MATH 148 Calculus 2, Solutions to the Exercises for Chapter 2

1: (a) Approximate $\displaystyle\int_1^3 \frac{dx}{x}$ using Simpson's Approximatoion S_4 and find a bound on the error.

Solution: Let $f(x) = 1/x$. Then we have

$$\int_1^3 \frac{dx}{x} \cong S_4 = \frac{2}{12}\big(f(1) + 4f\big(\tfrac{3}{2}\big) + 2f(2) + 4f\big(\tfrac{5}{2}\big) + f(3)\big) = \tfrac{1}{6}\big(1 + \tfrac{8}{3} + 1 + \tfrac{8}{5} + \tfrac{1}{3}\big) = \tfrac{1}{6}\big(\tfrac{33}{5}\big) = \tfrac{11}{10}\,.$$

We have $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = -\frac{6}{x^4}$ and $f''''(x) = \frac{24}{x^5}$, so for $1 \le x \le 3$ we have $\big|f''''(x)\big| \le 24$. Thus the error is $S_n \le \frac{2^5 \cdot 24}{180 \cdot 4^4} = \frac{1}{60}$.

(b) Find a value of n such that if we approximate $\displaystyle\int_0^1 \frac{4\,dx}{1+x^2}$ by M_n, the error is $E_n \le \frac{1}{300}$.

Solution: Let $f(x) = \dfrac{4}{1+x^2}$. When we approximate $\displaystyle\int_0^1 f(x)\,dx$ using M_n, the absolute error is

$$|E_n| \le \frac{1(1-0)^3}{24\,n^2} \cdot K = \frac{K}{24\,n^2}\ \text{, where } K = \max_{0 \le x \le 1} \big|f''(x)\big|\,.$$

Verify that

$$f'(x) = \frac{-8\,x}{(1+x^2)^2}\ ,\ f''(x) = \frac{8(3x^2-1)}{(1+x^2)^3}\ \text{ and }\ f'''(x) = \frac{-96\,x(x^2-1)}{(1+x^2)^4}\,.$$

Since $f'''(x) = \frac{-96x(x-1)(x+1)}{(1+x^2)^4}$, we have $f'''(x) > 0$ for $x \in (0,1)$ and so $f''(x)$ is increasing in $(0,1)$. Since $f''(x) = \frac{8(3x^2-1)}{(1+x^2)^3}$ we have $f''(0) = -8$ and $f''(1) = 2$ so $K = \max_{0 \le x \le 1} \big|f''(x)\big| = 8$. Thus the absolute error is

$$|E_n| \le \frac{K}{24\,n^2} = \frac{8}{24\,n^2} = \frac{1}{3\,n^2}\,.$$

To get $|E_n| \le \frac{1}{300}$ we can choose n so that $\frac{1}{3\,n^2} \le \frac{1}{300}$, that is $n^2 \ge 100$ so $n \ge 10$.

(c) Let f be differentiable on $[a,b]$ with $|f'(x)| \le M$ for all $x \in [a,b]$. Let $I = \displaystyle\int_a^b f$ and let L_n be the Left Endpoint Approximation for I on n equal-sized subintervals. Prove that

$$|I - L_n| \le \frac{M(b-a)^2}{2n}\,.$$

Solution: Fix n and let $X = X_n = \big\{x_0, x_1, \cdots, x_n\big\}$ be the partition of $[a,b]$ into n equal-sized intervals. Note that, by the Mean Value Theorem, given $t \in [x_{k-1}, x_k]$, we can choose $s \in [x_{k-1}, t]$ so that $f(t) - f(x_{k-1} = f'(s)(t - x_{k-1})$, so we have $|f(t) - f(x_{k-1})| \le M(t - x_{k-1})$. Thus

$$\begin{aligned}
|I - L_n| &= \left|\int_a^b f(t)\,dt - \sum_{k=1}^n f(x_{k-1})\Delta_k x\right| = \left|\sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t)\,dt - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x_{k-1})\,dt\right| \\
&= \left|\sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t) - f(x_{k-1})\ dt\right| \le \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \big|f(t) - f(x_{k-1})\big|\,dt \le \sum_{k=1}^n \int_{x_{k-1}}^{x_k} M(t - x_{k-1})\,dt \\
&= \sum_{k=1}^n \Big[\tfrac{M}{2}(t - x_{k-1})^2\Big]_{x_{k-1}}^{x_k} = \sum_{k=1}^n \tfrac{M}{2}\,(\Delta_k x)^2 = \sum_{k=1}^n \tfrac{M}{2}\,\big(\tfrac{b-a}{n}\big)^2 = \tfrac{M(b-a)^2}{2n}\,.
\end{aligned}$$

2: Find each of the following indefinite integrals (antiderivatives).

(a) $\displaystyle\int \frac{dx}{x\,(\ln x)^3}$, where $x > 1$

Solution: Let $u = \ln x$ so $du = \dfrac{1}{x}\,dx$. Then $\displaystyle\int \frac{dx}{x(\ln x)^3} = \int u^{-3}\,du = -\tfrac{1}{2}\,u^{-2} + c = \frac{-1}{2\,(\ln x)^2} + c.$

(b) $\displaystyle\int e^{-x}\cos(2x)\,dx$

Solution: Write $I = \displaystyle\int e^{-x}\cos(2x)\,dx$. Integrate by parts using $u = e^{-x}$, $du = -e^{-x}\,dx$, $v = \tfrac{1}{2}\sin(2x)$ and $dv = \cos(2x)\,dx$ to get

$$I = \tfrac{1}{2}\,e^{-x}\sin(2x) + \tfrac{1}{2}\int e^{-x}\sin(2x)\,dx\,.$$

To find $\displaystyle\int e^{-x}\sin(2x)\,dx$, integrate by parts again, this time using $u = e^{-x}$, $du = -e^{-x}\,dx$, $v = -\tfrac{1}{2}\cos(2x)$ and $dv = \sin(2x)\,dx$ to get

$$I = \tfrac{1}{2}\,e^{-x}\sin(2x) + \tfrac{1}{2}\left(-\tfrac{1}{2}e^{-x}\cos(2x) - \tfrac{1}{2}\int e^{-x}\,\sin(2x)\,dx\right) = \tfrac{1}{2}\,e^{-x}\sin(2x) - \tfrac{1}{4}\,e^{-x}\cos(2x) - \tfrac{1}{4}\,I\,.$$

Add $\tfrac{1}{4}\,I$ to both sides to get $\tfrac{5}{4}\,I = \tfrac{1}{2}\,e^{-x}\sin(2x) - \tfrac{1}{4}\,e^{-x}\cos(2x) + \tfrac{5}{4}\,c$ and so we have

$$I = \tfrac{2}{5}\,e^{-x}\sin(2x) - \tfrac{1}{5}\,e^{-x}\cos(2x) + c\,.$$

(c) $\displaystyle\int \sin(\sqrt{x})\,dx$, where $x > 0$

Solution: Make the substitution $t = \sqrt{x}$ or $t^2 = x$ so $2t\,dt = dx$, and then integrate by parts using $u = 2t$, $du = 2\,dt$, $v = -\cos t$ and $dv = \sin t\;dt$ to get

$$\int \sin\sqrt{x}\;dx = \int 2t\;\sin t\;dt = -2t\;\cos t + \int 2\cos t\,dt = 2\sin t - 2t\;\cos t + c = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + c\,.$$

(d) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$, where $0 \le x < 1$

Solution: We have

$$\int \sqrt{\frac{1+x}{1-x}}\;dx = \int \sqrt{\frac{1+x}{1-x}}\sqrt{\frac{1+x}{1+x}}\;dx = \int \frac{1+x}{\sqrt{1-x^2}}\;dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}\,dx$$

We know that $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c_1$, and to solve $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}}$ we can let $u = 1 - x^2$ so $du = -2x\,dx$ to get $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} = \int -\tfrac{1}{2}\,u^{-1/2}\,du = -u^{1/2} + c = -\sqrt{1-x^2} + c_2$, and so we have

$$\int \sqrt{\frac{1+x}{1-x}}\;dx = \sin^{-1}x - \sqrt{1-x^2} + c\,.$$

3: Find each of the following indefinite integrals (antiderivatives).

(a) $\displaystyle\int x^2\sqrt{4-x^2}\;dx$

Solution: Let $2\sin\theta=x$ so $2\cos\theta=\sqrt{4-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$\begin{aligned}\int x^2\sqrt{4-x^2}\,dx&=16\int\sin^2\theta\cos^2\theta\,d\theta=4\int\sin^2 2\theta\,d\theta=2\int 1-\cos 4\theta\,d\theta=2(\theta-\tfrac{1}{4}\sin 4\theta)+c\\&=2\theta-\sin 2\theta\cos 2\theta+c=2\theta-2\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)+c\\&=2\sin^{-1}\tfrac{x}{2}-2\tfrac{x}{2}\tfrac{\sqrt{4-x^2}}{2}\left(\tfrac{4-x^2}{4}-\tfrac{x^2}{4}\right)+c=2\sin^{-1}\tfrac{x}{2}-\tfrac{1}{4}\,x\sqrt{4-x^2}(2-x^2)+c\,.\end{aligned}$$

(b) $\displaystyle\int\big(x\ln x\big)^2\,dx$

Solution: Let $I=\displaystyle\int(x\ln x)^2\,dx=\int x^2(\ln x)^2\,dx$. Integrate by parts with $u=(\ln x)^2$, $du=\frac{2}{x}\ln x\,dx$, $v=\frac{1}{3}x^3$ and $dv=x^2\,dx$ to get $I=\frac{1}{3}x^3(\ln x)^2-\frac{2}{3}\int x^2\ln x\,dx$. Integrate by parts again with $u=\ln x$, $du=\frac{1}{x}\,dx$, $v=\frac{1}{3}x^3$ and $dv=x^2\,dx$ to get

$$I=\tfrac{1}{3}x^3(\ln x)^2-\tfrac{2}{3}\Big[\tfrac{1}{3}x^3\ln x-\tfrac{1}{3}\textstyle\int x^2\,dx\Big]=\tfrac{1}{3}x^3(\ln x)^2-\tfrac{2}{9}x^3\ln x+\tfrac{2}{27}x^3+c\,.$$

