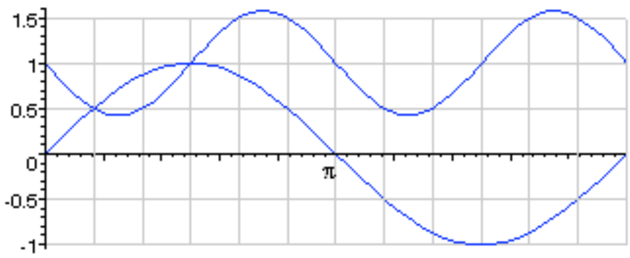
MATH 148 Calculus 2, Solutions to the Exercises for Chapter 3

1: (a) Find the area of the region given by $0 < x < 2\pi$ and $1 - \frac{1}{\sqrt{3}}\sin 2x \le y \le \sin x$.

Solution: First sketch the graphs of $y = \sin x$ and $y = 1 - \frac{1}{\sqrt{3}}\sin 2x$.

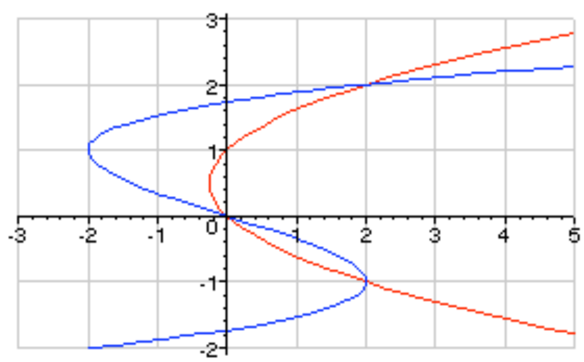


From the graph (or by doing a little algebra) we see that the points of intersection are at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$, and that the curve $y = \sin x$ lies above the other curve between the points of intersection. So the area is

$$A = \int_{\pi/6}^{\pi/2} \sin x - 1 + \tfrac{1}{\sqrt{3}}\sin 2x \, dx = \Big[-\cos x - x - \tfrac{1}{2\sqrt{3}}\cos 2x \Big]_{\pi/6}^{\pi/2}$$
$$= -\big(0 + \tfrac{\pi}{2} - \tfrac{1}{2\sqrt{3}}\big) + \big(\tfrac{\sqrt{3}}{2} + \tfrac{\pi}{6} + \tfrac{1}{4\sqrt{3}}\big) = \tfrac{3\sqrt{3}}{4} - \tfrac{\pi}{3}\,.$$

(b) Find the area of the region which is bounded by the curves $x = y^3 - 3y$ and $x = y^2 - y$.

Solution: First make a sketch.

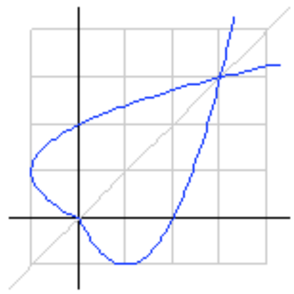


From the graph (or by doing some algebra) we can see the points of intersection, and we can see which curve is greater (farther to the right). The area is given by

$$A = \int_{-1}^{0} (y^3 - 3y) - (y^2 - y)\, dy + \int_0^2 (y^2 - y) - (y^3 - 3y)\, dy$$
$$= \int_{-1}^{0} y^3 - y^2 - 2y\, dy + \int_0^2 -y^3 + y^2 + 2y\, dy$$
$$= \Big[\tfrac{1}{4}y^4 - \tfrac{1}{3}y^3 - y^2\Big]_{-1}^{0} + \Big[-\tfrac{1}{4}y^4 + \tfrac{1}{3}y^3 + y^2\Big]_0^2$$
$$= -\big(\tfrac{1}{4} + \tfrac{1}{3} + 1\big) + \big(-4 + \tfrac{8}{3} + 4\big) = \tfrac{5}{12} + \tfrac{8}{3} = \tfrac{37}{12}\,.$$

(c) Find the area of the region between the curve $y = x(x-2)$ with $x \geq 0$ and the curve $x = y(y-2)$ with $y \geq 0$.

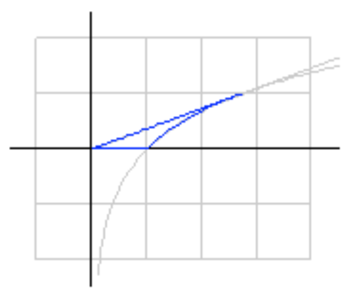
Solution: First we sketch the region.



We note that the region is symmetric in the line $y = x$ so the area is twice the area of the region which lies under $y = x$ and over $y = x(x-2)$: $A = 2\int_0^3 x - x(x-2)\,dx = \int_0^3 6x - 2x^2\,dx = \left[3x^2 - \frac{2}{3}x^3\right]_0^3 = 27 - 18 = 9$.

(d) Find the area of the region bounded by the x-axis, the graph of $y = \ln x$, and by the tangent line to $y = \ln x$ which passes through the origin.

Solution: The equation of the tangent line to $y = \ln x$ at the point $(a, \ln a)$ is $y - \ln a = \frac{1}{a}(x - a)$. Putting in $(x, y) = (0, 0)$ gives $-\ln a = -1$ so that $a = e$, and so the tangent line which passes through the origin is the tangent line at the point $(e, 1)$ and it has equation $y = \frac{1}{e}x$. Now we can sketch the region.



To find the area, it is convenient to treat y as the variable, so the region lies to the left of $x = e^y$ and to the right of $x = e\,y$. The area is $A = \int_{y=0}^1 e^y - e\,y\,dy = \left[e^y - \frac{e}{2}y^2\right]_0^1 = \left(e - \frac{e}{2}\right) - 1 = \frac{e}{2} - 1$.

2: (a) Let R be the region given by $0 \leq x \leq \pi$ and $0 \leq y \leq \sin^2 x$. Find the volume of the solid obtained by revolving R about the x-axis, and find the volume of the solid obtained by revolving R about the y-axis.

Solution: The volume of the solid obtained by revolving R about the x-axis is

$$\begin{aligned} V &= \int_0^\pi \pi \sin^4 x\,dx = \pi \int_0^\pi \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)^2 dx = \pi \int_0^\pi \left(\tfrac{1}{4} - \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x\right) dx \\ &= \pi \int_0^\pi \tfrac{1}{4} - \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right)\ dx = \pi \int_0^\pi \tfrac{3}{8} - \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x\ dx \\ &= \pi \Big[\tfrac{3}{8}\,x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x\Big]_0^\pi = \tfrac{3\pi^2}{8}\,. \end{aligned}$$

The volume of the solid obtained by revolving R about the y-axis is

$$V = \int_0^\pi 2\pi\, x\, \sin^2 x\ dx = \pi \int_0^\pi x \cdot 2\sin^2 x\ dx.$$

We integrate by parts using $u = x$, $du = dx$, $v = x - \frac{1}{2}\sin 2x$ and $dv = (1 - \cos 2x)dx = 2\sin^2 x\,dx$ to get

$$\begin{aligned} V &= \pi \int_0^\pi x \cdot 2\sin^2 x\,dx = \Big[x\left(x - \tfrac{1}{2}\sin 2x\right) - \int x - \tfrac{1}{2}\sin 2x\ dx\Big]_0^\pi \\ &= \pi\Big[x^2 - \tfrac{1}{2}\,x\sin 2x - \tfrac{1}{2}\,x^2 - \tfrac{1}{4}\cos 2x\Big]_0^\pi = \pi\left(\left(\pi^2 - \tfrac{1}{2}\pi^2 - \tfrac{1}{4}\right) - \left(-\tfrac{1}{4}\right)\right) = \tfrac{\pi^3}{2}\,. \end{aligned}$$

(b) Let R be the region $1 \leq x \leq 2$, $0 \leq y \leq \dfrac{1}{x\sqrt{x^2+2x}}$. Find the volume of the solid obtained by revolving R about the x-axis, and find the volume of the solid obtained by revolving R about the y-axis.

