t^2+4t-4)^3}$, and so when $t=1$, we have $\dfrac{d^2y}{dx^2} = \dfrac{2\cdot 3 - 3\cdot 10}{3^3} = -\frac{8}{9}$.

(d) Eliminate the parameter to find an implicit cartesian equation for the curve.

Solution: We have

$$\begin{aligned} y &= t^2+t \\ x &= t^3+2t^2-4t \\ y^2 &= t^4+2t^3+t^2 \\ xy &= t^5+3t^4-2t^3-4t^2 \\ x^2 &= t^6+4t^5-4t^4-16t^3+16t^3 \\ y^3 &= t^6+3t^5+3t^4+t^3 \end{aligned}$$

and so

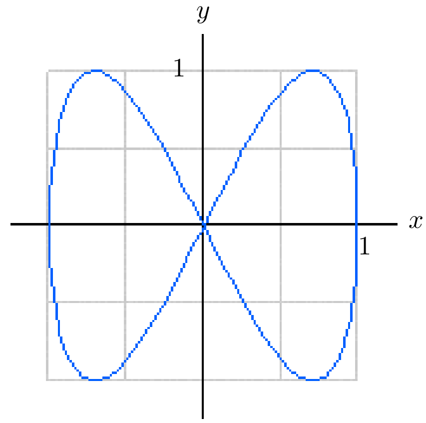
$$\begin{aligned} y^3-x^2 &= -t^5+7t^4+17t^3-16t^2 \\ y^3-x^2+xy &= 10t^4+15t^3-20t^2 \\ y^3-x^2+xy-10y^2 &= -5t^3-30t^2 \\ y^3-x^2+xy-10y^2+5x &= -20t^2-20t = -20y \end{aligned}$$

and so the curve satisfies the Cartesian equation $y^3-x^2+xy-10y^2+5x+20y=0$

2: (a) Sketch the curve $(x, y) = (\cos t, \sin 2t)$, showing all horizontal and vertical points.

Solution: We have $x'(t) = -\sin t$ and $y'(t) = 2\cos 2t$. The curve is horizontal when $y'(t) = 0$, that is when $t = \frac{\pi}{4} + \frac{\pi}{2}k$ with $k \in \mathbb{Z}$, and the curve is vertical when $x'(t) = 0$, that is when $t = \pi k$ with $k \in \mathbb{Z}$. We make a table of values and sketch the curve.

t	x	y
0	1	0
$\pi/6$	$\sqrt{3}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	1
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	0
$2\pi/3$	$-1/2$	$-\sqrt{3}/2$
$3\pi/4$	$-\sqrt{2}/2$	-1
$5\pi/6$	$-\sqrt{3}/2$	$-\sqrt{3}/2$
π	-1	0
etc		



(b) Find the angle inside the loop at the origin.

Solution: The curve passes through the origin when $t = \frac{\pi}{2}$ and then again when $t = \frac{3\pi}{2}$. We have $x'\left(\frac{\pi}{2}\right) = -1$ and $y'\left(\frac{\pi}{2}\right) = -2$, so the slope of the tangent line at $t = \frac{\pi}{2}$ is equal to 2. The angle from the positive x-axis to this tangent line is $\alpha = \tan^{-1} 2 = \sin^{-1} \frac{2}{5} = \cos^{-1} \frac{1}{\sqrt{5}}$. By symmetry, the angle inside the loop at the origin is $\theta = 2\alpha = 2\tan^{-1} 2$. Since $0 \le \theta \le \pi$ and $\cos\theta = \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{1}{5} = \frac{4}{5} = -\frac{3}{5}$, we can also write the angle as $\theta = \cos^{-1}\left(-\frac{3}{5}\right)$.

(c) Find the total area of the enclosed region.

Solution: The total area is 4 times the area of the portion in the first quadrant, so $A = 4\int_{x=0}^{1} y\,dx$. To find this we use $x = \cos t$ so $dx = -\sin t\,dt$ and $y = \sin 2t = 2\sin t\cos t$. Then

$$A = 4\int_{x=0}^{1} y\,dx = 4\int_{t=\pi/2}^{0} (2\sin x\cos x)(-\sin x)\,dx = \int_{0}^{\pi/2} 8\sin^2 x\cos x\,dx.$$

Now we use $u = \sin t$ so $du = \cos t\,dt$ to get

$$\int_{t=0}^{\pi/2} 8\sin^2 t\cos t\,dt = \int_{u=0}^{1} 8u^2\,du = \left[\tfrac{8}{3}u^3\right]_0^1 = \tfrac{8}{3}\,.$$

(d) Find an implicit cartesian equation for this curve.