(c) $\displaystyle\int\cos^4 x\;dx$

Solution: We have

$$\begin{aligned}\int\cos^4 x\;dx&=\int\tfrac{1}{4}(1+\cos 2x)^2\,dx=\int\tfrac{1}{4}\big(1+2\cos 2x+\cos^2 2x\big)\,dx=\int\tfrac{1}{4}\big(1+2\cos 2x+\tfrac{1}{2}(1+\cos 4x)\big)\,dx\\&=\int\tfrac{3}{4}+\tfrac{1}{2}\cos 2x+\tfrac{1}{8}\cos 4x\;dx=\tfrac{3}{4}\,x+\tfrac{1}{4}\sin 2x+\tfrac{1}{32}\sin 4x+c.\end{aligned}$$

(d) $\displaystyle\int\frac{x^4+3x^3-x^2+1}{x^4+2x^3+x^2}\;dx$

Solution: Let $I=\displaystyle\int\frac{x^4+3x^3-x^2+1}{x^4+2x^3+x^2}\,dx=\int\frac{(x^4+2x^3+x^2)+(x^3-2x^2+1)}{x^4+2x^3+x^2}\,dx=\int 1+\frac{x^3-2x^2+1}{x^2(x+1)^2}\,dx$
$=\displaystyle\int 1+\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}\;dx$, where $Ax(x+1)^2+B(x+1)^2+Cx^2(x+1)+Dx^2=x^3-2x^2+1$. Equate coefficients to get the 4 equations $B=1$, $A+2B=0$, $2A+B+C+D=-2$ and $A+C=1$. Solve these (easily) to get $B=1$, $A=-2$, $C=3$ and $D=-2$, so

$$I=\int 1-\frac{2}{x}+\frac{1}{x^2}+\frac{3}{x+1}-\frac{2}{(x+1)^2}\;dx=x-2\ln|x|-\frac{1}{x}+3\ln|x+1|+\frac{2}{x+1}+c\,.$$

4: Find each of the following definite integrals.

(a) $\displaystyle\int_0^4 |x^2-3x+2|\,dx$

Solution: Since $x^2-3x+2=(x-1)(x-2)$, we have $x^2-3x+2\geq 0$ on $[0,1]$, $x^2-3x+2\leq 0$ on $[1,2]$, and $x^2-3x+2\geq 0$ on $[2,4]$, and so

$$\begin{aligned}\int_0^4 \big|x^2-3x+2\big|\,dx &= \int_0^1 x^2-3x+2\;dx-\int_1^2 x^2-3x+2\;dx+\int_2^4 x^2-3x+2\;dx\\ &=\Big[\tfrac{1}{3}\,x^3-\tfrac{3}{2}\,x^2+2x\Big]_0^1-\Big[\tfrac{1}{3}\,x^3-\tfrac{3}{2}\,x^2+2x\Big]_1^2+\Big[\tfrac{1}{3}\,x^3-\tfrac{3}{2}\,x^2+2x\Big]_2^4\\ &=\big(\tfrac{1}{3}-\tfrac{3}{2}+2\big)-\big(\tfrac{8}{3}-6+4\big)+\big(\tfrac{1}{3}-\tfrac{3}{2}+2\big)+\big(\tfrac{64}{3}-24+8\big)-\big(\tfrac{8}{3}-6+4\big)\\ &=\tfrac{17}{3}\,.\end{aligned}$$

(b) $\displaystyle\int_0^{\ln 3}\frac{e^x\,dx}{1+e^x}$

Solution: Make the substitution $u=e^x$ so $du=e^x\,dx$. Then

$$\int_0^{\ln 3}\frac{e^x\,dx}{1+e^x}=\int_1^3\frac{du}{1+u}=\Big[\ln(1+u)\Big]_1^3=\ln 4-\ln 2=\ln 2\,.$$

(c) $\displaystyle\int_0^{\pi/3}\frac{\sin^3 x}{\cos^2 x}\,dx$

Solution: Make the substitution $u=\cos x$ so $du=-\sin x\;dx$. Then

$$\begin{aligned}\int_0^{\pi/3}\frac{\sin^3 x}{\cos^2 x}\,dx &= \int_0^{\pi/3}\frac{(1-\cos^2 x)}{\cos^2 x}\,\sin x\;dx=\int_1^{1/2}-\frac{1-u^2}{u^2}\,du=\int_1^{1/2}-\tfrac{1}{u^2}+1\;du\\ &=\Big[\tfrac{1}{u}+u\Big]_1^{1/2}=\big(2+\tfrac{1}{2}\big)-(1+1)=\tfrac{1}{2}\,.\end{aligned}$$

(d) $\displaystyle\int_0^2 x\,e^{-x/2}\,dx$

Solution: Integrate by parts using $u=x$, $du=dx$, $v=-2\,e^{-x/2}$ and $dv=e^{-x/2}\,dx$ to get

$$\int_0^2 x\,e^{-x/2}\,dx=\Big[-2x\,e^{-x/2}+\int 2\,e^{-x/2}\,dx\Big]_0^2=\Big[-2x\,e^{-x/2}-4\,e^{-x/2}\Big]_0^2=-4e^{-1}-4e^{-1}+4=4-\frac{8}{e}\,.$$

5: Find each of the following definite integrals.

(a) $\displaystyle\int_1^e (\ln x)^2\,dx$

Solution: Integrate by parts using $u = (\ln x)^2$, $du = \dfrac{2\ln x}{x}\,dx$, $v = x$ and $dv = dx$ to get

$$\int_1^e (\ln x)^2\,dx = \left[x(\ln x)^2 - \int 2\ln x\ dx\right]_1^e = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e = (e - 2e + 2e) - 2 = e - 2\,.$$

(b) $\displaystyle\int_1^3 \frac{\sqrt{x}}{x+1}\,dx$

Solution: Make the substitution $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\begin{aligned}\int_1^3 \frac{\sqrt{x}\ dx}{x+1} &= \int_1^{\sqrt{3}} \frac{2u^2\ du}{u^2+1} = \int_1^{\sqrt{3}} 2 - \frac{2}{u^2+1}\ du \\ &= \left[2u - 2\tan^{-1} u\right]_1^{\sqrt{3}} = \left(2\sqrt{3} - \tfrac{2\pi}{3}\right) - \left(2 - \tfrac{\pi}{2}\right) = 2\sqrt{3} - 2 - \tfrac{\pi}{6}\,.\end{aligned}$$

(c) $\displaystyle\int_0^{\pi/2} \sqrt{1-\sin x}\,dx$

Solution: Let $u = 1 + \sin x$ so $du = \cos x\ dx$. Then we have

$$\int_0^{\pi/2} \sqrt{1-\sin x}\ dx = \int_0^{\pi/2} \frac{\sqrt{1-\sin x}\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\ dx = \int_0^{\pi/2} \frac{\cos x\ dx}{\sqrt{1+\sin x}} = \int_1^2 u^{-1/2}\,du = \left[2\,u^{1/2}\right]_1^2 = 2\sqrt{2} - 2\,.$$

(d) $\displaystyle\int_1^{27} \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Solution: Let $u = \sqrt[6]{x}$ so $u^2 = \sqrt[3]{x}$, $u^3 = \sqrt{x}$, $u^6 = x$ and $6\,u^5\,du = dx$. Then

$$\begin{aligned}\int_1^{27} \frac{dx}{\sqrt{x}+\sqrt[3]{x}} &= \int_1^{\sqrt{3}} \frac{6\,u^5\,du}{u^3+u^2} = \int_1^{\sqrt{3}} \frac{6\,u^3\,du}{u+1} = \int_1^{\sqrt{3}} \frac{6((u+1)(u^2-u+1)-1)}{u+1}\,du \\ &= \int_1^{\sqrt{3}} 6u^2 - 6u + 6 - \frac{6}{u+1}\ du = \left[2u^3 - 3u^2 + 6u - 6\ln(u+1)\right]_1^{\sqrt{3}} \\ &= \left(6\sqrt{3} - 9 + 6\sqrt{3} - 6\ln(\sqrt{3}+1)\right) - (2 - 3 + 6 - 6\ln 2) \\ &= 12\sqrt{3} - 14 + 6\ \ln\left(\tfrac{2}{\sqrt{3}+1}\right) \\ &= 12\sqrt{3} - 14 + 6\ \ln(\sqrt{3}-1)\,.\end{aligned}$$

6: Find each of the following definite integrals.

(a) $\displaystyle\int_0^3 \frac{x^2\,dx}{(x+1)^{3/2}}$

Solution: Make the substitution $u = x+1$ so $x = u-1$ and $dx = du$. Then

$$\int_0^3 \frac{x^2\,dx}{(x+1)^{3/2}} = \int_1^4 \frac{(u-1)^2}{u^{3/2}}\,du = \int_1^4 \frac{u^2-2u+1}{u^{3/2}}\,du = \int_1^4 u^{1/2} - 2u^{-1/2} + u^{-3/2}\,du$$
$$= \Big[\tfrac{2}{3}\,u^{3/2} - 4u^{1/2} - 2u^{-1/2}\Big]_1^4 = \left(\tfrac{16}{3} - 8 - 1\right) - \left(\tfrac{2}{3} - 4 - 2\right) = \tfrac{5}{3}\,.$$

(b) $\displaystyle\int_0^{\pi/4} \tan^5 x\;dx$

Solution: Make the substitution $u = \sec x$ so $du = \sec x \tan x\;dx$. Then

$$\int_0^{\pi/4} \tan^5 x\;dx = \int_0^{\pi/4} \frac{(\sec^2 x - 1)^2}{\sec x}\,\sec x \tan x\;dx = \int_1^{\sqrt{2}} \frac{(u^2-1)^2}{u}\,du = \int_1^{\sqrt{2}} u^3 - 2u + \frac{1}{u}\;du$$
$$= \Big[\tfrac{1}{4}\,u^4 - u^2 + \ln u\Big]_1^{\sqrt{2}} = \left(1 - 2 + \ln\sqrt{2}\right) - \left(\tfrac{1}{4} - 1\right) = \tfrac{1}{2}\ln 2 - \tfrac{1}{4}\,.$$