Solution: Using cross-sections, the volume of the solid obtained by revolving R about the x-axis is

$$V = \int_{x=1}^2 \pi\left(\frac{1}{x\sqrt{x^2+2x}}\right)^2 dx = \pi \int_1^2 \frac{dx}{x^2(x^2+2x)} = \pi \int_1^2 \frac{dx}{x^3(x+2)}\,.$$

To get $\dfrac{1}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+2}$ we need $Ax^2(x+2) + Bx(x+2) + C(x+2) + Dx^3 = 1$. Equate coefficients to get $A + d = 0$, $2A + B = 0$, $2B + C = 0$ and $2C = 1$. Solve these to get $C = \frac{1}{2}$, $B = -\frac{1}{4}$, $A = \frac{1}{8}$ and $D = -\frac{1}{8}$. Thus we have

$$\begin{aligned} V &= \pi \int_1^2 \frac{\frac{1}{8}}{x} - \frac{\frac{1}{4}}{x^2} + \frac{\frac{1}{2}}{x^3} - \frac{\frac{1}{8}}{x+2}\ dx = \pi\left[\tfrac{1}{8}\ln x + \frac{1}{4x} - \frac{1}{4x^2} - \tfrac{1}{8}\ln(x+2)\right]_1^2 \\ &= \pi\Big(\left(\tfrac{1}{8}\ln 2 + \tfrac{1}{8} - \tfrac{1}{16} - \tfrac{1}{8}\ln 4\right) - \left(\tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{8}\ln 3\right)\Big) = \pi\left(\tfrac{1}{16} + \tfrac{1}{8}\ln\tfrac{3}{2}\right)\,. \end{aligned}$$

Using cylindrical shells, the volume of the solid obtained by revolving R about the y-axis is

$$V = \int_{x=1}^2 2\pi\, x\left(\frac{1}{x\sqrt{x^2+2x}}\right) dx = \int_1^2 \frac{2\pi\,dx}{\sqrt{x^2+2x}}\,.$$

Note that $x^2 + 2x = (x+1)^2 - 1$. Make the substitution $\sec\theta = x + 1$, $\tan\theta = \sqrt{x^2+2x}$, $\sec\theta\tan\theta\,d\theta$ to get

$$\begin{aligned} \int \frac{2\pi\,dx}{\sqrt{x^2+2x}} &= \int \frac{2\pi\,\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int 2\pi\sec\theta\,d\theta \\ &= 2\pi\ln\big|\sec\theta + \tan\theta\big| + c = 2\pi\ln\big|(x+1) + \sqrt{x^2+2x}\big| + c \end{aligned}$$

and so

$$V = \int_1^2 \frac{2\pi\,dx}{\sqrt{x^2+2x}} = 2\pi\Big[\ln\left((x+1) + \sqrt{x^2+2x}\right)\Big]_1^2 = 2\pi\big(\ln(3+\sqrt{8}) - \ln(2+\sqrt{3})\big)\,.$$

3: (a) Let R be the (infinitely long) region given by $0 \le x < \infty$ and $0 \le y \le (x^2+1)^{-3/2}$. Find the area of the region R, and find the volume of the solid obtained by revolving R about the y-axis.

Solution: The area is given by $A = \displaystyle\int_0^\infty \frac{dx}{(x^2+1)^{3/2}}$. Let $\tan\theta = x$ so $\sec\theta = \sqrt{x^2+1}$ and $\sec^2\theta\,d\theta = dx$. Then we have

$$A = \int_0^\infty \frac{dx}{(x^2+1)^{3/2}} = \int_0^{\pi/2} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int_0^{\pi/2} \cos\theta\,d\theta = \Big[\sin\theta\Big]_0^{\pi/2^-} = \lim_{\theta\to\pi/2^-} \sin\theta = 1\,.$$

The volume is $V = \displaystyle\int_0^\infty \frac{2\pi x\,dx}{(x^2+1)^{3/2}}$. Let $u = x^2+1$ so $du = 2x\,dx$. Then we have

$$V = \int_0^\infty \frac{2\pi x\,dx}{(x^2+1)^{3/2}} = \int_1^\infty \frac{\pi\,du}{u^{3/2}} = \int_1^\infty \pi\,u^{-3/2}\,du = \Big[-2\pi\,u^{-1/2}\Big]_1^\infty = \lim_{u\to\infty}\left(-2\pi\,u^{-1/2}\right) + 2\pi = 2\pi\,.$$

(b) Let R be the (infinitely long) region $0 \le x < \infty$, $0 \le y \le \dfrac{2\sqrt{x}}{4+x^2}$. Find the area of R and find the volume of the solid obtained by revolving R about the x-axis.

Solution: The area is

$$A = \int_0^\infty \frac{2\sqrt{x}}{4+x^2}\,dx\,.$$

Make the substitution $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$ to get

$$\begin{aligned}
A &= \int_{u=0}^\infty \frac{2u\cdot 2u\,du}{4+u^4} = \int_0^\infty \frac{4u^2\,du}{(u^2-2u+2)(u^2+2u+2)} \\
&= \int_0^\infty \frac{u}{u^2-2u+2} - \frac{u}{u^2+2u+2}\,du = \int_0^\infty \frac{(u-1)+1}{u^2-2u+2} - \frac{(u+1)-1}{u^2+2u+2}\,du \\
&= \int_0^\infty \frac{u-1}{u^2-2u+2} + \frac{1}{u^2-2u+2} - \frac{u+1}{u^2+2u+2} + \frac{1}{u^2+2u+2}\,du \\
&= \Big[\tfrac12\ln(u^2-2u+2) + \tan^{-1}(u-1) - \tfrac12\ln(u^2+2u+2) + \tan^{-1}(u+1)\Big]_0^\infty \\
&= \left[\tfrac12\ln\left(\frac{u^2-2u+2}{u^2+2u+2}\right) + \tan^{-1}(u-1) + \tan^{-1}(u+1)\right]_0^\infty \\
&= \left(0 + \tfrac{\pi}{2} + \tfrac{\pi}{2}\right) - \left(0 - \tfrac{\pi}{4} + \tfrac{\pi}{4}\right) = \pi\,.
\end{aligned}$$

Using cross-sections, the volume is

$$V = \int_{x=0}^\infty \pi\left(\frac{2\sqrt{x}}{4+x^2}\right)^2 dx = \int_0^\infty \frac{4\pi\,x}{(4+x^2)^2}\,dx\,.$$

Make the substitution $2\tan\theta = x$, $2\sec\theta = \sqrt{4+x^2}$, $2\sec\theta\tan\theta\,d\theta$ to get

$$V = \int_{\theta=0}^{\pi/2} \frac{4\pi\cdot 2\tan\theta\cdot 2\sec^2\theta\,d\theta}{16\sec^4\theta} = \pi\int_0^{\pi/2} \frac{\tan\theta}{\sec^2\theta}\,d\theta = \pi\int_0^{\pi/2} \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta}$$

Make the substitution $u = \sec\theta$, $du = \sec\theta\tan\theta\,d\theta$ to get

$$V = \pi\int_{u=1}^\infty \frac{du}{u^3} = \pi\left[\frac{-1}{2u^2}\right]_1^\infty = \tfrac{\pi}{2}\,.$$

4: (a) Let S be the solid $0 \le x \le 2$, $-x \le y \le x$, $0 \le z \le x^2 - y^2$. Find the volume of S.