Solution: If $x = \cos t$ and $y = \sin 2t$ then $x^2 = \cos^2 t$ and $y^2 = 4\sin^2 t\cos^2 t = 4\cos^2 t(1-\cos^2 t) = 4x^2(1-x^2)$. This shows that the parametric curve is contained in the implicit cartesian curve $y^2 = 4x^2(1-x^2)$.

Let us verify not that our parametric curve is not only contained in the implicit curve $y^2 = 4x^2(1-x^2)$ but is in fact equal to that implicit curve. Suppose that $y^2 = 4x^2(1-x^2)$. Then we have $y = \pm 2x\sqrt{1-x^2}$ with $-1 \le x \le 1$. If $y = 2x\sqrt{1-x^2}$ then we can let $t = \cos^{-1} x \in [0, \pi]$, and then $\cos t = x$ and, since $\sin t \ge 0$,

$$\sin 2t = 2\sin t\cos t = 2\cos t\sqrt{\sin^2 t} = 2\cos t\sqrt{1-\cos^2 t} = 2x\sqrt{1-x^2} = y\,.$$

If $y = -2x\sqrt{1-x^2}$ then we can let $t = -\cos^{-1} x \in [-\pi, 0]$, and then $\cos t = x$ and, since $\sin t \le 0$,

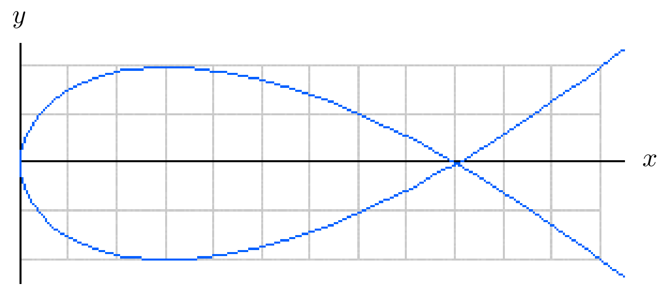
$$\sin 2t = 2\sin t\cos t = -2\cos t\sqrt{\sin^2 t} = -2\cos t\sqrt{1-\cos^2 t} = -2x\sqrt{1-x^2} = y\,.$$

This shows that the implicit curve is contained in the parametric curve.

3: (a) Sketch the curve $(x,y) = (3t^2, 3t - t^3)$, showing all horizontal and vertical points.

Solution: We have $x'(t) = 6t$ and $y'(t) = 3 - 3t^2 = -3(t-1)(t+1)$. The curve is horizontal when $y'(t) = 0$, that is when $t = \pm 1$, and the curve is vertical when $x'(t) = 0$, that is when $t = 0$. We make a table of values and sketch the curve.

t	x	y
-2	12	2
$-\sqrt{3}$	9	0
-1	3	-2
0	0	0
1	3	2
$\sqrt{3}$	9	0
2	12	-2



(b) Find the arclength of the loop in this curve.

Solution: The total arclength of the loop is equal to twice the arclength of the portion above the x-axis, so we have

$$L = 2\int_{t=0}^{\sqrt{3}} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{\sqrt{3}} 2\sqrt{(6t)^2 + (3-3t^2)^2}\,dt = \int_0^{\sqrt{3}} 6\sqrt{(2t)^2 + (1-t^2)^2}\,dt$$

$$= \int_0^{\sqrt{3}} 6\sqrt{1 + 2t^2 + t^4}\,dt = \int_0^{\sqrt{3}} 6(1+t^2)\,dt = \Big[6t + 2t^3\Big]_0^{\sqrt{3}} = 6\sqrt{3} + 2\cdot 3\sqrt{3} = 12\sqrt{3}\,.$$

(c) Find the area of the surface obtained by revolving the loop about the x-axis.