(c) $\displaystyle\int_0^{\pi/2} e^{2x}\cos x\;dx$

Solution: Let $I = \displaystyle\int e^{2x}\cos x\;dx$. Integrate by parts twice, first using $u_1 = e^{2x}$, $du_1 = 2e^{2x}\,dx$, $v_1 = \sin x$ and $dv_1 = \cos x\,dx$, and then using $u_2 = 2e^{2x}$, $du_2 = 4e^{2x}\,dx$, $v_2 = -\cos x$ and $dv_2 = \sin x\,dx$ to get

$$\int e^{2x}\cos x\;dx = u_1v_1 - \int v_1du_1 = e^{2x}\sin x - \int 2e^{2x}\sin x\,dx = e^{2x}\sin x - \left(u_2v_2 - \int v_2\,du_2\right)$$
$$= e^{2x}\sin x - \left(-2e^{2x}\cos x + \int 4e^{2x}\cos x\,dx\right) = e^{2x}\left(\sin x + 2\cos x\right) - 4I\,.$$

and so $5I = e^{2x}\left(\sin x + 2\cos x\right) + c$, that is $I = \frac{1}{5}\,e^{2x}\left(\sin x + 2\cos x\right) + d$. Thus

$$\int_0^{\pi/2} e^{2x}\cos x\;dx = \Big[\tfrac{1}{5}\,e^{2x}\left(\sin x + 2\cos x\right)\Big]_0^{\pi/2} = \tfrac{1}{5}\,e^{\pi} - \tfrac{2}{5}$$

(d) $\displaystyle\int_0^1 x^2\sin^{-1} x\;dx$

Solution: Integrate by parts using $u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$, $v = \frac{1}{3}x^3$ and $dv = x^2\,dx$, and then make the substitution $w = 1-x^2$ so $dw = -2x\,dx$ to get

$$\int x^2\sin^{-1} x\;dx = \tfrac{1}{3}x^3\sin^{-1} x - \int \frac{\frac{1}{3}x^3}{\sqrt{1-x^2}}\,dx = \tfrac{1}{3}x^3\sin^{-1} x - \int \frac{-\frac{1}{6}(1-w)dw}{\sqrt{w}}$$
$$= \tfrac{1}{3}x^3\sin^{-1} x + \int \tfrac{1}{6}w^{-1/2} - \tfrac{1}{6}w^{1/2}\;dw = \tfrac{1}{3}x^3\sin^{-1} x + \tfrac{1}{3}w^{1/2} - \tfrac{1}{9}w^{3/2} + c$$
$$= \tfrac{1}{3}x^3\sin^{-1} x + \tfrac{1}{3}(1-x^2)^{1/2} - \tfrac{1}{9}(1-x^2)^{3/2} + c$$

and so

$$\int_0^1 x^2\sin^{-1} x\;dx = \Big[\tfrac{1}{3}x^3\sin^{-1} x + \tfrac{1}{3}(1-x^2)^{1/2} - \tfrac{1}{9}(1-x^2)^{3/2}\Big]_0^1 = \left(\tfrac{\pi}{6}\right) - \left(\tfrac{1}{3} - \tfrac{1}{9}\right) = \tfrac{\pi}{6} - \tfrac{2}{9}\,.$$

7: Find each of the following definite integrals.

(a) $\displaystyle\int_0^{\sqrt{3}} \frac{x^2\,dx}{(4-x^2)^{3/2}}$

Solution: Make the substitution $2\sin\theta = x$ so $2\cos\theta = \sqrt{4-x^2}$ and $2\cos\theta\,d\theta = dx$ to get

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{x^2\,dx}{(4-x^2)^{3/2}} &= \int_0^{\pi/3} \frac{(2\sin\theta)^2 2\cos\theta\,d\theta}{(2\cos\theta)^3} = \int_0^{\pi/3} \tan^2\theta\,d\theta \\ &= \int_0^{\pi/3} \sec^2\theta - 1\ d\theta = \Big[\tan\theta - \theta\Big]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}\,.\end{aligned}$$

(b) $\displaystyle\int_0^1 x^2\sqrt{2-x^2}\,dx$

Solution: Make the substitution $\sqrt{2}\sin\theta = x$ so that $\sqrt{2}\cos\theta = \sqrt{2-x^2}$ and $\sqrt{2}\cos\theta\,d\theta = dx$ to get

$$\begin{aligned}\int_0^1 x^2\sqrt{2-x^2}\,dx &= \int_0^{\pi/4} (\sqrt{2}\sin\theta)^2\cdot\sqrt{2}\cos\theta\cdot\sqrt{2}\cos\theta\,d\theta = \int_0^{\pi/4} 4\sin^2\theta\cos^2\theta\ d\theta \\ &= \int_0^{\pi/4} \sin^2 2\theta\ d\theta = \int_0^{\pi/4} \tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta\ d\theta = \Big[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta\Big]_0^{\pi/4} = \tfrac{\pi}{8}\end{aligned}$$

(c) $\displaystyle\int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x}\,dx$

Solution: We first make the substitution $\tan\theta = x$ so $\sec\theta = \sqrt{1+x^2}$ and $\sec^2\theta\,d\theta = dx$ and then later we make the substitution $u = \sec\theta$ so $du = \sec\theta\tan\theta\,d\theta$. Note that $u = \sec\theta = \sqrt{1+x^2}$, so when $x = \sqrt{3}$ we have $u = 2$ and when $x = \sqrt{8}$ we have $u = 3$. We have

$$\begin{aligned}\int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x}\,dx &= \int_{\tan^{-1}\sqrt{3}}^{\tan^{-1}\sqrt{8}} \frac{\sec\theta\cdot\sec^2\theta\,d\theta}{\tan\theta} = \int_{\tan^{-1}\sqrt{3}}^{\tan^{-1}\sqrt{8}} \frac{\sec^2\theta}{\sec^2\theta - 1}\sec\theta\tan\theta\,d\theta \\ &= \int_2^3 \frac{u^2}{u^2-1}\,du = \int_2^3 1 + \frac{1}{u^2-1}\,du = \int_2^3 1 + \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\,du \\ &= \left[u + \tfrac{1}{2}\ln\frac{u-1}{u+1}\right]_2^3 = \left(3 + \tfrac{1}{2}\ln\tfrac{1}{2}\right) - \left(2 + \tfrac{1}{2}\ln\tfrac{1}{3}\right) = 1 + \tfrac{1}{2}\ln\tfrac{3}{2}\,.\end{aligned}$$

(d) $\displaystyle\int_3^5 \frac{\sqrt{x^2-9}}{x^2}\,dx$

Solution: Let $I = \displaystyle\int_3^5 \frac{\sqrt{x^2-9}}{x^2}\,dx$. Let $3\sec\theta = x$, $3\tan\theta = \sqrt{x^2-4}$, $3\sec\theta\tan\theta\,d\theta = dx$. Then

$$\begin{aligned}I &= \int_{\theta=0}^{\alpha} \frac{\sec\theta\tan^2\theta}{\sec^2\theta}\,d\theta\ ,\text{ where } \alpha = \sec^{-1}5/3 = \cos^{-1}3/5 \\ &= \int_{\theta=0}^{\alpha} \frac{\sin^2\theta}{\cos\theta}\,d\theta = \int_{\theta=0}^{\alpha} \frac{\sin^2\theta\cos\theta}{1-\sin^2\theta}\,d\theta \\ &= \int_{u=0}^{4/5} \frac{u^2}{1-u^2}\ ,\text{ where } u = \sin x \\ &= \int_{u=0}^{4/5} \frac{1}{1-u^2} - 1\ du = \int_{u=0}^{4/5} \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} - 1\ du \\ &= \left[\tfrac{1}{2}\ln\left|\tfrac{1+u}{1-u}\right| - u\right]_0^{4/5} = \tfrac{1}{2}\ln\left(\tfrac{9/5}{1/5}\right) - \tfrac{4}{5} = \ln 3 - \tfrac{4}{5}\end{aligned}$$

8: Find each of the following definite integrals, where n is a positive integer.

(a) $\displaystyle\int_{-\pi/6}^{\pi/6} \sin(2x)\sin(3x)\,dx$

Solution: Note first that $\cos 5x = \cos(2x+3x) = \cos 2x\cos 3x - \sin 2x \sin 3x$ and $\cos x = \cos(3x-2x) = \cos 3x\cos 2x + \sin 3x\sin 2x$, so we have $\cos x - \cos 5x = 2\sin 2x\sin 3x$. Thus

$$\begin{aligned}\int_{-\pi/6}^{\pi/6} \sin 2x\sin 3x\;dx = 2\int_0^{\pi/6} \sin 2x \sin 3x\;dx &= \int_0^{\pi/6} \cos x - \cos 5x\;dx\\ = \Big[\sin x - \tfrac{1}{5}\sin 5x\Big]_0^{\pi/6} &= \tfrac{1}{2} - \tfrac{1}{10} = \tfrac{2}{5}\,.\end{aligned}$$

(b) $\displaystyle\int_0^{\pi^2} \sin^2\sqrt{x}\;dx$

Solution: First we make the substitution $y=\sqrt{x}$ so $y^2 = x$ and $2y\,dy = dx$, and then we integrate by parts using $u=y$, $du=dy$, $v=\frac{1}{2}\sin 2y$ and $dv = \cos 2y\,dy$ to get

$$\begin{aligned}\int_0^{\pi^2}\sin^2\sqrt{x}\;dx &= \int_0^\pi 2y\sin^2 y\;dy = \int_0^\pi 2y\left(\tfrac12 - \tfrac12\cos 2y\right)\,dy = \int_0^\pi y - y\cos 2y\;dy\\ &= \left[\tfrac12 y^2 - \left(uv - \int v\,du\right)\right]_0^\pi = \left[\tfrac12 y^2 - \tfrac12 y\sin 2y + \int \tfrac12\sin 2y\;dy\right]_0^\pi\\ &= \Big[\tfrac12 y^2 - \tfrac12 y\sin 2y - \tfrac14\cos 2y\Big]_0^\pi = \left(\tfrac{\pi^2}{2} - \tfrac14\right) - \left(-\tfrac14\right) = \tfrac{\pi^2}{2}\,.\end{aligned}$$

(c) $\displaystyle\int_0^{\pi/2}\cos^{2n} x\;dx$

Solution: Let $I_n = \displaystyle\int_0^{\pi/2}\cos^n x\;dx$. Integrate by parts using $u=\cos^{n-1}x$, $du = -(n-1)\sin x\cos^{n-2}x\;dx$, $v=\sin x$ and $dv = \cos x\;dx$ to get

$$\begin{aligned}I_n &= \left[\sin x\,\cos^{n-1}x + \int (n-1)\sin^2 x\,\cos^{n-2}x\;dx\right]_0^{\pi/2}\\ &= \int_0^{\pi/2}(n-1)(1-\cos^2 x)\cos^{n-2}x\;dx = (n-1)I_{n-1} - (n-1)I_n\,,\end{aligned}$$

and so $n\,I_n = (n-1)I_{n-1}$ that is $I_n = \frac{n-1}{n}I_{n-1}$. Thus we have $I_0 = \frac{\pi}{2}$, $I_2 = \frac12\cdot\frac{\pi}{2}$, $I_4 = \frac{1\cdot 3}{2\cdot 4}\,\frac{\pi}{4}$, $I_6 = \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{\pi}{2}$ and so on, and in general (by induction)

$$\int_0^{\pi/2}\cos^{2n}x\;dx = I_{2n} = \tfrac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot2n}\,\tfrac{\pi}{2} = \tfrac{(2n)!}{(2^n\,n!)^2}\,\tfrac{\pi}{2} = \tfrac{1}{4^n}\tbinom{2n}{n}\tfrac{\pi}{2}.$$