Solution: We provide two solutions. The cross-section at x perpendicular to the x-axis is shaped like the region in the yz-plane given by $-x \le y \le x$, $0 \le z \le x^2 - y^2$. The area of this cross-section is

$$A(x) = \int_{y=-x}^{x} x^2 - y^2 \, dy = \Big[x^2 y - \tfrac{1}{3}y^3\Big]_{y=-x}^{x} = \big(x^3 - \tfrac{1}{3}x^3\big) - \big(-x^3 + \tfrac{1}{3}x^3\big) = \tfrac{4}{3}\,x^3\,,$$

and so the volume is

$$V = \int_{x=0}^{2} A(x)\, dx = \int_0^2 \tfrac{4}{3}x^3\, dx = \Big[\tfrac{1}{3}x^4\Big]_0^2 = \tfrac{16}{3}\,.$$

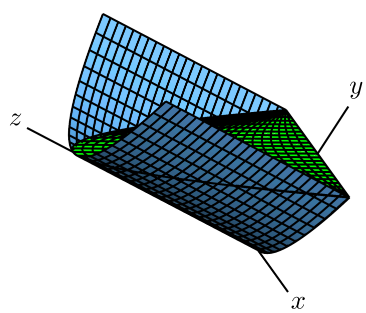
The cross-section at y perpendicular to the y-axis is shaped like the region in the xz-plane given by $|y| \le x \le 2$, $0 \le z \le x^2 - y^2$. The area of this cross-section is

$$A(y) = \int_{x=|y|}^{2} x^2 - y^2 \, dx = \Big[\tfrac{1}{3}x^3 - y^2 x\Big]_{x=|y|}^{2} = \big(\tfrac{8}{3} - 2y^2\big) - \big(\tfrac{1}{3}y^2|y| - y^2|y|\big) = \tfrac{8}{3} - 2y^2 + \tfrac{2}{3}y^2|y|\,,$$

and so the volume is

$$\begin{aligned} V &= \int_{y=-2}^{2} \tfrac{8}{3} - 2y^2 + \tfrac{2}{3}y^2|y| \, dy = \int_{y=-2}^{2} \tfrac{8}{3} - 2y^2 = \tfrac{2}{3}y^3 \, dy + \int_0^2 \tfrac{8}{3} - 2y^2 + \tfrac{2}{3}y^3 \, dy \\ &= \Big[\tfrac{8}{3}y - \tfrac{2}{3}y^3 - \tfrac{1}{6}y^4\Big]_{-2}^{0} + \Big[\tfrac{8}{3} - \tfrac{2}{3}y^3 + \tfrac{1}{6}y^4\Big]_0^2 = -\big(-\tfrac{16}{3} + \tfrac{16}{3} - \tfrac{8}{3}\big) + \big(\tfrac{16}{3} - \tfrac{16}{3} + \tfrac{8}{3}\big) = \tfrac{16}{3}\,. \end{aligned}$$

(b) A scoop is in the shape of the parabolic surface $-1 \le x \le 1$, $y = x^2$, $0 \le z \le 2$, with one end covered by the region $-1 \le x \le 1$, $x^2 \le y \le 1$, $z = 0$. How much water can the scoop hold?



Solution: We provide 3 solutions. The cross-section at x perpendicular to the x-axis is a triangle with base $1 - x^2$ and height $2 - 2x^2$, so the area of the cross-section is $A(x) = \frac{1}{2}(1 - x^2)(2 - 2x^2) = 1 - 2x^2 + x^4$. The volume is

$$V = 2\int_0^1 A(x)\, dx = 2\int_0^1 1 - 2x^2 + x^4 \, dx = 2\Big[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\Big]_0^1 = 2\big(1 - \tfrac{2}{3} + \tfrac{1}{5}\big) = \tfrac{16}{5}\,.$$

The cross-section at y perpendicular to the y-axis is a rectangle with base $2\sqrt{y}$ and height $2 - 2y$, so the area of the cross-section is $A(y) = 4y^{1/2} - 4y^{3/2}$, and hence the volume is

$$V = \int_0^1 4y^{1/2} - 4y^{3/2} \, dy = 8\Big[\tfrac{1}{3}y^{3/2} - \tfrac{1}{5}y^{5/2}\Big]_0^1 = 8\big(\tfrac{1}{3} - \tfrac{1}{5}\big) = \tfrac{16}{15}\,.$$

The cross-section at z perpendicular to the z-axis is the region which lies between the parabola $y = x^2$ and the line $y = 1 - \frac{1}{2}z$, so the area of the cross-section is

$$A(z) = 2\int_0^{\sqrt{1-z/2}} \big(1 - \tfrac{1}{2}z\big) - x^2 \, dx = 2\Big[\big(1 - \tfrac{1}{2}z\big)x - \tfrac{1}{3}x^3\Big]_0^{\sqrt{1-z/2}} = \tfrac{4}{3}\big(1 - \tfrac{1}{2}z\big)^{3/2}\,.$$

Thus the volume is

$$V = \int_0^2 \tfrac{4}{3}\big(1 - \tfrac{1}{2}z\big)^{3/2} dx = -\tfrac{8}{3}\int_1^0 u^{3/2}\, du = \tfrac{8}{3}\Big[\tfrac{2}{5}u^{5/2}\Big]_0^1 = \tfrac{16}{15}\,,$$

where we made the substitution $u = 1 - \frac{1}{2}z$.

5: (a) Find the length of the curve $y=\sqrt{4x-x^2}$ with $0\le x\le 3$.

Solution: We have $y'=\dfrac{2-x}{\sqrt{4x-x^2}}$ so the arclength is given by

$$L=\int_0^3\sqrt{1+\left(\frac{2-x}{\sqrt{4x-x^2}}\right)^2}\,dx=\int_0^3\sqrt{1+\frac{4-4x+x^2}{4x-x^2}}\,dx=\int_0^3\sqrt{\frac{4}{4x-x^2}}\,dx=\int_0^3\frac{2\,dx}{\sqrt{4x-x^2}}\,.$$

Note that $4x-x^2=4-(x-2)^2$. Let $2\sin\theta=x-2$ so that $2\cos\theta=\sqrt{4x-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$L=\int_0^3\frac{2\,dx}{\sqrt{4x-x^2}}=\int_{-\pi/2}^{\pi/6}\frac{4\cos\theta\,d\theta}{2\cos\theta}=\int_{-\pi/2}^{\pi/6}2\,d\theta=\Big[2\theta\Big]_{-\pi/2}^{\pi/6}=\tfrac{\pi}{3}+\pi=\tfrac{4\pi}{3}\,.$$

(b) Find the length of the curve $0\le x\le 8$, $y=3\,x^{2/3}$.