Solution: The surface is obtained by revolving the portion of the curve with $0 \le t \le \sqrt{3}$ about the x-axis. As shown above, we have $\sqrt{x'(t)^2 + y'(t)^2} = 3(1+t^2)$ and so the area is

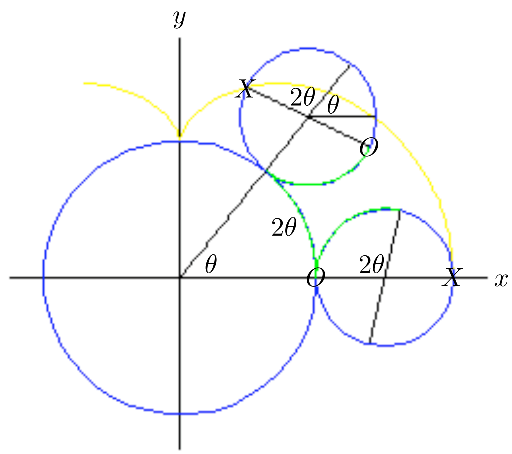
$$A = \int_{t=0}^{\sqrt{3}} 2\pi\, y(t)\sqrt{x'(t)^2 + y'(t)^2}\;dt = \int_0^{\sqrt{3}} 2\pi(3t - t^3)\cdot 3(1+t^2) = \pi\int_0^{\sqrt{3}} 18t + 12t^3 - 6t^5\;dt$$

$$= \pi\Big[9t^2 + 3t^4 - t^5\Big]_0^{\sqrt{3}} = \pi(27 + 27 - 27) = 27\pi\,.$$

4: A circular hoop of radius 1, initially centered at $(3,0)$, rolls without slipping once, counterclockwise, around a circular hoop of radius 2 which remains stationary, centered at $(0,0)$. Consider the curve which is followed by the point on the moving hoop which is initially at the position $(4,0)$.

(a) Show that the curve is given parametrically by $(x,y) = \big(3\cos\theta + \cos 3\theta, 3\sin\theta + \sin 3\theta\big)$, with $0 \le \theta \le 2\pi$.

Solution: When the center of the small hoop rotates through an angle θ, the hoop must spin through an angle 3θ in order to roll without slipping. (This can be seen with the help of the following picture, in which the green arcs all have length 2θ). After rotating through an angle θ, the center of the small hoop is at $(3\cos\theta, 3\sin\theta)$, so the point labeled by X is at $(3\cos\theta, 3\sin\theta) + (\sin 3\theta, \cos 3\theta)$.



(b) Find the area enclosed by the curve.

Solution: Using the identity $2\sin A\sin B = \cos(A-B) - \cos(A+B)$ with $A = 3\theta$ and $B = \theta$, we get $2\sin\theta\sin 3\theta = \cos 2\theta - \cos 4\theta$, so the area is

$$
\begin{aligned}
A &= 4\int_{\theta=0}^{\pi/2} |y(\theta)x'(\theta)|d\theta = -4\int_0^{\pi/2} y(\theta)x'(\theta)\,d\theta = -4\int_0^{\pi/2}(3\sin\theta + \sin 3\theta)(-3\sin\theta - 3\sin 3\theta)\,d\theta \\
&= \int_0^{\pi/2} 36\sin\theta + 48\sin\theta\sin 3\theta + 12\sin^2 3\theta\ d\theta \\
&= \int_0^{\pi/2} (18 - 18\cos 2\theta) + 24(\cos 2\theta - \cos 4\theta) + (6 - 6\cos 6\theta)\ d\theta \\
&= \int_0^{\pi/2} 24 + 6\cos 2\theta - 24\cos 4\theta - 6\cos 6\theta\ d\theta = \Big[24\theta + 3\sin 2\theta - 6\sin 4\theta - \sin 6\theta\Big]_0^{\pi/2} = 12\pi\,.
\end{aligned}
$$

(c) Find the distance travelled by the point on the moving hoop.

