MATH 148 Calculus 2, Solutions to the Exercises for Appendix 1

1: Let p_n be equal to one half of the perimeter of a regular polygon with 2^{n+1} sides which is circumscribed around the unit circle (note that $\{p_n\}_{n\geq 1}$ is decreasing and tends to π). Find p_1, p_2, p_3 and p_4. Express your answers as algebraic numbers (that is in terms of integers and radicals, not in terms of values of trigonometric functions).

Solution: Let $\theta_n = \frac{\pi}{2^{n+1}}$. Then the regular polygon with 2^{n+1} sides which is circumscribed around the unit circle $x^2+y^2=1$, and touches the circle at the point $(1,0)$, will have a vertex at the point $(1,\tan\theta_n)$. Note that $\tan\theta_n$ is the distance from $(1,0)$ to $(1,\tan\theta_n)$ which is one half of the length of one side of the polygon. Thus

$$p_n = 2^{n+2}\tan\theta_n\,.$$

Since $\theta_1 = \frac{\pi}{4}$ we have $\cos\theta_1 = \frac{1}{2}\sqrt{2}$. Since $\theta_{n+1} = \frac{\theta_n}{2}$ we have $\cos\theta_{n+1} = \frac{1}{2}\sqrt{2+2\cos\theta}$, and so we obtain

$$\big\{\cos\theta_n\big\}_{n\geq 1} = \tfrac{1}{2}\sqrt{2},\ \tfrac{1}{2}\sqrt{2+\sqrt{2}},\ \tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},\ \tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},\cdots$$

$$\big\{\cos^2\theta_n\big\}_{n\geq 1} = \tfrac{1}{4}(2),\ \tfrac{1}{4}(2+\sqrt{2}),\ \tfrac{1}{4}(2+\sqrt{2+\sqrt{2}}),\ \tfrac{1}{4}(2+\sqrt{2+\sqrt{2+\sqrt{2}}}),\cdots$$

$$\big\{1-\cos^2\theta_n\big\}_{n\geq 1} = \tfrac{1}{4}(2),\ \tfrac{1}{4}(2-\sqrt{2}),\ \tfrac{1}{4}(2-\sqrt{2+\sqrt{2}}),\ \tfrac{1}{4}(2-\sqrt{2+\sqrt{2+\sqrt{2}}}),\cdots$$

$$\big\{\sin\theta_n\big\}_{n\geq 1} = \tfrac{1}{2}\sqrt{2},\ \tfrac{1}{2}\sqrt{2-\sqrt{2}},\ \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}},\ \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}},\cdots$$

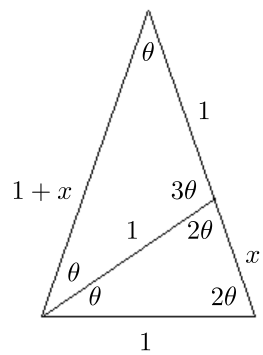
$$\big\{\tan\theta_n\big\}_{n\geq 1} = 1,\ \sqrt{\tfrac{2-\sqrt{2}}{2+\sqrt{2}}},\ \sqrt{\tfrac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ \sqrt{\tfrac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}},\cdots$$

$$\big\{p_n\big\}_{n\geq 1} = 4,\ 8\sqrt{\tfrac{2-\sqrt{2}}{2+\sqrt{2}}},\ 16\sqrt{\tfrac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ 32\sqrt{\tfrac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}},\cdots$$

If you go to the trouble of rationalizing the denominators, you will find that $p_1 = 4$, $p_2 = 8\big(\sqrt{2}-1\big)$, $p_3 = 16\Big(\sqrt{2}\sqrt{2+\sqrt{2}}-\sqrt{2}-1\Big)$ and $p_4 = 32\left(\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{2+\sqrt{2+\sqrt{2}}}-\sqrt{2}\sqrt{2+\sqrt{2}}-\sqrt{2}-1\right)$.

2: (a) Find the value of $\cos\left(\frac{\pi}{5}\right)$. Express your answer in terms of integers and radicals.

Solution: Let $\theta = \frac{\pi}{5}$. Consider an isosceles triangle with base $b = 1$ and with angles θ, 2θ and 2θ, with another similar inscribed triangle with base x, as shown below.



Since the two triangles are similar we have $\dfrac{1+x}{1} = \dfrac{1}{x}$ and so $x + x^2 = 1$ or equivalently $x^2 + x - 1 = 0$. From the quadratic formula, $x = \frac{-1+\sqrt{5}}{2}$. Note also that $x^2 = 1 - x = \frac{3-\sqrt{5}}{2}$. From the Law of Cosines, applied to the smaller triangle, we obtain

$$\cos\theta = \frac{2 - x^2}{2} = 1 - \tfrac{1}{2}x^2 = \tfrac{1+\sqrt{5}}{4}\,.$$

(b) Find the area of a regular decagon with sides of length 1 (a decagon has 10 sides).