Solution: We have $y'=2\,x^{-1/3}$, so the arclength is given by

$$L=\int_0^8\sqrt{1+(2x^{-1/3})^2}\,dx=\int_0^8\sqrt{1+\frac{4}{x^{2/3}}}\,dx=\int_0^8\frac{\sqrt{x^{2/3}+4}}{x^{1/3}}\,dx\,.$$

Let $u=x^{2/3}+4$ so that $du=\frac{2}{3}\,x^{-1/3}\,dx$, that is $\frac{3}{2}\,du=\dfrac{1}{x^{1/3}}\,dx$. Then we have

$$L=\int_0^8\frac{\sqrt{x^{2/3}+4}}{x^{1/3}}\,dx=\int_4^8\sqrt{u}\cdot\tfrac{3}{2}\,du=\int_4^8\tfrac{3}{2}\,u^{1/2}\,du=\Big[u^{3/2}\Big]_4^8=16\sqrt{2}-8\,.$$

(c) Find the length of the curve $0\le x\le\ln 2$, $y=e^x$.

Solution: Since $y'=e^x$, the length is $L=\displaystyle\int_{x=0}^{\ln 2}\sqrt{1+e^{2x}}\,dx$. Let $u^2=1+e^{2x}$, $2u\,du=2e^{2x}\,dx$, and $dx=\dfrac{u}{e^{2x}}\,du=\dfrac{u}{u^2-1}\,du$ to get $L=\displaystyle\int_{u=\sqrt{2}}^{\sqrt{5}}\frac{u^2\,du}{u^2-1}=\int_{\sqrt{2}}^{\sqrt{5}}1+\frac{1}{u^2-1}\,du=\int_{\sqrt{2}}^{\sqrt{5}}1+\frac{\frac12}{u-1}-\frac{\frac12}{u+1}\,du$
$=\left[u+\frac12\,\ln\left|\dfrac{u-1}{u+1}\right|\right]_{\sqrt{2}}^{\sqrt{5}}=\sqrt{5}+\frac12\ln\left(\frac{\sqrt5-1}{\sqrt5+1}\right)-\sqrt2-\frac12\ln\left(\frac{\sqrt2-1}{\sqrt2+1}\right)=\sqrt5-\sqrt2+\ln\left(\frac{\sqrt5-1}{2(\sqrt2-1)}\right).$

(d) Find the length of the curve $y=\ln x$ with $1\le x\le\sqrt3$.

Solution: We have $y'=\frac{1}{x}$ so the length is

$$L=\int_{x=1}^{\sqrt3}\sqrt{1+\frac{1}{x^2}}\,dx=\int_1^{\sqrt3}\frac{\sqrt{x^2+1}}{x}\,dx\,.$$

Make the substitution $\tan\theta=x$, $\sec\theta=\sqrt{x^2+1}$, $\sec\theta\,d\theta=dx$, and then make the substitution $u=\sec\theta$, $du=\sec\theta\tan\theta\,d\theta$ to get

$$\begin{aligned}L&=\int_{\theta=\pi/4}^{\pi/3}\frac{\sec^3\theta}{\tan\theta}\,d\theta=\int_{\pi/4}^{\pi/3}\frac{\sec^2\theta}{\sec^2\theta-1}\,\sec\theta\tan\theta\,d\theta=\int_{u=\sqrt2}^{2}\frac{u^2\,du}{u^2-1}=\int_{\sqrt2}^{2}1+\frac{1}{u^2-1}\,du\\&=\int_{\sqrt2}^{2}1+\frac{\frac12}{u-1}-\frac{\frac12}{u+1}\,du=\left[u+\tfrac12\ln\frac{u-1}{u+1}\right]_{\sqrt2}^{2}=\left(2+\tfrac12\ln\tfrac13\right)-\left(\sqrt2+\tfrac12\ln\tfrac{\sqrt2-1}{\sqrt2+1}\right)\\&=2-\sqrt2+\ln\tfrac{\sqrt2+1}{\sqrt3}\,.\end{aligned}$$

6: (a) Find the area of the surface which is obtained by revolving the curve $y = \tan x$ with $0 \le x \le \frac{\pi}{6}$ about the x-axis.

Solution: We have $y' = \sec^2 x$ and so the surface area is $A = \int_0^{\pi/6} 2\pi \tan x \sqrt{1+\sec^4 x}\, dx$. We solve this integral by making the following sequence of substitutions: first, we let $u = \sec x$ so $du = \sec x \tan x\, dx$; then we let $v = u^2$ so $dv = 2u\, du$; next we let $\tan\theta = v$ so $\sec\theta = \sqrt{1+v^2}$ and $\sec^2\theta\, d\theta = dv$; and finally we let $w = \sec\theta$ so $dw = \sec\theta\tan\theta\, d\theta$. (All of this could be accomplished in a single step by making the substitution $w = \sqrt{1+\sec^4 x}$). Also note that when $\theta = \tan^{-1}\frac{4}{3}$, we have $\sec\theta = \frac{5}{3}$ (as can be seen using a right-angled triangle with sides of lengths 3, 4 and 5). We have

$$\begin{aligned}
A &= \int_0^{\pi/6} 2\pi \tan x \sqrt{1+\sec^4 x}\, dx = \int_0^{\pi/6} \frac{2\pi\sqrt{1+\sec^4 x}}{\sec x} \sec x \tan x\, dx \\
&= \int_1^{2/\sqrt{3}} \frac{2\pi\sqrt{1+u^4}}{u}\, du = \int_1^{2/\sqrt{3}} \frac{\pi\sqrt{1+u^4}}{u^2}\, 2u\, du = \int_1^{4/3} \frac{\pi\sqrt{1+v^2}}{v}\, dv \\
&= \int_{\pi/4}^{\tan^{-1}(4/3)} \frac{\pi\sec\theta\cdot\sec^2\theta\, d\theta}{\tan\theta} = \int_{\pi/4}^{\tan^{-1}(4/3)} \frac{\pi\sec^3\theta\tan\theta\, d\theta}{\tan^2\theta} = \int_{\pi/4}^{\tan^{-1}(4/3)} \frac{\pi\sec^2\theta}{\sec^2\theta - 1}\sec\theta\tan\theta\, d\theta \\
&= \int_{\sqrt{2}}^{5/3} \frac{\pi\, w^2}{w^2-1}\, dw = \int_{\sqrt{2}}^{5/3} \pi\left(1 + \frac{1}{w^2-1}\right) dw = \int_{\sqrt{2}}^{5/3} \pi\left(1 + \frac{\frac{1}{2}}{w-1} - \frac{\frac{1}{2}}{w+1}\right) dw \\
&= \pi\Big[w + \tfrac{1}{2}\ln(w-1) - \tfrac{1}{2}\ln(w+1)\Big]_{\sqrt{2}}^{5/3} = \pi\left[w + \tfrac{1}{2}\ln\left(\frac{w-1}{w+1}\right)\right]_{\sqrt{2}}^{5/3} \\
&= \pi\Big(\left(\tfrac{5}{3} + \tfrac{1}{2}\ln\tfrac{1}{4}\right) - \left(\sqrt{2} + \tfrac{1}{2}\ln\tfrac{\sqrt{2}-1}{\sqrt{2}+1}\right)\Big) = \pi\left(\tfrac{5}{3} - \sqrt{2} + \ln\left(\tfrac{\sqrt{2}+1}{2}\right)\right),
\end{aligned}$$

since $\frac{1}{2}\ln\frac{1}{4} = -\ln 2$ and $-\frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = -\frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = -\frac{1}{2}\ln\left(\frac{1}{(\sqrt{2}+1)^2}\right) = \ln(\sqrt{2}+1)$.

(b) Find the area of the surface which is obtained by revolving the curve $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $y = \cos x$ about the x-axis.