Solution: The distance travelled by the point is the arclenth of the curve, which is

$$
\begin{aligned}
L &= 4\int_0^{\pi/2} \sqrt{x'(\theta)^2 + y'(\theta)^2}\ d\theta = 4\int_0^{\pi/2} \sqrt{(-3\sin\theta + 3\sin 3\theta)^2 + (3\cos\theta + 3\cos 3\theta)^2}\ d\theta \\
&= \int_0^{\pi/2} 12\sqrt{(\sin\theta + \sin 3\theta)^2 + (\cos\theta + \cos 3\theta)^2}\ d\theta \\
&= \int_0^{\pi/2} 12\sqrt{\sin^2\theta + 2\sin\theta\sin 3\theta + \sin^2 3\theta + \cos^2\theta + 2\cos\theta\cos 3\theta + \cos^2 3\theta}\ d\theta \\
&= \int_0^{\pi/2} 12\sqrt{2 + 2\sin\theta\sin 3\theta + 2\cos\theta\cos 3\theta} d\theta \quad (\text{since } \sin^2\theta + \cos^2\theta = \sin^2 3\theta + \cos^2 3\theta = 1) \\
&= \int_0^{\pi/2} 12\sqrt{2 + 2\cos 2\theta}\ d\theta \quad (\text{since } \cos 2\theta = \cos(3\theta - \theta) = \cos 3\theta\cos\theta + \sin 3\theta\sin\theta) \\
&= \int_0^{\pi/2} 12\sqrt{4\cos^2\theta}\ d\theta = \int_0^{\pi/2} 24\cos\theta\ d\theta = \Big[24\sin\theta\Big]_0^{\pi/2} = 24\,.
\end{aligned}
$$

5: Consider the polar curve $r = \dfrac{3}{2-\cos\theta}$.

(a) Sketch the curve, showing all horizontal and vertical points.

Solution: We have $x = r\cos\theta = \dfrac{3\cos\theta}{2-\cos\theta}$ so

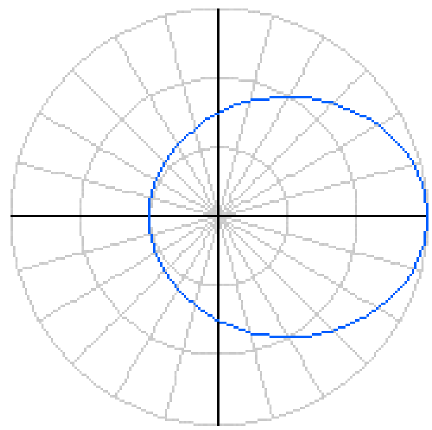
$$x'(\theta) = \frac{(-3\sin\theta)(2-\cos\theta) - (3\sin\theta)(\sin\theta)}{(2-\cos\theta)^2} = \frac{-6\sin\theta}{(2-\cos\theta)^2}\,,$$

and we have $y = r\sin\theta = \dfrac{3\sin\theta}{2-\cos\theta}$ and so

$$y'(\theta) = \frac{(3\cos\theta)(2-\cos\theta) - (3\sin\theta)(\sin\theta)}{(2-\cos\theta)^2} = \frac{6\cos\theta - 3\cos^2\theta - 3\sin^2\theta}{(2-\cos\theta)^2} = \frac{3(2\cos\theta - 1)}{(2-\cos\theta)^2}\,.$$

The curve is horizontal when $y'(\theta) = 0$, that is when $\cos\theta = \frac{1}{2}$ or when $\theta = \pm\frac{\pi}{3} + 2\pi k$ with $k \in \mathbb{Z}$, and the curve is vertical when $x'(\theta) = 0$, that is when $\sin\theta = 0$, that is $\theta = \pi k$ with $k \in \mathbb{Z}$. We make a table of values (r, θ) and plot the curve on a polar grid.

θ	r
0	3
$\pi/3$	2
$\pi/2$	$3/2$
$2\pi/3$	$6/5$
π	1
etc.	



(b) Find the Cartesian equation of this curve.

Solution: Note that $2 - \cos\theta \geq 1$ so $r \leq 3$ hence $3 + r\cos\theta \geq 0$, so we have

$$\begin{aligned} r = \frac{3}{2-\cos\theta} &\iff r\,(2-\cos\theta) = 3 \iff 2\,r = 3 + r\cos\theta \\ &\iff 4\,r^2 = (3 + r\cos\theta)^2 \iff 4(x^2+y^2) = (3+x)^2 \end{aligned}$$

so the Cartesian equation is $4(x^2+y^2) = (3+x)^2$.

