MATH 148 Calculus 2, Exercises for Chapter 4

1: Consider the parametric curve $(x,y) = \big(t^3 + 2t^2 - 4t, t^2 + t\big)$ with $-3 \le t \le 2$.
(a) Sketch the curve showing all of the horizontal and vertical points.
(b) Find the equation of the tangent line at the point where $t = 1$.
(c) Find $\frac{d^2y}{dx^2}$ at the point where $t = 1$.
(d) Eliminate the parameter to find an implicit equation for the curve.

2: (a) Sketch the curve $(x,y) = (\cos t, \sin 2t)$, showing all horizontal and vertical points.
(b) Find the angle inside the loop at the origin.
(c) Find the total area of the enclosed region.
(d) Find an implicit cartesian equation for this curve.

3: (a) Sketch the curve $(x,y) = (3t^2, 3t - t^3)$, showing all horizontal and vertical points.
(b) Find the arclength of the loop in this curve.
(c) Find the area of the surface obtained by revolving the loop about the x-axis.

4: A circular hoop of radius 1, initially centered at $(3,0)$, rolls without slipping once, counterclockwise, around a circular hoop of radius 2 which remains stationary, centered at $(0,0)$. Consider the curve which is followed by the point on the moving hoop which is initially at the position $(4,0)$.
(a) Show that the curve is given parametrically by $(x,y) = \big(3\cos\theta + \cos 3\theta, 3\sin\theta + \sin 3\theta\big)$, with $0 \le \theta \le 2\pi$.
(b) Find the area enclosed by the curve.
(c) Find the distance travelled by the point on the moving hoop.

5: Consider the polar curve $r = \frac{3}{2 - \cos\theta}$.
(a) Sketch the curve, showing all horizontal and vertical points.
(b) Find the Cartesian equation of this curve.

6: Consider the polar curve $r = \frac{2\pi}{\theta}$.
(a) Sketch the portion of the curve with $\frac{\pi}{2} \le \theta \le 4\pi$.
(b) Find the arclength of the portion of the curve with $\pi \le \theta \le 2\pi$.

7: Consider the two the polar curves $r = 3\sin\theta$ and $r = 1 + \cos 2\theta$.
(a) Sketch both polar curves on the same grid, showing all points of intersection.
(b) Find the area of the region which lies inside both curves.

8: Let R be the region which lies inside the polar curve $r = 1 + \cos\theta$, and let S be the solid obtained by revolving R about the x-axis.
(a) Find the volume of S.
(b) Find the surface area of S.

9: (a) Show that our two methods for finding areas in polar coordinates yield the same value, as follows: Let $r(\theta)$ be differentiable for $\theta \in [\theta_1, \theta_2]$. Let $x(\theta) = r(\theta)\cos\theta$ and $y(\theta) = r(\theta)\sin\theta$, and for $k = 1,2$ write $x_k = x(\theta_k)$ and $y_k = y(\theta_k)$. Show that

$$\int_{\theta_1}^{\theta_2} \tfrac{1}{2} r(\theta)^2 \, d\theta + \int_{\theta_1}^{\theta_2} y(\theta)\, x'(\theta)\, d\theta = \tfrac{1}{2}(x_2 y_2 - x_1 y_1)\,.$$

(b) For a point $p = (a,b) \in \mathbb{R}^2$ and a continuous curve $\alpha(t) = (a,b) + \big(x(t), y(t)\big)$ with $\alpha(t) \ne p$ for any t, we define the **winding number** $\mathrm{W}(\alpha, p)$ of α about p as follows. We write $\alpha(t)$ parametrically in polar coordinates as $\alpha(t) = (a,b) + r(t)\big(\cos\theta(t), \sin\theta(t)\big)$ where $r(t)$ and $\theta(t)$ are continuous with $r(t) > 0$ and $\theta(a) \in [0, 2\pi)$. Then

$$\mathrm{W}(\alpha, p) = \tfrac{1}{2\pi}\big(\theta(b) - \theta(a)\big).$$

Suppose that $x(t)$, $y(t)$, $r(t)$ and $\theta(t)$ are all differentiable and their derivatives ar continuous. Show that

$$\mathrm{W}(\alpha, p) = \tfrac{1}{2\pi} \int_a^b \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2}\, dt.$$

(c) Let $\alpha(t) = (\cos t, \sin t) + 2(\cos 3t, \sin 3t)$ with $0 \le t \le 2\pi$. Sketch the loop α, then use the sketch (intuitively) to find the winding number of α about each of the points $(0,0)$, $(2,0)$, $(4,0)$ and $(0,2)$.