Solution: We have $y' = -\sin x$. By symmetry, the surface area is given by

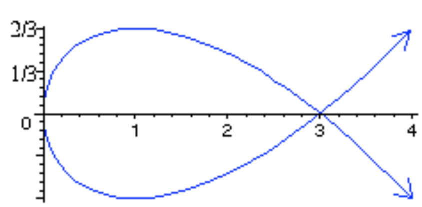
$$A = 2\int_0^{\pi/2} 2\pi\cos x\sqrt{1+\sin^2 x}\, dx = \int_0^{\pi/2} 4\pi\cos x\sqrt{1+\sin^2 x}\, dx = \int_0^1 4\pi\sqrt{1+u^2}\, du\,,$$

where $u = \sin x$ so $du = \cos x\, dx$. Now let $\tan\theta = u$ so that $\sec\theta = \sqrt{1+u^2}$ and $\sec^2\theta\, d\theta = du$. Then

$$A = \int_0^{\pi/4} 4\pi\sec^3\theta\, d\theta = \Big[2\pi\big(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\big)\Big]_0^{\pi/4} = 2\pi\big(\sqrt{2} + \ln(\sqrt{2}+1)\big)\,.$$

(c) The curve $9y^2 = x(x-3)^2$ has a loop in it. Find the area of the surface obtained by rotating the loop about the x-axis.

Solution: First we make a sketch.



The equation of the top half of the loop is $y = \frac{1}{3}\sqrt{x}(3-x) = x^{1/2} - \frac{1}{3}x^{3/2}$, and $y' = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} = \dfrac{1-x}{2\sqrt{x}}$. So we have $dL = \sqrt{1 + \dfrac{(1-x)^2}{4x}}\, dx = \dfrac{1+x}{2\sqrt{x}}\, dx$. The surface area is

$$\begin{aligned}
A &= \int_0^3 2\pi y\, dL = \tfrac{\pi}{3}\int_0^3 (3-x)(1+x)\, dx = \tfrac{\pi}{3}\int_0^3 3 + 2x - x^2\, dx \\
&= \tfrac{\pi}{3}\Big[3x + x^2 - \tfrac{1}{3}x^3\Big]_0^3 = \tfrac{\pi}{3}\big(9 + 9 - 9\big) = 3\pi\,.
\end{aligned}$$

7: Let $S = \{u \in \mathbb{R}^3 \,|\, |u| = R\}$ be the sphere of radius R centered at the origin in $\mathbb{R}^3$.

(a) Find the area of a slice of thickness h on the surface of the sphere S (that is the area of the portion of the sphere which lies between two parallel planes separated by a distance of h).

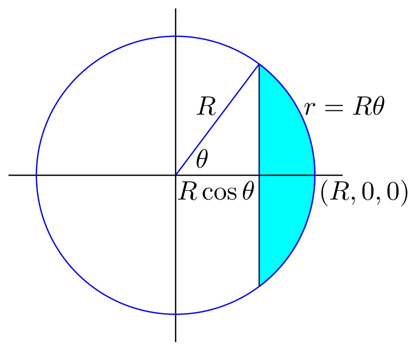
Solution: We take the two planes to be perpendicular to the x-axis, say with one plane at $x = a$ and the other at $x = b$ with $-R \le a \le b \le R$, so the thickness of the slice is $h = b - a$. The portion of the sphere between the planes is obtained by revolving the curve $z = \sqrt{R^2 - x^2}$ (in the xz-plane) with $a \le x \le b$ about the x-axis. We have $z' = \frac{x}{\sqrt{R^2 - x^2}}$ so the area is

$$A = \int_{x=a}^{b} 2\pi z(x)\sqrt{1 + z'(x)^2}\,dx = \int_a^b 2\pi\sqrt{R^2 - x^2}\,\sqrt{1 + \frac{x^2}{R^2 - x^2}}\,dx = \int_a^b 2\pi R\,dx = 2\pi R(b-a) = 2\pi Rh.$$

We remark that it might be surprising to see that the area only depends on the thickness $h = b - a$ and not on the position of the slice (that is not on the value of a). We also remark that the entire sphere is a slice of thickness $h = 2R$ so its area is $A = 4\pi R$.

(b) Find the circumference of, and the area inside, a spherical circle of radius r on S (the spherical circle of radius r centred at the point $u \in S$ is the set of points $v \in S$ such that $R\theta = r$ where θ is the angle between the vectors $u, v \in \mathbb{R}^3$ which is given by $\theta = \cos^{-1} \frac{u \bullet v}{|u|\,|v|}$).

Solution: Rotate the sphere so that u (the center of the disc) is at $(R, 0, 0)$.



The spherical disc is the slice between the planes $x = a$ and $x = b$ with $a = R\cos\theta$ and $b = R$, and the spherical circle is an ordinary Euclidean circle, in the plane $x = a$, of Euclidean radius $r = R\sin\theta$, and so the circumference and area are given by

$$C = 2\pi R\sin\theta = 2\pi \sin\tfrac{r}{R}$$
$$A = 2\pi h = 2\pi(b-a) = 2\pi(R - R\cos\theta) = 2\pi R\big(1 - \cos\tfrac{r}{R}\big).$$

(c) Find the area of a spherical triangle with interior angles α, β and γ (given three non-coplanar points $u, v, w \in S$, the spherical triangle with vertices at $u, v, w \in S$ has three spherical edges; the spherical edge through u and v is the set of points on S which lie on the plane through the origin which contains u and v, and similarly for the other two edges; the angle at u in the spherical triangle is the angle between the two spherical edges at u, which is the same as the angle between the corresponding planes through the origin).

Solution: Let T be the spherical triangle with vertices at u, v and w with angles α, β and γ. Let H_α be the hemisphere which contains u whose boundary is the line through v and w, and let $-H_\alpha$ be the opposite hemisphere. Define H_β, $-H_\beta$, H_γ and $-H_\gamma$ similarly. Let $W_\alpha = (H_\beta \cap H_\gamma) \cup (-H_\beta \cap -H_\gamma)$, and define W_β and W_γ similarly. By looking at the sphere with the vector u pointing towards us, we can see that the double wedge W_α covers $\frac{\alpha}{\pi}$ of the entire sphere and so its area is $A_\alpha = \frac{\alpha}{\pi} \cdot 4\pi R^2 = 4R^2\alpha$. Similarly the areas of the double wedges W_β and W_γ are equal to $A_\beta = 4R^2\beta$ and $A_\gamma = 4R^2\gamma$. Notice that when we shade each of the double wedges W_α, W_β and W_γ, the triangle T and its opposite triangle $-T$ are each shaded three times while the rest of the sphere is shaded once. It follows that if we let $O = 4\pi R$ be the area of the entire sphere and we let Δ be the area of the triangle T (which is equal to the area of the opposite triangle $-T$) then we have $A_\alpha + A_\beta + A_\gamma = O + 4\Delta$, that is $4R^2\alpha + 4R^2\beta + 4R^2\gamma = 4\pi R^2 + 4T$, hence $T = \big((\alpha + \beta + \gamma) - \pi\big)R^2$.

8: (a) A tank, shaped like a lower hemisphere of radius R, is filled with a liquid of density ρ. Find the work done when the tank is emptied by pumping the liquid to the level of the top of the tank.