Remark: we can rewrite the above Cartesian equation as follows

$$\begin{aligned} 4(x^2+y^2) = (3+x)^2 &\iff 4x^2 + 4y^2 = 9 + 6x + x^2 \iff 3x^2 - 6x + 4y^2 = 9 \\ &\iff 3(x-1)^2 + 4y^2 = 12 \iff \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1 \end{aligned}$$

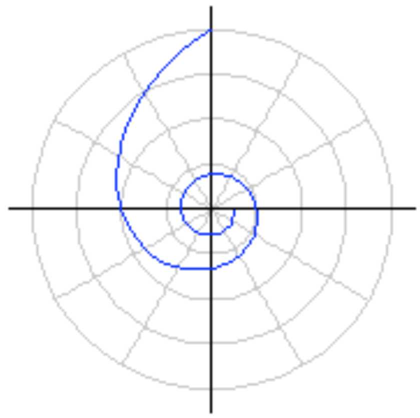
which shows that the curve is an ellipse.

6: Consider the polar curve $r = \dfrac{2\pi}{\theta}$.

(a) Sketch the portion of the curve with $\frac{\pi}{2} \le \theta \le 4\pi$.

Solution: We make a table of values and sketch the curve.

θ	r
$\frac{\pi}{2}$	4
π	2
$\frac{3\pi}{2}$	$\frac{4}{3}$
2π	1
$\frac{5\pi}{2}$	$\frac{4}{5}$
3π	$\frac{2}{3}$
$\frac{7\pi}{2}$	$\frac{4}{7}$
4π	$\frac{1}{2}$



(b) Find the arclength of the portion of the curve with $\pi \le \theta \le 2\pi$.

Solution: The arclength of the portion of the curve with $\pi \le \theta \le 2\pi$ is

$$L = \int_{\theta=\pi}^{2\pi} \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta = \int_{\pi}^{2\pi} \sqrt{\left(\frac{2\pi}{\theta}\right)^2 + \left(-\frac{2\pi}{\theta^2}\right)^2}\, d\theta = \int_{\pi}^{2\pi} 2\pi\sqrt{\frac{1}{\theta^2} + \frac{1}{\theta^4}}\, d\theta = \int_{\pi}^{2\pi} \frac{2\pi\sqrt{\theta^2+1}}{\theta^2}\, d\theta\,.$$

Let $\tan\phi = \theta$ so that $\sec\phi = \sqrt{1+\theta^2}$ and $\sec^2\phi\, d\phi = d\theta$. Then

$$L = \int_{\theta=\pi}^{2\pi} \frac{2\pi \sec^3\phi\, d\phi}{\tan^2\phi} = \int_{\theta=\pi}^{2\pi} \frac{2\pi\, d\phi}{\cos\phi \sin^2\phi} = \int_{\theta=\pi}^{2\pi} \frac{2\pi\, \cos\phi\, d\phi}{(1-\sin^2\phi)\sin^2\phi}\,.$$