(d) $\displaystyle\int_0^\pi \frac{\sin(nx)}{\sin x}\,dx$

Solution: Write $I_n = \displaystyle\int_0^\pi\frac{\sin(nx)}{\sin x}\,dx$. Then

$$\begin{aligned}I_{n+2} - I_n &= \int_0^\pi \frac{\sin((n+2)x) - \sin(nx)}{\sin x}\,dx = \int_0^\pi\frac{\sin(nx)\cos(2x) - \cos(nx)\sin(2x) - \sin(nx)}{\sin x}\,dx\\ &= \int_0^\pi \frac{\sin(nx)(\cos(2x)-1) - \cos(nx)\sin(2x)}{\sin x}\,dx = \int_0^\pi\frac{\sin(nx)(-2\sin^2 x) - \cos(nx)(2\sin x\cos x)}{\sin x}\,dx\\ &= \int_0^\pi 2\big(\sin(nx)\sin x - \cos(nx)\cos x\big)\,dx = \int_0^\pi -2\cos((n+1)x)\,dx = \Big[-\tfrac{2}{n+1}\sin((n+1)x)\Big]_0^\pi = 0\end{aligned}$$

and so we have $I_{n+2} = I_n$ for all $n\ge 0$. Since $I_0 = \displaystyle\int_0^\pi 0\,dx = 0$ and $I_1 = \displaystyle\int_0^\pi 1\,dx = \pi$, we have $I_n = 0$ for all even values of n and $I_n = \pi$ for all odd values of n.

9: Find each of the following definite integrals.

(a) Find $\displaystyle\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$.

Solution: In order to get $\displaystyle\frac{x-7}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$ we need

$$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = x-7\,.$$

Equating coefficients gives $A+C=0$, $A+B-2C=1$ and $-2A+2B+C=-7$. Solving these three equations gives $A=1$, $B=-2$ and $C=-1$, and so we have

$$\begin{aligned}\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx &= \int_2^3 \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} = \int_2^3 \frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2}\,dx \\ &= \Big[\ln(x-1) + \tfrac{2}{x-1} - \ln(x+2)\Big]_2^3 = (\ln 2 + 1 - \ln 5) - (2 - \ln 4) = \ln\tfrac{8}{5} - 1\,.\end{aligned}$$

(b) Find $\displaystyle I = \int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$.

Solution: To get $\displaystyle\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2-2x+5} + \frac{Ex+F}{(x^2-2x+5)^2} = \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}$ we need to have $Ax(x^2-2x+5)^2 + B(x^2-2x+5)^2 + (Cx+D)(x^2)(x^2-2x+5) + (Ex+F)(x^2) = x^5+x^4-2x^3-2x^2-5x-25$. Expanding the left hand side then equating coefficients gives the 5 equations

$$\begin{aligned}&A+C=1\ ,\ -4A+B-2C+D=1\ ,\ 14A-4B+5C-2D+E=-2\\ &-20A+14B+5D+F=-2\ ,\ 25A-20B=-5\ ,\ 25B=-25\end{aligned}$$

Solving these equations gives $A=-1$, $B=-1$, $C=2$, $D=2$, $E=2$ and $F=-18$, so

$$\begin{aligned}I &= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x+2}{x^2-2x+5} + \frac{2x-18}{(x^2-2x+5)^2}\,dx \\ &= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x-2+4}{x^2-2x+5} + \frac{2x-2-16}{(x^2-2x+5)^2}\,dx \\ &= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x-2}{x^2-2x+5} + \frac{4}{x^2-2x+5} + \frac{2x-2}{(x^2-2x+5)^2} - \frac{16}{(x^2-2x+5)^2}\,dx\end{aligned}$$

We have $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$ and $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$. Letting $u = x^2-2x+5$ so $du = (2x-2)\,dx$ gives $\displaystyle\int \frac{(2x-2)\,dx}{x^2-2x+5} = \int \frac{du}{u} = \ln u + c = \ln(x^2-2x+5) + c$ and $\displaystyle\int \frac{(2x-2)\,dx}{(x^2-2x+5)^2} = \int \frac{du}{u^2} = \frac{-1}{u} + c = \frac{-1}{x^2-2x+5} + c.$

Letting $2\tan\theta = x-1$ so that $2\sec\theta = \sqrt{x^2-2x+5}$ and $2\sec^2\theta\,d\theta = dx$ gives

$$\int \frac{4\,dx}{x^2-2x+5} = \int \frac{4\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^2} = \int 2\,d\theta = 2\theta + c = 2\tan^{-1}\left(\tfrac{x-1}{2}\right) + c$$

and

$$\begin{aligned}\int \frac{16\,dx}{(x^2-2x+5)^2} &= \int \frac{16\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^4}\,d\theta = \int \frac{2\,d\theta}{\sec^2\theta} = \int 2\cos^2\theta\,d\theta = \int 1 + \cos 2\theta\ d\theta \\ &= \theta + \tfrac{1}{2}\sin 2\theta + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\tfrac{x-1}{2}\right) + \tfrac{2(x-1)}{x^2-2x+5} + c\,.\end{aligned}$$

Thus

$$\begin{aligned}I &= \left[-\ln x + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\frac{x-1}{2} - \frac{1}{x^2-2x+5} - \tan^{-1}\frac{x-1}{2} - \frac{2(x-1)}{x^2-2x+5}\right]_1^2 \\ &= \left[\ln\frac{x^2-2x+5}{x} + \frac{1}{x} - \frac{2x-1}{x^2-2x+5} + \tan^{-1}\frac{x-1}{2}\right]_1^2 \\ &= \left(\ln\tfrac{5}{2} + \tfrac{1}{2} - \tfrac{3}{5} + \tan^{-1}\tfrac{1}{2}\right) - \left(\ln 4 + 1 - \tfrac{1}{4}\right) \\ &= \ln\tfrac{5}{8} - \tfrac{17}{20} + \tan^{-1}\tfrac{1}{2}\,.\end{aligned}$$

(c) $\displaystyle\int_1^3 \frac{x^4+3x^3+6}{x^3+4x^2+3x}\,dx$

Solution: We use long division to obtain

$$\begin{array}{r|l} & x-1 \\ \hline x^3+4x^2+3x & x^4+3x^3+0\,x^2+0x+6 \\ & \underline{x^4+4x^3+3x^2} \\ & \quad -x^3-3x^2+0x+6 \\ & \quad \underline{-x^3-4x^2-3x} \\ & \qquad\qquad x^2+3x+6 \end{array}$$

This shows that $\dfrac{x^4+3x^3+6}{x^3+4x^2+3x}=x-1+\dfrac{x+3x+6}{x^3+4x^2+3x}$. Note that $x^3+4x^2+3x=x(x+1)(x+3)$. In order to get $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+3}=\dfrac{x+3x+6}{x(x+1)(x+3)}$ for all x we need $A(x+1)(x+3)+Bx(x+3)+Cx(x+1)=x^2+3x+6$ for all x. Put in $x=0$ to get $3A=6$ so $A=2$, put in $x=-1$ to get $-2B=4$ so $B=-2$, and put in $x=-3$ to get $6C=6$ so $C=1$. Thus

$$\begin{aligned}\int_1^3 \frac{x^4+3x^3+6}{x^3+4x^2+3x}\,dx &= \int_1^3 x-1+\frac{2}{x}-\frac{2}{x+1}+\frac{1}{x+3}\,dx \\ &= \Big[\tfrac12 x^2-x+2\ln x-2\ln(x+1)+\ln(x+3)\Big]_1^3 \\ &= \big(\tfrac92-3+2\ln 3-2\ln 4+\ln 6\big)-\big(\tfrac12-1-2\ln 2+\ln 4\big) \\ &= 2+3\ln 3-3\ln 2 = 2+3\ln\tfrac32\,.\end{aligned}$$

(d) $\displaystyle\int_1^2 \frac{x^3+2}{x^5+2x^3+x}\,dx$

Solution: To begin with, we find $\displaystyle\int \frac{dx}{(x^2+1)^2}$, which we need later. Let $\tan\theta=x$ so that $\sec\theta=\sqrt{x^2+1}$ and $\sec^2\theta\,d\theta=dx$. Then

$$\begin{aligned}\int \frac{dx}{(x^2+1)^2} &= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta\,d\theta = \int \tfrac12+\tfrac12\cos 2\theta\;d\theta \\ &= \tfrac12\theta+\tfrac14\sin 2\theta+c = \tfrac12+\tfrac12\sin\theta\cos\theta+c = \tfrac12\tan^{-1}x+\tfrac12\frac{x}{x^2+1}+c\,.\end{aligned}$$

Note that $x^5+2x^3+x=x(x^2+1)^2$. In order to get $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}=\dfrac{x^3+2}{x(x^2+1)^2}$ we need $A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+F)(x^2+1)=x^3+2$. Equate coefficients to get the 5 equations $A+B=0$, $C=1$, $2A+B+D=0$, $C+E=0$ and $A=2$. Solve these to get $A=2$, $B=-2$, $C=1$, $D=-2$ and $E=-1$, and so

$$\begin{aligned}\int_1^2 \frac{x^3+2}{x^5+2x^3+x}\,dx &= \int_1^2 \frac{2}{x}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2x}{(x^2+1)^2}-\frac{1}{(x^2+1)^2}\,dx \\ &= \left[2\ln x-\ln(x^2+1)+\tan^{-1}x+\frac{1}{x^2+1}-\left(\tfrac12\tan^{-1}x+\tfrac12\frac{x}{x^2+1}\right)\right]_1^2 \\ &= \left[2\ln x-\ln(x^2+1)+\tfrac12\tan^{-1}x+\frac{2-x}{2(x^2+1)}\right]_1^2 \\ &= \big(2\ln 2-\ln 5+\tfrac12\tan^{-1}2\big)-\big(-\ln 2+\tfrac{\pi}{8}+\tfrac14\big) \\ &= \ln\tfrac85+\tfrac12\tan^{-1}2-\tfrac{\pi}{8}-\tfrac14\,.\end{aligned}$$

10: Find each of the following integrals.

