MATH 148 Calculus 2, Exercises for Chapter 3

1: (a) Find the area of the region given by $0 < x < 2\pi$ and $1 - \frac{1}{\sqrt{3}}\sin 2x \le y \le \sin x$.

(b) Find the area of the region which is bounded by the curves $x = y^3 - 3y$ and $x = y^2 - y$.

(c) Find the area of the region between the curve $y = x(x-2)$ with $x \ge 0$ and the curve $x = y(y-2)$ with $y \ge 0$.

(d) Find the area of the region bounded by the x-axis, the graph of $y = \ln x$, and by the tangent line to $y = \ln x$ which passes through the origin.

2: (a) Let R be the region given by $0 \le x \le \pi$ and $0 \le y \le \sin^2 x$. Find the volume of the solid obtained by revolving R about the x-axis, and find the volume of the solid obtained by revolving R about the y-axis.

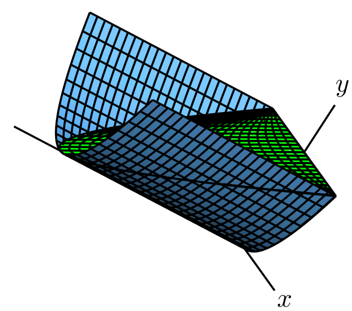
(b) Let R be the region $1 \le x \le 2$, $0 \le y \le \dfrac{1}{x\sqrt{x^2+2x}}$. Find the volume of the solid obtained by revolving R about the x-axis, and find the volume of the solid obtained by revolving R about the y-axis.

3: (a) Let R be the (infinitely long) region given by $0 \le x < \infty$ and $0 \le y \le (x^2+1)^{-3/2}$. Find the area of the region R, and find the volume of the solid obtained by revolving R about the y-axis.

(b) Let R be the (infinitely long) region $0 \le x < \infty$, $0 \le y \le \dfrac{2\sqrt{x}}{4+x^2}$. Find the area of R and find the volume of the solid obtained by revolving R about the x-axis.

4: (a) Let S be the solid $0 \le x \le 2$, $-x \le y \le x$, $0 \le z \le x^2 - y^2$. Find the volume of S.

(b) A scoop is in the shape of the parabolic surface $-1 \le x \le 1$, $y = x^2$, $0 \le z \le 2$, with one end covered by the region $-1 \le x \le 1$, $x^2 \le y \le 1$, $z = 0$. How much water can the scoop hold?



5: (a) Find the length of the curve $y = \sqrt{4x - x^2}$ with $0 \le x \le 3$.

(b) Find the length of the curve $0 \le x \le 8$, $y = 3\,x^{2/3}$.

(c) Find the length of the curve $0 \le x \le \ln 2$, $y = e^x$.

(d) Find the length of the curve $y = \ln x$ with $1 \le x \le \sqrt{3}$.

6: (a) Find the area of the surface which is obtained by revolving the curve $y = \tan x$ with $0 \le x \le \frac{\pi}{6}$ about the x-axis.

(b) Find the area of the surface which is obtained by revolving the curve $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $y = \cos x$ about the x-axis.

(c) The curve $9y^2 = x(x-3)^2$ has a loop in it. Find the area of the surface obtained by rotating the loop about the x-axis.

7: Let $S = \left\{ u \in \mathbb{R}^3 \,\middle|\, |u| = R \right\}$ be the sphere of radius R centered at the origin in $\mathbb{R}^3$.

(a) Find the area of a slice of thickness h on the surface of the sphere S (that is the area of the portion of the sphere which lies between two parallel planes separated by a distance of h).

(b) Find the circumference of, and the area inside, a spherical circle of radius r on S (the spherical circle of radius r centred at the point $u \in S$ is the set of points $v \in S$ such that $R\theta = r$ where θ is the angle between the vectors $u, v \in \mathbb{R}^3$ which is given by $\theta = \cos^{-1} \frac{u \bullet v}{|u|\,|v|}$).

(c) Find the area of a spherical triangle with interior angles α, β and γ (given three non-coplanar points $u, v, w \in S$, the spherical triangle with vertices at $u, v, w \in S$ has three spherical edges; the spherical edge through u and v is the set of points on S which lie on the plane through the origin which contains u and v, and similarly for the other two edges; the angle at u in the spherical triangle is the angle between the two spherical edges at u, which is the same as the angle between the corresponding planes through the origin).

8: (a) A tank, shaped like a lower hemisphere of radius R, is filled with a liquid of density ρ. Find the work done when the tank is emptied by pumping the liquid to the level of the top of the tank.

(b) A chain, of length π and mass M, lies in the xy-plane. Find the work required to lift the chain and lie it along the top half of the circle $x = 0$, $y^2 + (z-1)^2 = 1$.