Solution: Choose the x-axis to point downwards with the origin at the center of the top of the hemisphere. The cross-section at x is a circle of radius $r(x) = \sqrt{R^2 - x^2}$ and the cross-ssectional area is

$$A(x) = \pi\, r(x)^2 = \pi(R^2 - r^2)$$

A thin slice of thickness Δx has volume $\Delta V = A(x)\,\Delta x$ and mass $\Delta M = \rho\, A(x)\,\Delta x$. The work done to lift the liquid in this slice to the top of the tank (where $x = 0$) is $\Delta W = gx\,\Delta M = \rho g x\, A(x)\,\Delta x$. The total work done is

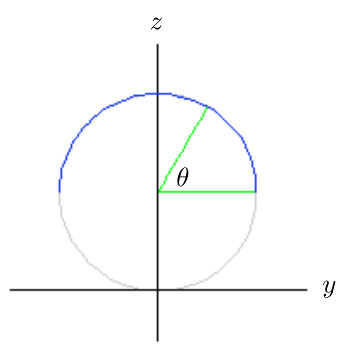
$$W = \int_0^R \rho g\, x\, A(x)\, dx = \int_0^R \pi\rho g\, x(R^2 - x^2)\, dx = \pi\rho g \int_0^R R^2 x - x^3\, dx$$

$$= \pi\rho g\Big[\tfrac{1}{2}\, R^2x^2 - \tfrac{1}{4}\, x^4\Big]_0^R = \pi\rho g\left(\tfrac{1}{2}\, R^4 - \tfrac{1}{4}\, R^4\right) = \tfrac{1}{4}\,\pi\rho g\, R^4\,.$$

(b) A chain, of length π and mass M, lies in the xy-plane. Find the work required to lift the chain and lie it along the top half of the circle $x = 0$, $y^2 + (z-1)^2 = 1$.

Solution: Let θ be as shown below. For a thin slice of the chain (when it is lying on the top half of the circle) at position θ of thickness $\Delta\theta$, the mass of the slice is $\Delta M = \frac{M}{\pi}\Delta\theta$, and the height of the slice above the xy-plane is $z = 1 + \sin\theta$, so the work done in lifting the slice from the xy-plane is $\Delta W = gz\Delta M = \frac{gM}{\pi}(1 + \sin\theta)\Delta\theta$. The total work is

$$W = \int_{\theta=0}^{\theta=\pi} \tfrac{gM}{\pi}(1 + \sin\theta)\, d\theta = \tfrac{gM}{\pi}\Big[\theta - \cos\theta\Big]_{\theta=0}^{\pi} = \tfrac{gM}{\pi}(\pi + 2)\,.$$



9: (a) A trough, in the shape of the bottom half of a cylinder, of radius r and length l, lying on its side, is filled with a liquid of density ρ. Find the force exerted by the liquid on each of the two semi-circular ends of the tank, and find the outward force exerted by the liquid on the semi-cylindrical base of the tank.

Solution: Choose the x-axis to point downwards with the origin at the center of the top of one of the semi-circulsr ends of the tank. Consider a thin horizontal strip along one of the ends of the tank at a depth x of thickness Δx. The pressure at all points at a depth x is $P = \rho g x$, and the area of the strip is $\Delta A = 2\sqrt{r^2 - x^2}\,\Delta x$, and so the force exerted by the liquid on this strip is

$$\Delta F = P\Delta A = 2\rho g\, x\sqrt{r^2 - x^2}\Delta x\,.$$

The total force on the end of the trough is

$$F = \int_{x=0}^{r} 2\rho g\, x\sqrt{r^2 - x^2}\, dx = \rho g\Big[-\tfrac{2}{3}(r^2 - x^2)^{3/2}\Big]_0^r = \tfrac{2}{3}\,\rho g r^3\,.$$

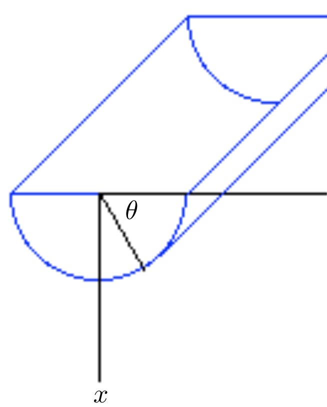
$\Big($If you cannot solve the integral $\displaystyle\int x\sqrt{r^2 - x^2}\, dx$ by inspection, then try the substitution $u = r^2 - x^2$, or try the substitution $r\sin\theta = x\Big)$.

Now, let θ be the angle as shown below. Consider a thin horizontal strip along the semi-cylindrical wall of the tank, at a depth $x = r\sin\theta$, and of thickness $r\Delta\theta$. The pressure at all points at this depth is $P = \rho g x = \rho g r\sin\theta$, and the area of the strip is $\Delta A = lr\Delta\theta$, and so the outward force exerted by the liquid on this strip is

$$\Delta F = P\Delta A = \rho g l r^2 \sin\theta\,\Delta\theta\,.$$

The total outward force is

$$F = \int_{\theta=0}^{\pi} \rho g l r^2 \sin\theta\, d\theta = \rho g l r^2\Big[-\cos\theta\Big]_0^\pi = 2\rho g l r^2\,.$$



(b) A flat circular disc of radius $\sqrt{5}$ lies in the xy-plane in the region $x^2+y^2 \leq 5$, $z=0$. The disc has varying density. The planar density (mass per unit area) at points which lie a distance r from the center is given by $\rho(r) = \dfrac{1}{3+r^2}$. A small object of mass m lies above the disc at the point $(0,0,2)$. Find the gravitational force exerted by the disc on the object.

Solution: By symmetry, the horizontal component of the force is zero, so we only consider the vertical component of the force. Consider a thin ring (annulus) of radius r and thickness Δr. The area of the ring is $\Delta A = 2\pi r \Delta r$ so its mass is $\Delta M = \rho(r)\Delta A = \frac{2\pi r}{3+r^2}\Delta r$. The distance from all points along this ring to the small object is $d = \sqrt{4+r^2}$ and the angle θ, between a line from a point on the ring to the object and the vertical axis, is given by $\cos\theta = \frac{2}{d}$, so the contribution made by the points on the ring to the vertical component of the force is $\Delta F = \frac{Gm\Delta M}{d^2}\cos\theta = \frac{4\pi G m\, r}{(3+r^2)(4+r^2)^{3/2}}\Delta r$. The total force is

$$F = \int_{r=0}^{\sqrt{5}} \frac{4\pi G m\, r\, dr}{(3+r^2)(4+r^2)^{3/2}}\,.$$

Make the substitution $u = \sqrt{4+r^2}$, $u^2 = 4+r^2$, $2u\,du = 2r\,dr$ to get

$$\begin{aligned}
F &= \int_{u=2}^{3} \frac{4\pi G m\, u\, du}{(u^2-1)u^3} = 2\pi G m \int_{u=2}^{3} \frac{2\,du}{(u-1)(u+1)u^2} = 2\pi G m \int_{u=2}^{3} \frac{2}{u^2-1} - \frac{2}{u^2}\,du \\
&= 2\pi G m \int_{u=2}^{3} \frac{1}{u-1} - \frac{1}{u+1} - \frac{2}{u^2}\,du = 2\pi G m \left[\ln\left(\frac{u-1}{u+1}\right) + \frac{2}{u}\right]_{u=2}^{3} \\
&= 2\pi G m \left(\ln\tfrac{1}{2} + \tfrac{2}{3} - \ln\tfrac{1}{3} - 1\right) = 2\pi G m \left(\ln\tfrac{3}{2} - \tfrac{1}{3}\right)\,.
\end{aligned}$$

10: In many situations, we can find a length or an area or a volume either by integrating with respect to one variable, say x, or by integrating with respect to another variable, say y. We expect to obtain the same answer using either method. Let us show that our expectation is justified in a few circumstances. Let $f:[a,b]\to[c,d]$ be strictly decreasing with $f(a)=d$ and $f(b)=c$, and let $g=f^{-1}:[c,d]\to[a,b]$.