(a) $\displaystyle\int_0^{\pi/2} \frac{\sin x\ dx}{\sin x + \cos x}$

Solution: We provide three solutions. For the first solution, write $I = \displaystyle\int_0^{\pi/2} \frac{\sin x\ dx}{\sin x + \cos x}$. Make the substitution $u = \frac{\pi}{2} - x$, $du = -dx$, and then make the substitution $x = u$, $dx = du$ to get

$$I = \int_{-\pi/2}^{0} -\frac{\sin\left(u - \frac{\pi}{2}\right)}{\sin\left(u - \frac{\pi}{2}\right) + \cos\left(u - \frac{\pi}{2}\right)}\ du = \int_0^{\pi/2} \frac{\cos u\ du}{\cos u + \sin u} = \int_0^{\pi/2} \frac{\cos x\ dx}{\cos x + \sin x}.$$

Thus we have

$$2I = \int_0^{\pi/2} \frac{\sin x\ dx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\cos x\ dx}{\sin x + \cos x} = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\ dx = \int_0^{\pi/2} 1\ dx = \tfrac{\pi}{2}$$

and so $I = \frac{\pi}{4}$.

For the second solution, write $I = \displaystyle\int \frac{\sin x\ dx}{\sin x + \cos x}$ and $J = \displaystyle\int \frac{\cos x\ dx}{\sin x + \cos x}$. Then we have

$$I + J = \int \frac{\sin x + \cos x}{\sin x + \cos x}\ dx = \int 1\ dx = x + c_1\ ,\ \text{and}$$

$$J - I = \int \frac{\cos x - \sin x}{\sin x + \cos x}\ dx = \ln(\sin x + \cos x) + c_2\,.$$

Subtract the second of these equations from the first to get $2I = x - \ln(\sin x + \cos x) + c$ so

$$I = \tfrac{1}{2}x - \tfrac{1}{2}\ln(\sin x + \cos x) + d\,.$$

Thus

$$\int_0^{\pi/2} \frac{\sin x\ dx}{\sin x + \cos x} = \Big[\tfrac{1}{2}x - \tfrac{1}{2}\ln(\sin x + \cos x)\Big]_0^{\pi/2} = \tfrac{\pi}{4}\,.$$

For the third solution we use the Weierstrass substitution $u = \tan(x/2)$, $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$, $dx = \frac{2}{1+u^2}\ du$. We obtain

$$\int_{x=0}^{\pi/2} \frac{\sin x\ dx}{\sin x + \cos x} = \int_{u=0}^{1} \frac{\frac{2u}{1+u^2}\,\frac{2}{1+u^2}}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}}\ du = \int_0^1 \frac{-4u\ du}{(u^2+1)(u^2-2u-1)}$$

To get $\displaystyle\frac{Au+B}{u^2+1} + \frac{Cu+D}{u^2-2u-1} = \frac{-4u}{(u^2+1)(u^2-2u+1)}$ we need $(Au+B)(u^2-2u-1)+(cu+d)(u^2+1) = -4u$. Equate coefficients to get $A + C = 0$, $-2A + B + D = 0$, $-A - 2B + C = -4$ and $-B + D = 0$. Solve these to get $A = 1$, $B = 1$, $C = -1$ and $D = 1$ and hence

$$\begin{aligned}\int_{x=0}^{\pi/2} \frac{\sin x\ dx}{\sin x + \cos x} &= \int_{u=0}^{1} \frac{-4u\ du}{(u^2+1)(u^2-2u-1)} = \int_0^1 \frac{u+1}{u^2+1} - \frac{u-1}{u^2-2u-1}\ du \\ &= \Big[\tan^{-1} u + \tfrac{1}{2}\ln(u^2+1) + \tfrac{1}{2}\ln|u^2-2u-1| + \Big]_0^1 = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln 2 = \tfrac{\pi}{4}.\end{aligned}$$

(b) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{x\ dx}{(\sin x + \cos x)^2}$

Solution: Note that $\sin x + \cos x = \sqrt{2}\cos\left((x - \frac{\pi}{4}\right)$. Integrate by parts using $u = x$, $du = dx$, $v = \tan\left(x - \frac{\pi}{4}\right)$ and $dv = \sec^2\left(x - \frac{\pi}{4}\right)$ to get

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{x\ dx}{(\sin x + \cos x)^2} &= \tfrac{1}{2}\int_{\pi/4}^{\pi/2} x\sec^2\left(x - \tfrac{\pi}{4}\right)\ dx = \tfrac{1}{2}\left[x\tan\left(x - \tfrac{\pi}{4}\right) - \int \tan\left(x - \tfrac{\pi}{4}\right)\ dx\right]_{\pi/4}^{\pi/2} \\ &= \left[\tfrac{1}{2}x\tan\left(x - \tfrac{\pi}{4}\right) - \tfrac{1}{2}\ln\left(\sec\left(x - \tfrac{\pi}{4}\right)\right)\right]_{\pi/4}^{\pi/2} = \tfrac{\pi}{4} - \tfrac{1}{2}\ln\sqrt{2} = \tfrac{1}{4}(\pi - \ln 2)\,.\end{aligned}$$

(c) $\displaystyle\int_{-\pi/4}^{\pi/2} \frac{dx}{(3\sin x + 4\cos x)^2}$

Solution: We give two solutions. For the first solution, we let $u = 3\tan x + 4$ so $du = 3\sec^2 x\, dx$ to get

$$\int_{-\pi/4}^{\pi/2} \frac{dx}{(3\sin x + 4\cos x)^2} = \int_{-\pi/4}^{\pi/2} \frac{\sec^2 x\; dx}{(3\tan x + 4)^2} = \int_{-1}^{\infty} \frac{\frac{1}{3}\, du}{u^2} = \left[\frac{-1}{3u}\right]_1^{\infty} = \tfrac{1}{3}\,.$$

For the second solution, we use the Weirstrass substitution $u = \tan(x/2)$, $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$, $dx = \frac{2}{1+u^2}\, du$ to get

$$\int \frac{dx}{(3\sin x + 4\cos x)^2} = \int \frac{\frac{2}{1+u^2}\, du}{\left(\frac{6u}{1+u^2} + \frac{4-4u^2}{1+u^2}\right)^2} = \int \frac{2(1+u^2)\, du}{(4+6u-4u^2)^2} = \int \frac{\frac{1}{2}(1+u^2)\, du}{(2u+1)^2(u-2)^2}\,.$$

To get $\dfrac{\frac{1}{2}(1+u^2)}{(2u+1)^2(u-2)^2} = \dfrac{A}{(2u+1)^2} + \dfrac{B}{(2u+1)^2} + \dfrac{C}{(u-2)} + \dfrac{D}{(u-2)^2}$ we need

$$A(2u+1)(u-2)^2 + B(u-2)^2 + C(2u+1)^2(u-2) + D(2u+1)^2 = \tfrac{1}{2}\, u^2 + \tfrac{1}{2}\,.$$

Put in $u = 2$ on both sides to get $25D = \frac{5}{2}$ so $D = \frac{1}{10}$, and put in $u = -\frac{1}{2}$ to get $\frac{25}{4}B = \frac{5}{8}$ so $B = \frac{1}{10}$. Equate the coefficients of u^3 on both sides to get $2A + 4C = 0$, and equate the coefficients of u^0 to get $4A + 4B - 2C + D = \frac{1}{2}$, that is $4A + \frac{4}{10} - 2C + \frac{1}{10} = \frac{1}{2}$, so $4A - 2C = 0$. From $2A + 4C = 0$ and $4A - 2C = 0$ we see that $A = C = 0$. Also, note that $\tan\left(-\frac{\pi}{8}\right) = \sqrt{2} - 1$, and so we have

$$\begin{aligned}\int_{-\pi/4}^{\pi/2} \frac{dx}{(3\sin x + 4\cos x)^2} &= \int_{\sqrt{2}-1}^{1} \frac{\frac{1}{10}}{(2u+1)^2} + \frac{\frac{1}{10}}{(u-2)^2}\; du = \left[\frac{-\frac{1}{20}}{2u+1} - \frac{\frac{1}{10}}{u-2}\right]_{\sqrt{2}-1}^{1} \\ &= \left(-\tfrac{1}{60} + \tfrac{1}{10}\right) - \left(-\tfrac{1}{20(3-2\sqrt{2})} + \tfrac{1}{10(1+\sqrt{2})}\right) \\ &= \tfrac{1}{12} + \tfrac{1}{20}\big((3+2\sqrt{2}) - 2(\sqrt{2}-1)\big) = \tfrac{1}{12} + \tfrac{1}{4} = \tfrac{1}{3}\,.\end{aligned}$$

(d) $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\sin^2 x}\, dx$

Solution: Make the substitution $u = \pi - x$, $du = -dx$ to get

$$I = \int_{\pi}^{0} -\frac{(\pi - u)(-\sin u)}{1+\sin^2 u}\, du = \int_0^{\pi} \frac{\pi\sin u}{1+\sin^2 u}\, du - \int_0^{\pi} \frac{u\sin u}{1+\sin^2 u}\, du = \int_0^{\pi} \frac{\pi\sin u}{1+\sin^2 u}\, du - I\,.$$

Add I to both sides to get $2I = \displaystyle\int_0^{\pi} \frac{\pi\sin u}{1+\sin^2 u}\, du$, and so $I = \frac{\pi}{2}\displaystyle\int_0^{\pi} \frac{\sin u}{1+\sin^2 u}\, du = \frac{\pi}{2}\int_0^{\pi} \frac{\sin u\, du}{2-\cos^2 u}$. Make the substitution $v = -\cos u$, $dv = \sin u\, du$ to get

$$\begin{aligned}I &= \tfrac{\pi}{2}\int_{-1}^{1} \frac{dv}{2-v^2} = \pi\int_0^1 \frac{dv}{2-v^2} = \tfrac{\pi}{2\sqrt{2}}\int_0^1 \frac{1}{\sqrt{2}+v} + \frac{1}{\sqrt{2}-v}\; dv \\ &= \tfrac{\pi}{2\sqrt{2}}\left[\ln\tfrac{\sqrt{2}+v}{\sqrt{2}-v}\right]_0^1 = \tfrac{\pi}{2\sqrt{2}}\ln\left((\sqrt{2}+1)^2\right) = \tfrac{\pi}{\sqrt{2}}\ln(\sqrt{2}+1)\,.\end{aligned}$$

11: Find each of the following definite integrals.