Now let $u = \sin\phi$ so that $du = \cos\phi\, d\phi$. Then we have

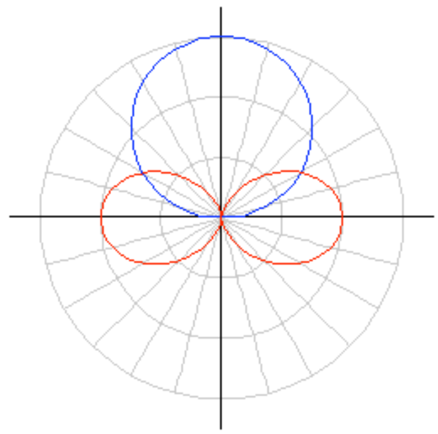
$$\begin{aligned}
L &= 2\pi \int_{\theta=\pi}^{2\pi} \frac{du}{(1-u^2)\,u^2} = 2\pi \int_{\theta=\pi}^{2\pi} \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} + \frac{1}{u}\, du = 2\pi\left[\tfrac{1}{2}\ln|1+u| - \tfrac{1}{2}\ln|1-u| - \frac{1}{u^2}\right]_{\theta=\pi}^{2\pi} \\
&= 2\pi\left[\tfrac{1}{2}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{u^2}\right]_{\theta=\pi}^{2\pi} = 2\pi\left[\tfrac{1}{2}\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right) - \frac{1}{\sin\phi}\right]_{\theta=\pi}^{2\pi} = 2\pi\left[\tfrac{1}{2}\ln\left(\frac{(1+\sin\phi)^2}{\cos^2\phi}\right) - \frac{1}{\sin\phi}\right]_{\theta=\pi}^{2\pi} \\
&= 2\pi\left[\ln\left(\frac{1+\sin\phi}{\cos\phi}\right) - \frac{1}{\sin\phi}\right]_{\theta=\pi}^{2\pi} = 2\pi\Big[\ln(\tan\phi + \sec\phi) - \csc\phi\Big]_{\theta=\pi}^{2\pi} \\
&= 2\pi\left[\ln\left(\theta + \sqrt{\theta^2+1}\right) - \frac{\sqrt{\theta^2+1}}{\theta}\right]_{\theta=\pi}^{2\pi} = 2\pi\ln\left(\frac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right) - \sqrt{4\pi^2+1} + 2\sqrt{\pi^2+1}\,.
\end{aligned}$$

7: Consider the two the polar curves $r = 3\sin\theta$ and $r = 1 + \cos 2\theta$.

(a) Sketch both polar curves on the same grid, showing all points of intersection.

Solution: To find the points of intersection, note that $3\sin\theta = 1 + \cos 2\theta \iff 3\sin\theta = 1 + (1 - 2\sin^2\theta) \iff 2\sin^2\theta + 3\sin\theta - 2 = 0 \iff (2\sin\theta - 1)(\sin\theta + 2) = 0 \iff \sin\theta = \frac{1}{2} \iff \theta = \cdots, \frac{\pi}{6}, \frac{5\pi}{6}, \cdots$. Now we make a table of values and plot the curves.

θ	$3\sin\theta$	$1+\cos 2\theta$
0	0	2
$\frac{\pi}{6}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	3	0
$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{5\pi}{6}$	$\frac{3}{2}$	$\frac{3}{2}$
π	0	2



(b) Find the area of the region which lies inside both curves.

Solution: The area is

$$
\begin{aligned}
A &= 2\left(\int_0^{\pi/6} \tfrac{1}{2}(3\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)^2\,d\theta\right) \\
&= \int_0^{\pi/6} 9\sin^2\theta\,d\theta + \int_{\pi/6}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\ d\theta \\
&= \int_0^{\pi/6} \tfrac{9}{2} - \tfrac{9}{2}\cos 2\theta\ d\theta + \int_{\pi/6}^{\pi/2} 1 + 2\cos 2\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 4\theta\ d\theta \\
&= \left[\tfrac{9}{2}\,\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/6} + \left[\tfrac{3}{2}\,\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_{\pi/6}^{\pi/2} \\
&= \left(\tfrac{3\pi}{4} - \tfrac{9\sqrt{3}}{8}\right) + \left(\tfrac{3\pi}{4}\right) - \left(\tfrac{\pi}{4} + \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{3}}{16}\right) \\
&= \tfrac{5\pi}{4} - \tfrac{27\sqrt{3}}{16}\,.
\end{aligned}
$$

8: Let R be the region which lies inside the polar curve $r = 1+\cos\theta$, and let S be the solid obtained by revolving R about the x-axis.

(a) Find the volume of S.

Solution: We have $x(\theta) = r(\theta)\cos\theta = (1+\cos\theta)\cos\theta = \cos\theta + \cos^2\theta$ and $y(\theta) = r(\theta)\sin\theta = (1+\cos\theta)\sin\theta$. Also $x'(\theta) = -\sin\theta - 2\sin\theta\cos\theta = -(1+2\cos\theta)\sin\theta$. Using the substitution $u = \cos\theta$ so $du = -\sin\theta\,d\theta$, the volume is