9: (a) A trough, in the shape of the bottom half of a cylinder, of radius r and length l, lying on its side, is filled with a liquid of density ρ. Find the force exerted by the liquid on each of the two semi-circular ends of the tank, and find the outward force exerted by the liquid on the semi-cylindrical base of the tank.

(b) A flat circular disc of radius $\sqrt{5}$ lies in the xy-plane in the region $x^2+y^2 \leq 5$, $z = 0$. The disc has varying density. The planar density (mass per unit area) at points which lie a distance r from the center is given by $\rho(r) = \dfrac{1}{3+r^2}$. A small object of mass m lies above the disc at the point $(0,0,2)$. Find the gravitational force exerted by the disc on the object.

10: In many situations, we can find a length or an area or a volume either by integrating with respect to one variable, say x, or by integrating with respect to another variable, say y. We expect to obtain the same answer using either method. Let us show that our expectation is justified in a few circumstances. Let $f : [a,b] \to [c,d]$ be strictly decreasing with $f(a) = d$ and $f(b) = c$, and let $g = f^{-1} : [c,d] \to [a,b]$.

(a) Suppose f and g are differentiable and consider the area of the region $a \leq x \leq b$, $c \leq y \leq f(x)$. Use Substitution and Integration by Parts to show that

$$\int_{x=a}^{b} \big(f(x) - c\big)\, dx = \int_{y=c}^{d} \big(g(y) - a\big)\, dy$$

.

(b) Suppose f and g are differentiable and consider the length of the curve $a \leq x \leq b$, $y = f(x)$. Show that

$$\int_{x=a}^{b} \sqrt{1 + f'(x)^2}\, dx = \int_{y=c}^{d} \sqrt{1 + g'(y)^2}\, dy.$$

(c) Suppose f and g are differentiable and consider the volume of the solid obtained by revolving the region $a \leq x \leq b$, $c \leq y \leq f(x)$ about the x-axis. Show that

$$\int_{x=a}^{b} \pi\big(f(x)^2 - c^2\big)\, dx = \int_{y=c}^{d} 2\pi\, y\, \big(g(y) - a\big)\, dy.$$

(d) Suppose f and g are continuous and consider the area of the region $a \leq x \leq b$, $c \leq y \leq f(x)$. Use properties of the Riemann integral to prove that

$$\int_{x=a}^{b} \big(f(x) - c\big)\, dx = \int_{y=c}^{d} \big(g(y) - a\big)\, dy$$

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11: We defined the length of the curve $y = f(x)$ on $[a,b]$ only in the case that f is differentiable. In fact we can give a more general definition. Let $f : [a,b] \to \mathbb{R}$ be any function. For a partition $X = \big(x_0, x_1, \cdots, x_n\big)$ of $[a,b]$, define

$$L(f,X) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

and define the **length** of the curve $y = f(x)$ on $[a,b]$ to be

$$L(f) = \sup\big\{L(f,X) \,\big|\, X \text{ is a partition of } [a,b]\big\}$$

(note that the supremum can be infinite). We say that f is **rectifiable** when $L(f)$ is finite.

(a) Show that if f is $\mathcal{C}^1$ (which means that f is differentiable and f' is continuous on $[a,b]$) then f is rectifiable and

$$L(f) = \int_{x=a}^{b} \sqrt{1 + f'(x)^2}\, dx.$$

(b) Define $f : [0,1] \to \mathbb{R}$ by $f(0) = 0$ and $f(x) = x^2 \cos \frac{\pi}{x^2}$ when $x \neq 0$. Show that f is differentiable on $[0,1]$ but not rectifiable on $[0,1]$.

12: Tonelli's version of Fubini's Theorem implies that when $f:(a,b)\times(c,d)\to[0,\infty)$ is continuous, where the endpoints a,b,c,d can be finite or infinite, we have

$$\int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)dx=\int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\,dx\right)dy$$

(so we can calculate the volume of the solid given by $a<x<b$, $c<y<d$, $0\le z\le f(x,y)$, either by integrating the cross-sectional area $A(x)=\int_c^d f(x,y)\,dy$ or by integrating the cross-sectional area $A(y)=\int_a^b f(x,y)\,dx$, and the two calculations will yield the same value for the volume). Assuming that Tonelli's Theorem is true (we have not developed the machinery needed to prove it) and assuming that elementary functions $f(x,y)$ are continuous, evaluate each of the following improper integrals.

(a) $\displaystyle\int_0^\infty \frac{\tan^{-1}2x-\tan^{-1}x}{x}\,dx$ $\left(\text{hint: use } f(x,y)=\frac{1}{1+x^2y^2}\right)$.

(b) $\displaystyle\int_0^1 \frac{x-1}{\ln x}\,dx$ $\left(\text{hint: use } f(x,y)=x^y\right)$.

(c) $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ $\left(\text{hint: use } f(x,y)=e^{-xy}\sin x\right)$.