(a) $\displaystyle\int_1^2 (1+2x^2)e^{x^2}\,dx$

Solution: Using Integration by Parts with $u=x$, $du=dx$, $v=e^{x^2}$ and $dv=2xe^{x^2}\,dx$ gives

$$\int 2x^2e^{x^2}\,dx = xe^{x^2} - \int e^{x^2}\,dx$$

and so we have

$$\int (1+2x^2)e^{x^2}\,dx = \int e^{x^2}\,dx + \int 2x^2e^{x^2}\,dx = \int e^{x^2}\,dx + \left(xe^{x^2} - \int e^{x^2}\,dx\right) = xe^{x^2} + c\,.$$

Thus $\displaystyle\int_1^2 (1+2x^2)e^{x^2}\,dx = \left[xe^{x^2}\right]_1^2 = 2e^4 - e.$

(b) $\displaystyle\int_1^3 \frac{\sqrt[3]{5-x}\,dx}{\sqrt[3]{5-x}+\sqrt[3]{1+x}}$

Solution: Write $I = \displaystyle\int_1^3 \frac{\sqrt[3]{5-x}\,dx}{\sqrt[3]{5-x}+\sqrt[3]{1+x}}$. First make the substitution $u=4-x$, $x=4-u$, $dx=-du$, and then make the substitution $x=u$ to get

$$I = \int_{u=3}^1 \frac{-\sqrt[3]{1+u}\,du}{\sqrt[3]{1+u}+\sqrt[3]{5-u}} = \int_{u=1}^3 \frac{\sqrt[3]{1+u}\,du}{\sqrt[3]{1+u}+\sqrt[3]{5-u}} = \int_{x=1}^3 \frac{\sqrt[3]{1+x}\,dx}{\sqrt[3]{1+x}+\sqrt[3]{5-x}}\,.$$

Thus we have

$$2I = \int_1^3 \frac{\sqrt[3]{5-x}\,dx}{\sqrt[3]{5-x}+\sqrt[3]{1+x}} + \int_1^3 \frac{\sqrt[3]{1+x}\,dx}{\sqrt[3]{1+x}+\sqrt[3]{5-x}} = \int_1^3 \frac{\sqrt[3]{1+x}+\sqrt[3]{5-x}}{\sqrt[3]{1+x}+\sqrt[3]{5-x}}\,dx = \int_1^3 1\,dx = 2$$

and so $I=1$.

(c) $\displaystyle\int_0^2 \sqrt{x^3+1}+\sqrt[3]{x^2+2x}\,dx$

Solution: Let $f(x)=\sqrt{x^3+1}-1$. For $x\geq -1$ we have

$$\begin{aligned} y=f(x) &\iff y+1=\sqrt{x^3+1} \iff (y+1)^2=x^3+1 \\ &\iff y^2+2x=x^3 \iff x=\sqrt[3]{y^2+2x} \end{aligned}$$

and so we have $f^{-1}(x)=\sqrt[3]{x^2+2x}$. We have $f(0)=0$ and $f(2)=2$. Since $\int_0^2 f$ is the area of the region under $y=f(x)$ and $\int_0^2 f^{-2}$ is the are of the region to the left of $y=f(x)$ we have

$$\int_0^2 f + \int_0^2 f^{-1} = 4$$

and so

$$\int_0^2 \sqrt{x^3+1}+\sqrt[3]{x^2+2x}\,dx = \int_0^2 f(x)+1+f^{-1}(x)\,dx = 4+\int_0^2 1\,dx = 6\,.$$

(d) $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

Solution: Write $I=\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$. Let $u=\frac{1-x}{1+x}$, $x=\frac{1-u}{1+u}$, $dx=\frac{-(1+u)-(1-u)}{(1+u)^2}$. Note also that $x+1=\frac{2}{1+u}$ and $x^2+1=\frac{2(u^2+1)}{(1+u)^2}$, so

$$I=\int_{u=1}^0 \frac{\ln\left(\frac{2}{1+u}\right)\cdot\frac{-2}{(1+u)^2}\,du}{\frac{2(1+u^2)}{(1+u)^2}} = \int_{u=0}^1 \frac{\ln 2-\ln(1+u)}{1+u^2}\,du = \int_0^1 \frac{\ln 2}{1+u^2}\,du - I$$

and so we have $2I=\displaystyle\int_0^1 \frac{\ln 2}{1+u^2}\,du = \Big[\ln 2\cdot\tan^{-1}u\Big]_0^1 = \frac{\pi}{4}\ln 2$, and so $I=\frac{\pi\ln 2}{8}$.

12: Find the following improper integrals.

(a) $\displaystyle\int_1^\infty \frac{dx}{x^3\sqrt{x^2-1}}$

Solution: Make the substitution $\sec\theta = x$ so $\tan\theta = \sqrt{x^2-1}$ and $\sec\theta\tan\theta\,d\theta = dx$. Note that as $x\to\infty$ we have $\theta\to\frac{\pi}{2}$ and so

$$\begin{aligned}\int_1^\infty \frac{dx}{x^3\sqrt{x^2-1}} &= \int_0^{\pi/2}\frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\tan\theta} = \int_0^{\pi/2}\frac{d\theta}{\sec^2\theta} = \int_0^{\pi/2}\cos^2\theta\,d\theta \\ &= \int_0^{\pi/2}\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\ d\theta = \Big[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\Big]_0^{\pi/2} = \tfrac{\pi}{4}\,.\end{aligned}$$

(b) $\displaystyle\int_0^\infty \frac{e^x+1}{e^{2x}+1}\,dx$

Solution: Make the substitution $u=e^x$ so $du = e^x\,dx$. Note that as $x\to\infty$ we have $u\to\infty$, so

$$\int_0^\infty \frac{e^x+1}{e^{2x}+1}\,dx = \int_0^\infty \frac{e^x+1}{e^x(e^{2x}+1)}\,e^x\,dx = \int_1^\infty \frac{u+1}{u(u^2+1)}\,du\,.$$

To get $\dfrac{A}{u}+\dfrac{Bu+C}{x^2+1} = \dfrac{u+1}{u(u^2+1)}$ we need $A(u^2+1)+(Bu+C)u = u+1$. Equate coefficients to get $A+B=0$, $C=1$ and $A=1$, so we have $A=1$, $B=-1$ and $C=1$, and so

$$\begin{aligned}\int_0^\infty \frac{e^x+1}{e^{2x}+1}\,dx &= \int_1^\infty \frac{u+1}{u(u^2+1)}\,du = \int_1^\infty \frac{1}{u}-\frac{u}{u^2+1}+\frac{1}{u^2+1}\ du = \Big[\ln u - \tfrac{1}{2}\ln(u^2+1)+\tan^{-1}u\Big]_1^\infty \\ &= \Big[\tfrac{1}{2}\ln\frac{u^2}{u^2+1}+\tan^{-1}u\Big]_1^\infty = \left(\tfrac{\pi}{2}\right)-\left(\tfrac{1}{2}\ln\tfrac{1}{2}+\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}+\tfrac{1}{2}\ln 2\,,\end{aligned}$$

since $\lim\limits_{u\to\infty}\tan^{-1}u = \frac{\pi}{2}$ and as $u\to\infty$ we have $\frac{u^2}{u^2+1}\to 1$ and so $\lim\limits_{u\to\infty}\frac{1}{2}\ln\frac{u^2}{u^2+1} = \frac{1}{2}\ln 1 = 0$.

(c) $\displaystyle\int_3^\infty \frac{dx}{(x^2-1)\sqrt{x}}$

Solution: Make the substitution $u=\sqrt{x}$ so $u^2=x$ and $2u\,du = dx$ to get

$$\int_3^\infty \frac{dx}{(x^2-1)\sqrt{x}} = \int_{\sqrt{3}}^\infty \frac{2u\,du}{(u^4-1)\,u} = \int_{\sqrt{3}}^\infty \frac{2\,du}{u^4-1}\,.$$

Note that $u^4-1 = (u-1)(u+1)(u^2+1)$. To get $\dfrac{A}{u-1}+\dfrac{B}{u+1}+\dfrac{Cx+D}{u^2+1} = \dfrac{2}{u^4-1}$ we need $A(u+1)(u^2+1)+B(u-1)(u^2+1)+C(u-1)(u+1) = 2$. Equate coefficients to get the 4 equations $A+B+C=0$ (1), $A-B+D=0$ (2), $A+B-C=0$ (3), and $A-B-D=2$ (4). Subtracting (3) from (1) gives $2C=0$ so $C=0$. Subtracting (4) from (2) gives $2D=-2$ so $D=-1$. Put $C=0$ into (1) to get $A+B=0$ (5), and put $D=-1$ into (2) to get $A-B=1$ (6). Adding (5) and (6) gives $2A=1$ so $A=\frac{1}{2}$, and subtracting (6) from (5) gives $2B=-1$ so $B=-\frac{1}{2}$. Thus we have

$$\begin{aligned}\int_3^\infty \frac{dx}{(x^2-1)\sqrt{x}} &= \int_{\sqrt{3}}^\infty \frac{2\,du}{u^4-1} = \int_{\sqrt{3}}^\infty \frac{\frac{1}{2}}{u-1}-\frac{\frac{1}{2}}{u+1}-\frac{1}{u^2+1}\ du \\ &= \Big[\tfrac{1}{2}\ln\frac{u-1}{u+1}-\tan^{-1}u\Big]_{\sqrt{3}}^\infty = \left(-\tfrac{\pi}{2}\right)-\left(\tfrac{1}{2}\ln\tfrac{\sqrt{3}-1}{\sqrt{3}+1}-\tfrac{\pi}{3}\right) = \tfrac{1}{2}\ln(2+\sqrt{3})-\tfrac{\pi}{6}\end{aligned}$$

since $\lim\limits_{u\to\infty}\frac{1}{2}\ln\dfrac{u-1}{u+1} = 0$ and $\lim\limits_{u\to\infty}\tan^{-1}u = \frac{\pi}{2}$, and also $-\ln\frac{\sqrt{3}-1}{\sqrt{3}+1} = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\,\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \ln(2+\sqrt{3})$.

13: Find each of the following improper integrals.