(a) Suppose f and g are differentiable and consider the area of the region $a\le x\le b$, $c\le y\le f(x)$. Use Substitution and Integration by Parts to show that

$$\int_{x=a}^{b}\big(f(x)-c\big)\,dx=\int_{y=c}^{d}\big(g(y)-a\big)\,dy$$

.

Solution: Making the substitution $y=f(x)$ and then integrating by parts using $u=x-a$ and $dv=f'(x)\,dx$ we have

$$\begin{aligned}\int_{y=c}^{d}(g(y)-a)\,dy&=\int_{x=g(c)}^{g(d)}\big(g(f(x))-a\big)f'(x)\,dx=\int_{x=b}^{a}(x-a)f'(x)\,dx\\&=\Big[(x-a)f(x)\Big]_{x=b}^{a}-\int_{x=b}^{a}f(x)\,dx=-(b-a)f(b)+\int_{x=a}^{b}f(x)\,dx\\&=-(b-a)c+\int_{x=a}^{b}f(x)\;dx=-\int_{x=a}^{b}cx\,dx+\int_{x=a}^{b}f(x)\,dx=\int_{x=a}^{b}(f(x)-c)\,dx.\end{aligned}$$

(b) Suppose f and g are differentiable and consider the length of the curve $a\le x\le b$, $y=f(x)$. Show that

$$\int_{x=a}^{b}\sqrt{1+f'(x)^2}\,dx=\int_{y=c}^{d}\sqrt{1+g'(y)^2}\,dy.$$

Solution: This is on Assignment 3.

(c) Suppose f and g are differentiable and consider the volume of the solid obtained by revolving the region $a\le x\le b$, $c\le y\le f(x)$ about the x-axis. Show that

$$\int_{x=a}^{b}\pi\big(f(x)^2-c^2\big)\,dx=\int_{y=c}^{d}2\pi\,y\,\big(g(y)-a\big)\,dy.$$

Solution: This is on Assignment 3

(d) Suppose f and g are continuous and consider the area of the region $a\le x\le b$, $c\le y\le f(x)$. Use properties of the Riemann integral to prove that

$$\int_{x=a}^{b}\big(f(x)-c\big)\,dx=\int_{y=c}^{d}\big(g(y)-a\big)\,dy$$

.

Solution: This is on Assignment 3

11: We defined the length of the curve $y = f(x)$ on $[a,b]$ only in the case that f is differentiable. In fact we can give a more general definition. Let $f : [a,b] \to \mathbb{R}$ be any function. For a partition $X = \big(x_0, x_1, \cdots, x_n\big)$ of $[a,b]$, define

$$L(f,X) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

and define the **length** of the curve $y = f(x)$ on $[a,b]$ to be

$$L(f) = \sup\big\{L(f,X)\,\big|\,X \text{ is a partition of } [a,b]\big\}$$

(note that the supremum can be infinite). We say that f is **rectifiable** when $L(f)$ is finite.

(a) Show that if f is $\mathcal{C}^1$ (which means that f is differentiable and f' is continuous on $[a,b]$) then f is rectifiable and

$$L(f) = \int_{x=a}^{b} \sqrt{1 + f'(x)^2}\,dx.$$

Solution: Suppose that f is $\mathcal{C}^1$. Given a partition $X = (x_0, x_1, \cdots, x_n)$ of $[a,b]$, by the MVT (the Mean Value Theorem) we can choose $c_k \in [x_{k-1}, x_k]$ such that $(f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1})$ and then, letting $M = \max\big\{|f'(x)|\,\big|\,x \in [a,b]\big\}$, we have

$$\begin{aligned} L(f,X) &= \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + f'(c_k)^2 (x_k - x_{k-1})^2} \\ &= \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2}\,(x_k - x_{k-1}) = \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2}\,\Delta_k x \le \sum_{k=1}^{n} M \Delta_k x = M(b-a). \end{aligned}$$

It follows that $L(f) = \sup\big\{L(f,X)\,\big|\,X \text{ is a partition of } [a,b]\big\} \le M(b-a)$, so f is rectifiable.

Let $X = (x_0, x_1, \cdots, x_n)$ be a partition of $[a,b]$, and let Y be a partition which is obtained by adding one more point, say $Y = (x_0, x_1, \cdots, x_{k-1}, y, x_k, \cdots, x_n)$. Recall that for a triangle in the plane with side lengths a, b and c, if θ is the angle opposite the side of length a, then by the Law of Cosines we have $a^2 = b^2 + c^2 - 2bc\cos\theta \le b^2 + c^2 + 2bc = (b+c)^2$ so that $a \le b + c$. Applying this to the triangle with vertices at $(x_{k-1}, f(x_{k-1}))$, $(y, f(y))$ and $(x_k, f(x_k))$ we obtain

$$\sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1})^2} \le \sqrt{(x_k - y)^2 + (f(x_k) - f(y))^2} + \sqrt{(y - x_{k-1})^2 + (f(y) - f(x_{k-1})^2}$$

and hence that $L(f,X) \le L(f,Y)$. Since $L(f,X) \le L(f,Y) \le L(f)$ whenever Y is obtained by adding one point to X, it follows (by induction) that if Z is any partition with $X \subseteq Z$ we have

$$L(f,X) \le L(f,Z) \le L(f).$$

Let $\epsilon > 0$. Since $f'(x)$ is continuous, the function $\sqrt{1 + f'(x)^2}$ is continuous, hence integrable. Choose $\delta > 0$ so that for every partition X of $[a,b]$ with $|X| < \delta$ and for all sample ponts $t_k \in [x_{k-1}, x_k]$ we have

$$\left| \int_a^b \sqrt{1 + f'(x)^2}\,dx - \sum_{k=1}^{n} \sqrt{1 + f'(t_k)^2}\,\Delta_k x \right| < \tfrac{\epsilon}{2}.$$

Choose a partition X_1 with $|X_1| < \delta$, and choose a partition X_2 such that $L(f) - \frac{\epsilon}{2} < L(f, X_2) \le L(f)$, and let $X = X_1 \cup X_2$. Since $X_1 \subseteq X$ we have $|X| < \delta$ and since $X_2 \subseteq X$ we have $L(f) - \frac{\epsilon}{2} < L(f, X_2) \le L(f,X) \le L(f)$. Say $X = (x_0, x_1, \cdots, x_n)$ and choose sample points $c_k \in [x_{k-1}, x_k]$ using the Mean Value Theorem, as above, so that $f(x_k) - f(x_{k-1}) = f'(c_k)\Delta_k x$. As shown above, we have $L(f,X) = \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2}\,\Delta_k x$, and so

$$\begin{aligned} \left| L(f) - \int_a^b \sqrt{1 + f'(x)^2}\,dx \right| &\le \left| L(f) - \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2}\,\Delta_k x \right| + \left| \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2}\,\Delta_k x - \int_a^b \sqrt{1 + f'(x)^2}\,dx \right| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

(b) Define $f : [0,1] \to \mathbb{R}$ by $f(0) = 0$ and $f(x) = x^2 \cos \frac{\pi}{x^2}$ when $x \neq 0$. Show that f is differentiable on $[0,1]$ but not rectifiable on $[0,1]$.