$$\begin{aligned}
V &= -\int_{\theta=0}^{\pi} \pi\, y(\theta)^2 x'(\theta)\,d\theta \\
&= \int_{\theta=0}^{\pi} \pi(1+\cos\theta)^2(\sin\theta)^2(1+2\cos\theta)\sin\theta\,d\theta \\
&= \int_{u=1}^{-1} -\pi(1+u)^2(1-u^2)(1+2u)\,du \\
&= \int_{-1}^{1} \pi(1+2u+u^2)(1+2u-u^2-2u^3)\,du \\
&= \int_{-1}^{1} \pi(1+4u+4u^2-2u^3-5u^4-2u^5)\,du \\
&= \pi\Big[u+2u^2+\tfrac{4}{3}u^3-\tfrac{1}{2}u^4-u^5-\tfrac{1}{3}u^6\Big]_{-1}^{1} \\
&= \pi\Big(\big(1+2+\tfrac{4}{3}-\tfrac{1}{2}-1-\tfrac{1}{3}\big)-\big(-1+2-\tfrac{4}{3}-\tfrac{1}{2}+1-\tfrac{1}{3}\big)\Big) \\
&= \tfrac{8\pi}{3}\,.
\end{aligned}$$

(b) Find the surface area of S.

Solution: Using the substitution $u = 1+\cos\theta$ so $du = -\sin\theta\,d\theta$, the surface area is

$$\begin{aligned}
A &= \int_0^{\pi} 2\pi\, y(\theta)\sqrt{r(\theta)^2+r'(\theta)^2}\;d\theta \\
&= \int_0^{\pi} 2\pi\,(1+\cos\theta)(\sin\theta)\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\;d\theta \\
&= \int_0^{\pi} 2\pi\,(1+\cos\theta)\sqrt{2+2\cos\theta}\;\sin\theta\,d\theta \\
&= \int_0^{\pi} 2\sqrt{2}\,\pi\,(1+\cos\theta)\sqrt{1+\cos\theta}\;\sin\theta\,d\theta \\
&= \int_{u=2}^{0} -2\sqrt{2}\,\pi\,u^{3/2}\,du \\
&= \Big[-\tfrac{4\sqrt{2}\,\pi}{5}\,u^{5/2}\Big]_2^0 \\
&= \tfrac{32\pi}{5}\,.
\end{aligned}$$

9: (a) Show that our two methods for finding areas in polar coordinates yield the same value, as follows: Let $r(\theta)$ be differentiable for $\theta\in[\theta_1,\theta_2]$. Let $x(\theta)=r(\theta)\cos\theta$ and $y(\theta)=r(\theta)\sin\theta$, and for $k=1,2$ write $x_k=x(\theta_k)$ and $y_k=y(\theta_k)$. Show that

$$\int_{\theta_1}^{\theta_2}\tfrac{1}{2}r(\theta)^2\,d\theta+\int_{\theta_1}^{\theta_2}y(\theta)\,x'(\theta)\,d\theta=\tfrac{1}{2}(x_2y_2-x_1y_1)\,.$$

Solution: Note that

$$\begin{aligned}\int y(\theta)x'(\theta)\,d\theta&=\int r(\theta)\sin\theta(r'(\theta)\cos\theta-r(\theta)\sin\theta)\,dx\\&=\int r(\theta)r'(\theta)\sin\theta\cos\theta\;d\theta-\int r(\theta)^2\sin^2\theta\;d\theta\end{aligned},$$

and using Integration by Parts with $u=\sin\theta\cos\theta$, $du=(\cos^2\theta-\sin^2\theta)d\theta=(1-2\sin^2\theta)d\theta$, $v=\frac{1}{2}r(\theta)^2$ and $dv=r(\theta)r'(\theta)\,d\theta$ gives

$$\begin{aligned}\int r(\theta)r'(\theta)\sin\theta\cos\theta\;d\theta&=\tfrac{1}{2}r(\theta)^2\sin\theta\cos\theta-\int\tfrac{1}{2}r(\theta)^2(1-2\sin^2\theta)d\theta\\&=\tfrac{1}{2}x(\theta)y(\theta)-\int\tfrac{1}{2}r(\theta)^2\,d\theta+\int r(\theta)\sin^2\theta\;d\theta\end{aligned}$$

and so $\int y(\theta)x'(\theta)\,d\theta=\frac{1}{2}x(\theta)y(\theta)-\int\frac{1}{2}r(\theta)^2\,d\theta$. Putting in the endpoints θ_1,θ_2 yields the desired result.