(a) $\displaystyle\int_{-\infty}^{0} \big((x+1)e^x\big)^2\,dx$

Solution: Let $I=\displaystyle\int (x+1)^2e^{2x}\,dx$. Integrate by parts with $u=(x+1)^2$, $du=2(x+1)\,dx$, $v=\frac{1}{2}\,e^{2x}$ and $dv=e^{2x}\,dx$ to get $I=\dfrac{1}{2}(x^2+2x+1)e^{2x}-\displaystyle\int (x+1)e^{2x}\,dx$. Integrate by parts again, this time with $u=x+1$, $du=dx$, $v=\frac{1}{2}\,e^{2x}$ and $dv=e^{2x}\,dx$, to get

$$I=\tfrac{1}{2}(x^2+2x+1)e^{2x}-\tfrac{1}{2}(x+1)e^{2x}+\tfrac{1}{4}e^{2x}+c=\tfrac{1}{4}(2x^2+2x+1)\,e^{2x}+c\,.$$

Thus $\displaystyle\int_{-\infty}^{0}(x+1)^2e^{2x}\,dx=\tfrac{1}{4}\Big[(2x^2+2x+1)e^{2x}\Big]_{-\infty}^{0}=\tfrac{1}{4}$.

(b) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{e^x-1}}$

Solution: Let $u=\sqrt{e^x-1}$, so $u^2=e^x-1$, $2u\,du=e^x\,dx=(1+u^2)\,dx$, and $dx=\frac{2u}{1+u^2}\,du$. Then

$$\int_{x=0}^\infty \frac{dx}{\sqrt{e^x-1}}=\int_0^\infty \frac{2\,du}{1+u^2}=\Big[2\tan^{-1}u\Big]_{0^+}^{\infty}=\pi\,.$$

(c) $\displaystyle\int_{-\infty}^\infty \frac{x^3+4x+9}{(x^2+1)^2(x^2+4)}\,dx$

Solution: Let $I=\displaystyle\int_{-\infty}^\infty \frac{x^3+4x+9}{(x^2+1)^2(x^2+4)}\,dx=\int_{-\infty}^\infty \frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{Ex+F}{(x^2+4)}\,dx$, where

$$(Ax+B)(x^2+1)(x^2+4)+(Cx+D)(x^2+4)+(Ex+F)(x^2+1)^2=x^3+4x+9\,,$$

that is $A(x^5+5x^3+4x)+B(x^4+5x^2+4)+C(x^3+4x)+D(x^2+4)+E(x^5+2x^3+x)+F(x^4+2x+1)=x^3+4x+9$. Equate coefficients to get the 6 equations $4B+4D+F=9$, $4A+4C+E=4$, $5B+D+2F=0$, $5A+C+2E=1$, $B+F=0$ and $A+E=0$. We consider this as two sets of 3 equations, and solve them together using some linear algebra:

$$\left(\begin{array}{ccc|cc}4&4&1&4&9\\5&1&2&1&0\\1&0&1&0&0\end{array}\right)\sim\left(\begin{array}{ccc|cc}1&0&1&0&0\\5&1&2&1&0\\4&4&1&4&9\end{array}\right)\sim\left(\begin{array}{ccc|cc}1&0&1&0&0\\0&1&-3&1&0\\0&0&1&0&1\end{array}\right)\sim\left(\begin{array}{ccc|cc}1&0&0&0&-1\\0&1&0&1&3\\0&0&1&0&1\end{array}\right)$$

We find that $A=0$, $C=1$, $E=0$, and $B=-1$, $D=3$ and $F=1$. Thus we have

$$I=\int_{-\infty}^\infty -\frac{1}{x^2+1}+\frac{x}{(x^2+1)^2}+\frac{3}{(x^2+1)^2}+\frac{1}{x^2+4}\,dx\,.$$

To find $\displaystyle\int\frac{dx}{(x^2+1)^2}$, let $\tan\theta=x$, so $\sec\theta=\sqrt{x^2+1}$ and $\sec^2\theta\,d\theta=dx$ to get $\displaystyle\int\frac{dx}{(x^2+1)^2}=\int\frac{\sec^2\theta}{\sec^4\theta}\,d\theta=\int\cos^2\theta\;d\theta=\int\frac{1+\cos2\theta}{2}\,d\theta=\tfrac{1}{2}+\tfrac{1}{4}\sin2\theta+c=\tfrac{1}{2}\theta+\tfrac{1}{2}\sin\theta\cos\theta+c=\tfrac{1}{2}\tan^{-1}x+\tfrac{1}{2}\tfrac{x}{x^2+1}+c$. Thus we have

$$\begin{aligned}I&=\Big[-\tan^{-1}x-\tfrac{1}{2}\tfrac{1}{x^2+1}+\tfrac{3}{2}\big(\tan^{-1}x+\tfrac{x}{x^2+1}\big)+\tfrac{1}{2}\tan^{-1}\tfrac{x}{2}\Big]_{-\infty}^{\infty}\\&=\tfrac{1}{2}\Big[\tan^{-1}x+\tan^{-1}\tfrac{x}{2}+\tfrac{3x-1}{x^2+1}\Big]_{-\infty}^{\infty}=\tfrac{1}{2}\big[\big(\tfrac{\pi}{2}+\tfrac{\pi}{2}+0\big)-\big(-\tfrac{\pi}{2}-\tfrac{\pi}{2}+0\big)\big]=\pi\,.\end{aligned}$$

14: Evaluate the following improper integrals.

(a) $\displaystyle\int_0^{\pi/2} \sqrt{2\tan x}\ dx$

Solution: Let $I = \displaystyle\int_0^{\pi/2} \sqrt{2\tan x}\, dx$. Make the substitution $u = \sqrt{2\tan x}$, $u^2 = 2\tan\theta$, $2u\,du = 2\sec^2\theta\,d\theta$, $4u\,du = (4\tan^2\theta + 4)d\theta = (u^4+4)d\theta$, $d\theta = \frac{4u}{u^4+4}\,du$ to get

$$\begin{aligned}
I &= \int_0^\infty \frac{4u^2\,du}{u^4+4} = \int_0^\infty \frac{4u^2\,du}{(u^2+2u+2)(u^2-2u+2)} = \int_0^\infty \frac{u}{u^2-2u+2} - \frac{u}{u^2+2u+2}\,du \\
&= \int_0^\infty \frac{u-1}{u^2-2u+2} + \frac{1}{u^2-2u+2} - \frac{u+1}{u^2+2u+2} + \frac{1}{u^2+2u+2}\,du \\
&= \Big[\tfrac12\ln(u^2-2u+2) + \tan^{-1}(u-1) - \tfrac12\ln(u^2+2u+2) + \tan^{-1}(u+1)\Big]_0^\infty \\
&= \Big[\tfrac12\ln\tfrac{u^2-2u+2}{u^2+2u+2} + \tan^{-1}(u+1) + \tan^{-1}(u-1)\Big]_0^\infty \\
&= \big(0 + \tfrac{\pi}{2} + \tfrac{\pi}{2}\big) - \big(0 + \tfrac{\pi}{4} - \tfrac{\pi}{4}\big) \ = \pi\,.
\end{aligned}$$

(b) $\displaystyle\int_2^{17/4} \sqrt{\frac{x+2}{x-2}}\,dx$

Solution: First we make the substitution $u = \sqrt{x-2}$ so $u^2 = x-2$ and $2u\,du = dx$, and then we make the substitution $2\tan\theta = u$ so $2\sec\theta = \sqrt{u^2+4}$ and $2\sec^2\theta\,d\theta = du$. We obtain

$$\begin{aligned}
\int_2^{17/4} \sqrt{\frac{x+2}{x-2}}\,dx &= \int_0^{3/2} \frac{\sqrt{u^2+4}}{u}2u\,du = \int_0^{3/2} 2\sqrt{u^2+4}\,du = \int_0^{\tan^{-1}(3/4)} 2\cdot 2\sec\theta\cdot 2\sec^2\theta\,d\theta \\
&= \int_0^{\tan^{-1}(3/4)} 8\sec^3\theta\,d\theta = \Big[4\sec\theta\tan\theta + 4\ln(\sec\theta+\tan\theta)\Big]_0^{\tan^{-1}(3/4)} \\
&= 4\cdot\tfrac54\cdot\tfrac34 + 4\ln\big(\tfrac54+\tfrac34\big) = \tfrac{15}{4} + 4\ln 2\,.
\end{aligned}$$

We used the fact that $\displaystyle\int \sec^3\theta\,d\theta = \tfrac12\sec\theta\tan\theta + \tfrac12\ln\big|\sec\theta+\tan\theta\big| + c$, which is shown in Example 2.22, and also that $\sec\big(\tan^{-1}\tfrac34\big) = \tfrac54$, which can be seen from a right-angled triangle with sides of lengths 3, 4 and 5.

(c) $\displaystyle\int_0^{\ln 2} \sqrt{\frac{e^x+1}{e^x-1}}\,dx$

Solution: Make the substitution $u = \sqrt{\dfrac{e^x+1}{e^x-1}}$ so $e^x = \dfrac{u^2+1}{u^2-1}$, $x = \ln(u^2+1) - \ln(u+1) - \ln(u-1)$, and $dx = \Big(\frac{2u}{u^2+1} - \frac{1}{u+1} - \frac{1}{u-1}\Big)du$ to get

$$\begin{aligned}
\int_0^{\ln 2} \sqrt{\frac{e^x+1}{e^x-1}}\,dx &= \int_\infty^{\sqrt3} u\left(\frac{2u}{u^2+1} - \frac{1}{u+1} - \frac{1}{u-1}\right)du = \int_{\sqrt3}^\infty -\frac{2u^2}{u^2+1} + \frac{u}{u+1} + \frac{u}{u-1}\,du \\
&= \int_{\sqrt3}^\infty -2 + \frac{2}{u^2+1} + 1 - \frac{1}{u+1} + 1 + \frac{1}{u-1}\,du = \int_{\sqrt3}^\infty \frac{2}{u^2+1} - \frac{1}{u+1} + \frac{1}{u-1}\,du \\
&= \Big[2\tan^{-1}u + \ln\tfrac{u-1}{u+1}\Big]_{\sqrt3}^\infty = (\pi) - \Big(\tfrac{2\pi}{3} + \ln\Big(\tfrac{\sqrt3-1}{\sqrt3+1}\Big)\Big) = \tfrac{\pi}{3} + \ln(2+\sqrt3)\,.
\end{aligned}$$

15: Test each of the following improper integrals for convergence.