Solution: When $x \neq 0$, f is differentiable at x with $f'(x) = 2x \cos \frac{\pi}{x^2} - \frac{2\pi}{x} \sin \frac{\pi}{x^2}$, and (from the definition of the derivative) we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \cos \tfrac{\pi}{x^2} = 0$$

by the Squeeze Theorem (since $\left| x \cos \frac{\pi}{x^2} \right| \leq |x| \to 0$ as $x \to 0$). Thus f is differentiable everywhere.

Let $n \in \mathbb{Z}^+$ and let X be the partition $X = (x_0, x_1, \cdots x_n)$ with $x_0 = 0$ and $x_k = \frac{1}{\sqrt{n-k+1}}$, that is

$$X = \left(0, \tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n-1}}, \cdots, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{2}}, 1\right).$$

We have $f(x_k) = \frac{1}{n-k+1} \cos(n-k+1)\pi = \frac{(-1)^{n-k+1}}{n-k+1}$ for $1 \leq k \leq n$, and hence

$$\left| f(x_k) - f(x_{k-1}) \right| = \left| \tfrac{1}{n-k+1} + \tfrac{1}{n-k+2} \right| \geq \tfrac{2}{n-k+2} \quad \text{for} \quad 2 \leq k \leq n.$$

Letting $j = n-k+2$ we have

$$L(f, X) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} \geq \sum_{k=2}^{n} \left| f(x_k) - f(x_{k-1}) \right| \geq \sum_{k=2}^{n} \tfrac{2}{n-k+2} = \sum_{j=2}^{n} \tfrac{2}{j}.$$

But note that we can choose n to make the sum $\sum\limits_{j=2}^{n} \frac{2}{j}$ arbitrarily large, indeed when $n = 2^m$ we have

$$\begin{aligned}\sum_{j=2}^{n} \tfrac{2}{j} &= \tfrac{2}{2} + \left(\tfrac{2}{3} + \tfrac{2}{4}\right) + \left(\tfrac{2}{5} + \cdots + \tfrac{2}{8}\right) + \left(\tfrac{2}{9} + \cdots + \tfrac{2}{16}\right) + \cdots + \left(\tfrac{2}{2^{m-1}+1} + \cdots + \tfrac{2}{2^m}\right) \\ &\geq \tfrac{2}{2} + 2 \cdot \tfrac{2}{4} + 4 \cdot \tfrac{2}{8} + 8 \cdot \tfrac{2}{16} + \cdots + 2^{m-1} \cdot \tfrac{2}{2^m} = m,\end{aligned}$$

and so $L(f) = \sup \left\{ L(f.X) \,\middle|\, X \text{ is a partition of } [0,1] \right\} = \infty$.

12: Tonelli's version of Fubini's Theorem implies that when $f:(a,b)\times(c,d)\to[0,\infty)$ is continuous, where the endpoints a,b,c,d can be finite or infinite, we have

$$\int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)dx=\int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\,dx\right)dy$$

(so we can calculate the volume of the solid given by $a<x<b$, $c<y<d$, $0\le z\le f(x,y)$, either by integrating the cross-sectional area $A(x)=\int_c^d f(x,y)\,dy$ or by integrating the cross-sectional area $A(y)=\int_a^b f(x,y)\,dx$, and the two calculations will yield the same value for the volume). Assuming that Tonelli's Theorem is true (we have not developed the machinery needed to prove it) and assuming that elementary functions $f(x,y)$ are continuous, evaluate each of the following improper integrals.

(a) $\displaystyle\int_0^\infty \frac{\tan^{-1}2x-\tan^{-1}x}{x}\,dx$ $\left(\text{hint: use } f(x,y)=\frac{1}{1+x^2y^2}\right)$.

Solution: We have

$$\begin{aligned}\int_{x=0}^{\infty}\frac{\tan^{-1}2x-\tan^{-1}x}{x}\,dx&=\int_{x=0}^{\infty}\left[\frac{\tan^{-1}xt}{x}\right]_{y=1}^{2}dx=\int_{x=0}^{\infty}\left(\int_{y=1}^{2}\frac{1}{1+x^2y^2}\,dy\right)dx\\&=\int_{y=1}^{2}\left(\int_{x=0}^{\infty}\frac{1}{x^2y^2}\,dx\right)dy=\int_{y=1}^{2}\left[\frac{\tan^{-1}xy}{y}\right]_{x=0}^{\infty}dy=\int_{y=1}^{2}\frac{\pi}{2y}\,dy\\&=\left[\tfrac{\pi}{2}\ln y\right]_1^2=\tfrac{\pi}{2}\ln 2\,.\end{aligned}$$

(b) $\displaystyle\int_0^1\frac{x-1}{\ln x}\,dx$ $\left(\text{hint: use } f(x,y)=x^y\right)$.

Solution: We have

$$\begin{aligned}\int_{x=0}^{1}\frac{x-1}{\ln x}\,dx&=\int_{x=0}^{1}\left[\frac{x^y}{\ln x}\right]_{y=0}^{1}dx=\int_{x=0}^{1}\left(\int_{y=0}^{1}x^y\,dy\right)dx=\int_{y=0}^{1}\left(\int_{x=0}^{1}x^y\,dx\right)dy\\&=\int_{y=0}^{1}\left[\frac{x^{y+1}}{y+1}\right]_{x=0}^{1}dy=\int_{y=0}^{1}\frac{1}{y+1}\,dy=\Big[\ln(y+1)\Big]_0^1=\ln 2\,.\end{aligned}$$

(c) $\displaystyle\int_0^\infty\frac{\sin x}{x}\,dx$ $\left(\text{hint: use } f(x,y)=e^{-xy}\sin x\right)$.

Solution: For $y\in(0,\infty)$, let $I(y)=\displaystyle\int e^{-xy}\sin x\;dx$. Integrate by parts twice, first using $u_1=e^{-xy}$ and $v_1=-\cos x$, and then again using $u_2=ye^{-xy}$ and $v_2=\sin x$, to get

$$\begin{aligned}I(y)=\int e^{-xy}\sin x\;dx&=-e^{-xy}\cos x-\int ye^{-xy}\cos x\;dx\\&=-e^{-xy}\cos x-\left(ye^{-xy}\sin x+\int y^2e^{-xy}\sin x\;dx\right)\\&=-e^{-xy}\big(\cos x+y\sin x\big)-y^2I(y)\end{aligned}$$

and so $I(y)=-\dfrac{1}{1+y^2}e^{-xy}\big(\cos x+y\sin x\big)$. Thus, using Tonelli's Theorem, we have

$$\begin{aligned}\int_{x=0}^{\infty}\frac{\sin x}{x}\,dx&=\int_{x=0}^{\infty}\left[-\frac{e^{-xy}\sin x}{x}\right]_{y=0}^{\infty}dx=\int_{x=0}^{\infty}\int_{y=0}^{\infty}e^{-xy}\sin x\;dy\,dx\\&=\int_{y=0}^{\infty}\int_{x=0}^{\infty}e^{-xy}\sin x\;dx\,dy=\int_{y=0}^{\infty}\Big[I(y)\Big]_{x=0}^{\infty}dy\\&=\int_{y=0}^{\infty}\left[-\frac{1}{1+y^2}e^{-xy}\big(\cos x+y\sin x\big)\right]_{x=0}^{\infty}\\&=\int_{y=0}^{\infty}\frac{dy}{1+y^2}=\Big[\tan^{-1}y\Big]_0^\infty=\tfrac{\pi}{2}\,.\end{aligned}$$