(b) For a point $p=(a,b)\in\mathbb{R}^2$ and a continuous curve $\alpha(t)=(a,b)+\big(x(t),y(t)\big)$ with $\alpha(t)\neq p$ for any t, we define the **winding number** $\mathrm{W}(\alpha,p)$ of α about p as follows. We write $\alpha(t)$ parametrically in polar coordinates as $\alpha(t)=(a,b)+r(t)\big(\cos\theta(t),\sin\theta(t)\big)$ where $r(t)$ and $\theta(t)$ are continuous with $r(t)>0$ and $\theta(a)\in[0,2\pi)$. Then

$$\mathrm{W}(\alpha,p)=\tfrac{1}{2\pi}\big(\theta(b)-\theta(a)\big).$$

Suppose that $x(t)$, $y(t)$, $r(t)$ and $\theta(t)$ are all differentiable and their derivatives ar continuous. Show that

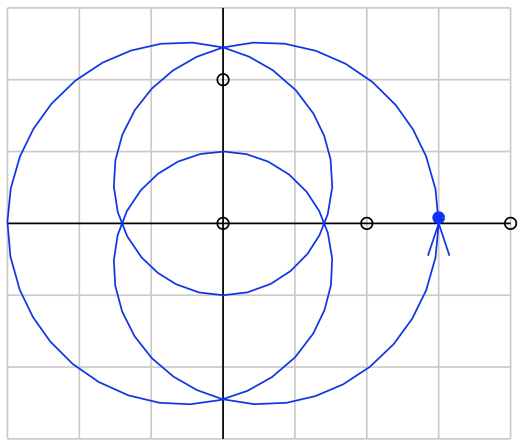
$$\mathrm{W}(\alpha,p)=\tfrac{1}{2\pi}\int_a^b\frac{x(t)y'(t)-y(t)x'(t)}{x(t)^2+y(t)^2}\,dt.$$

Solution: Write α in polar coordinates as $\alpha(t)=(a,b)+r(t)\big(\cos\theta(t),\sin\theta(t)\big)$, that is write $x=r\cos\theta$ and $y=r\sin\theta$ where $r=r(t)$ and $\theta=\theta(t)$ are continuous with $r(t)>0$ for all $t\in[a,b]$ and $\theta(a)\in[0,2\pi)$. Then

$$\begin{aligned}\int_a^b\frac{x\,y'-y\,x'}{x^2+y^2}\,dt&=\int_a^b\frac{(r\cos\theta)\big(r'\sin\theta+r\cos\theta\,\theta'\big)-(r\sin\theta)\big(r'\cos\theta-r\sin\theta\,\theta'\big)}{r^2\cos^\theta+r^2\sin^2\theta}\,dt\\&=\int_a^b\frac{r^2\cos^2\theta\,\theta'+r^2\sin^2\theta\,\theta'}{r^2}\,dt=\int_a^b\theta'\,dt\\&=\theta(b)-\theta(a)=2\pi\,\mathrm{W}(\alpha,p).\end{aligned}$$

(c) Let $\alpha(t) = (\cos t, \sin t) + 2(\cos 3t, \sin 3t)$ with $0 \leq t \leq 2\pi$. Sketch the loop α, then use the sketch (intuitively) to find the winding number of α about each of the points $(0,0)$, $(2,0)$, $(4,0)$ and $(0,2)$.

Solution: To sketch the loop by hand, it helps to treat the terms $2(\cos 3t, \sin 3t)$ and $(\cos t, \sin t)$ separately: choose values of t (say multiples of $\frac{\pi}{12}$), then for each value of t find the point $2(\cos 3t, \sin 3t)$ (which lies on the circle $r = 2$) then add the unit vector $(\cos t, \sin t)$ to the point.



With the help of the picture we find that $\mathrm{W}(\alpha, 0) = 3$, $\mathrm{W}(\alpha, 2) = 1$, $\mathrm{W}(\alpha, 4) = 0$ and $\mathrm{W}(\alpha, 2i) = 2$.