(a) $\displaystyle\int_1^\infty \frac{dx}{e^x \ln x}$

Solution: In order for $\displaystyle\int_1^\infty \frac{dx}{e^x \ln x}$ to converge, both of the integrals $\displaystyle\int_1^2 \frac{dx}{e^x \ln x}$ and $\displaystyle\int_2^\infty \frac{dx}{e^x \ln x}$ must converge. When $1 < x \leq 2$ we have $e < e^x \leq e^2$ and $0 < \ln x \leq x - 1$ and so $0 < \dfrac{1}{e^2(x-1)} \leq \dfrac{1}{e^x \ln x}$. Since $\displaystyle\int_1^2 \frac{dx}{e^2(x-1)} = \left[\tfrac{1}{e^2}\ln(x-1)\right]_{1^+}^2 = 0 - (-\infty) = \infty$, the integral $\displaystyle\int_1^2 \frac{dx}{e^x \ln x}$ also diverges.

(b) $\displaystyle\int_0^\infty 2^{1/x} - 1\; dx$

Solution: In order for $\displaystyle\int_0^\infty 2^{1/x} - 1\; dx$ to converge, both the integrals $\displaystyle\int_0^1 2^{1/x} - 1\; dx$ and $\displaystyle\int_1^\infty 2^{1/x} - 1\; dx$ must converge. For $u \geq 0$ we have $2^u \geq u \ln 2 + 1$, so for $x > 0$ we have $2^{1/x} \geq \frac{1}{x}\ln 2 + 1$, and so $2^{1/x} - 1 \geq \frac{\ln 2}{x}$. But $\displaystyle\int_1^\infty \frac{\ln 2}{x}\, dx$ diverges, and so $\displaystyle\int_1^\infty 2^{1/x} - 1\; dx$ diverges too.

(c) $\displaystyle\int_1^\infty \frac{dx}{(\ln x)^{\ln x}}$

Solution: This integral converges if and only if the integrals $\displaystyle\int_1^c \frac{dx}{(\ln x)^{\ln x}}$ and $\displaystyle\int_c^\infty \frac{dx}{(\ln x)^{\ln x}}$ both converge, where $c = e^{e^2}$ (this choice of c will become clear later). Since $\displaystyle\lim_{x \to 1^+} (\ln x)^{\ln x} = \lim_{u \to 0^+} u^u = 1$ (this is a well-known limit which can be found with the help of l'Hôpital's Rule), the function $f(x)$, defined by $f(0) = 1$ and $f(x) = \frac{1}{(\ln x)^{\ln x}}$ for $x > 0$, is continuous and hence integrable on $[1, c]$, and so $\displaystyle\int_1^c \frac{dx}{(\ln x)^{\ln x}}$ converges. Also, note that for $x \geq c = e^{e^2}$ we have $(\ln x)^{\ln x} = e^{\ln x \ln \ln x} \geq e^{\ln x \ln \ln e^{e^2}} = e^{2\ln x} = x^2$ and so $\dfrac{1}{(\ln x)^{\ln x}} \leq \dfrac{1}{x^2}$. Since $\displaystyle\int_c^\infty \frac{1}{x^2}$ converges, $\displaystyle\int_c^\infty \frac{dx}{(\ln x)^{\ln x}}$ converges too.

(d) $\displaystyle\int_0^\infty \sin(x)\sin(x^2)\, dx$

Solution: We shall show that this integral converges. Note that we can change the lower limit of integration to 1 without affecting convergence. Integrate by parts using $u = \dfrac{\sin x}{2x}$, $du = \dfrac{2x\cos x - 2\sin x}{4x^2}\, dx$, $v = -\cos(x^2)$ and $dv = 2x\sin(x^2)\, dx$ to get

$$\int \sin(x)\sin(x^2)\, dx = \frac{\sin(x)\cos(x^2)}{2x} + \int \frac{\cos(x)\cos(x^2)}{2x}\, dx - \int \frac{\sin(x)\cos(x^2)}{2x^2}\, dx\,.$$

Integrate by parts using $u = \dfrac{\cos x}{4x^2}$, $du = \dfrac{-x^2\sin x - 2x\cos x}{4x^4}\, dx$, $v = \sin x^2$ and $dv = 2x\cos x^2\; dx$ to get

$$\int \frac{\cos(x)\cos(x^2)}{2x}\, dx = \frac{\cos(x)\sin(x^2)}{4x^2} + \int \frac{\sin(x)\sin(x^2)}{4x^2}\, dx + \int \frac{\cos(x)\sin(x^2)}{2x^3}\, dx\,.$$

Thus we have

$$\begin{aligned}\int_1^r \sin(x)\sin(x^2)\, dx = &\left[-\frac{\sin(x)\cos(x^2)}{2x} + \frac{\cos(x)\sin(x^2)}{4x^2}\right]_1^r \\ &- \int_1^r \frac{\sin(x)\cos(x^2)}{2x^2}\, dx + \int_1^r \frac{\sin(x)\sin(x^2)}{4x^2}\, dx + \int_1^r \frac{\cos(x)\sin(x^2)}{2x^3}\, dx\,.\end{aligned}$$

Note that as $r \to \infty$, all of the limits on the right converge. For example, $\displaystyle\int_1^\infty \frac{\sin(x)\cos(x^2)}{2x^2}\, dx$ converges since for all r we have $\left|\dfrac{\sin(x)\cos(x^2)}{2x^2}\right| \leq \dfrac{1}{2x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{2x^2}$ converges.

16: (a) Let $f(x)=\displaystyle\int_0^x \frac{\sin(2x-t)}{2x-t}\,dt$. Find $f'\left(\frac{\pi}{2}\right)$.

Solution: Fix $x>0$ and let $u(t)=2x-t$ so $du=-dt$. Then we have

$$f(x)=\int_0^x \frac{\sin(2x-t)}{2x-t}\,dt=\int_{2x}^x -\frac{\sin u}{u}\,du=\int_{\pi/2}^{2x}\frac{\sin u}{u}\,du-\int_{\pi/2}^{x}\frac{\sin u}{u}\,du$$

and so by the Fundamental Theorem of Calculus we have $f'(x)=\dfrac{\sin 2x}{2x}-\dfrac{\sin x}{x}$. In particular, $f'\left(\frac{\pi}{2}\right)=-\frac{2}{\pi}$.

(b) Let $f(x)=\displaystyle\int_0^{x^3} e^{t^2}\,dt$. Show that $6\displaystyle\int_0^1 x^2 f(x)\,dx-2\int_0^1 e^{x^2}\,dx=1-e$.

Solution: Let $u(x)=x^3$ and let $G(u)=\displaystyle\int_0^u e^{t^2}\,dt$ so that $f(x)=G(u(x))$. Then by the Fundamental Theorem, $f'(x)=G'(u(x))u'(x)=3x^2\,e^{u^2}=3x^2\,e^{x^6}$. Now, we calculate $\displaystyle\int_0^1 x^2\,f(x)\,dx$ by integrating by parts using $u=f(x)$, $du=f'(x)\,dx=3x^2e^{x^6}$, $v=\frac{1}{3}\,x^3\,dx$ and $dv=x^2\,dx$ to get

$$\int_0^1 x^2\,f(x)\,dx=\left[\tfrac{1}{3}\,x^3 f(x)-\int x^5e^{x^6}\,dx\right]_0^1=\left[\tfrac{1}{3}\,x^3\,f(x)-\tfrac{1}{6}\,e^{x^6}\right]_0^1=\tfrac{1}{3}\,f(1)-\tfrac{1}{6}\,e+\tfrac{1}{6}=\tfrac{1}{3}\int_0^1 e^{t^2}\,dt-\tfrac{1}{6}\,e+\tfrac{1}{6}\,.$$

Since $\displaystyle\int_0^1 e^{t^2}\,dt=\int_0^1 e^{x^2}\,dt$, multiplying both sides gives $6\displaystyle\int_0^1 x^2 f(x)\,dx=2\int_0^1 e^{x^2}\,dx-e+1$, as required.

(c) Let f be continuous. Show that if $f\geq 0$ and $\displaystyle\int_1^3 f(x)\,dx=1$ then $\displaystyle\int_1^9 f(\sqrt{x})\,dx\leq 6$.

Solution: Let $u(x)=\sqrt{x}$ so that $du=\frac{1}{2\sqrt{x}}\,dx$. Since $\sqrt{x}$ is an increasing function, when $x\in[1,9]$ we have $\frac{1}{2}\geq u'(x)\geq\frac{1}{6}$. Since $f(u(x))\geq 0$ for all x, we have $f(u(x))\,u'(x)\geq\frac{1}{6}\,f'(u(x))$ for all $x\in[1,9]$. Since f is continuous, we can use the Change of Variables Rule to get

$$\int_1^3 f(u)\,du=\int_1^9 f(u(x))\,u'(x)\,dx\geq\int_1^9 \tfrac{1}{6}\,f(u(x))\,dx=\tfrac{1}{6}\int_1^9 f(\sqrt{x})\,dx\,.$$

Since $\displaystyle\int_1^3 f(u)\,du=\int_1^3 f(x)\,dx$, this gives $\displaystyle\int_1^3 f(x)\,dx\geq\frac{1}{6}\int_1^9 f(\sqrt{x})\,dx$. Thus if $\displaystyle\int_1^3 f(x)\,dx=1$ then we have $\displaystyle\int_1^9 f(\sqrt{x})\,dx\leq 6$, as required.

(d) Let f be integrable with $\displaystyle\int_0^1 f(x)\,dx=\int_0^1 xf(x)\,dx=1$. Show that $\displaystyle\int_0^1 f(x)^2\,dx\geq 4$.

Solution: Let $g(x)=6x-2$. This function g was chosen so that $\displaystyle\int_0^1 g=\int_0^1 6x-2\;dx=\Big[3x^2-2x\Big]_0^1=1$ and $\displaystyle\int_0^1 xg=\int_0^1 6x^2-2x\;dx=\Big[2x^3-x^2\Big]_0^1=1$. Note also that $\displaystyle\int_0^1 g^2=\int_0^1 36x^2-24x+4\;dx=\Big[12x^3-12x^2+4x\Big]_0^1=4$. Then we have

$$\begin{aligned}\int_0^1 f^2&=\int_0^1\big((f-g)+g\big)^2=\int_0^1 (f-g)^2+2(f-g)g+g^2=\int_0^1 (f-g)^2+2fg-g^2\\&=\int_0^1 (f-g)^2+2(6x-2)f-g^2=\int_0^1 (f-g)^2+12\int_0^1 xf-4\int_0^1 f-\int_0^1 g^2\\&=\int_0^1 (f-g)^2+12-4-4=\int_0^1 (f-g)^2+4\geq 4\,.\end{aligned}$$