Solution: A regular decagon with sides of length 1 can be cut into 10 copies of the above triangle. The trangle has base $b = 1$ and height

$$h = \tfrac{1}{2}\tan 2\theta = \frac{\frac{1}{2}\sin 2\theta}{\cos 2\theta} = \frac{\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\,.$$

Since $\cos\theta = \frac{1+\sqrt{5}}{4}$, we have $\cos^2\theta = \frac{6+2\sqrt{5}}{16} = \frac{3+\sqrt{5}}{8}$, $1 - \cos^2\theta = \frac{5-\sqrt{5}}{8}$ and $\sin\theta = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$, so

$$h = \frac{\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\cdot\frac{1+\sqrt{5}}{4}}{\frac{3+\sqrt{5}}{8} - \frac{5-\sqrt{5}}{8}} = \frac{\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\cdot\frac{1+\sqrt{5}}{4}}{\frac{\sqrt{5}-1}{4}} = \tfrac{1}{2}\sqrt{\tfrac{5-\sqrt{5}}{2}}\cdot\tfrac{\sqrt{5}+1}{\sqrt{5}-1}\,.$$

Since $\frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{(\sqrt{5}+1)^2}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$, we have

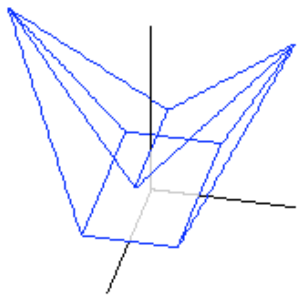
$$\begin{aligned} h &= \tfrac{1}{2}\sqrt{\tfrac{5-\sqrt{5}}{2}\cdot\tfrac{3+\sqrt{5}}{2}} = \tfrac{1}{4}\sqrt{\tfrac{(5-\sqrt{5})(3+\sqrt{5})^2}{2}} = \tfrac{1}{4}\sqrt{\tfrac{(5-\sqrt{5})(14+6\sqrt{6})}{2}} \\ &= \tfrac{1}{4}\sqrt{(5-\sqrt{5})(7+3\sqrt{5})} = \tfrac{1}{4}\sqrt{20+8\sqrt{5}} = \tfrac{1}{2}\sqrt{5+2\sqrt{5}}\,. \end{aligned}$$

Thus the area of one triangle is $\frac{1}{2}bh = \frac{1}{4}\sqrt{5+2\sqrt{5}}$, and so the area of the decagon is $A = \frac{5}{2}\sqrt{5+2\sqrt{5}}$.

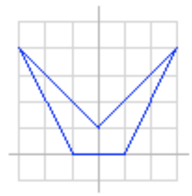
3: Let A be the rectangle-based cone with its base vertices at $(\pm 2, \pm 1, 0)$ and with its top vertex at $(0, 3, 4)$, and let B be the rectangle-based cone with the same base but with its top vertex at $(0, -3, 4)$. Find the volume and the surface area of the solid $A \cup B$.

Solution: First let us find the total surface area with the help of the following pictures.

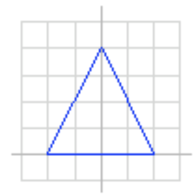
$A \cup B$



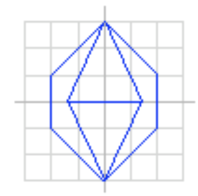
Front View



Side View



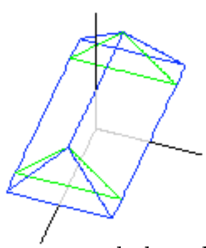
Top View



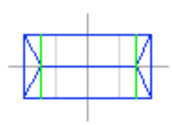
The base of the solid is a rectangle of area $A_1 = 2 \cdot 4 = 8$. The two triangles on the left and right sides (when the x-axis is pointing towards us) are congruent to each other. Their base is 4 and their height is $2\sqrt{5}$ (this height can best be seen from the front view), so they each have area $A_2 = \frac{1}{2} \cdot 4 \cdot 2\sqrt{5} = 4\sqrt{5}$. The two triangles on the top are congruent to each other. Their base is 3 (as seen from the top view) and their height is $3\sqrt{2}$ (from the front view), so they each have area $A_3 = \frac{1}{2} \cdot 3 \cdot 3\sqrt{2} = \frac{9\sqrt{2}}{2}$. Finally we consider the front and back faces of the solid. Each of the two faces is formed from two overlapping triangles. Each of the overlapping triangles has base 2 and height $2\sqrt{5}$ (best seen from the side view), so each triangle has area $\frac{1}{2} \cdot 2 \cdot 2\sqrt{5} = 2\sqrt{5}$. They overlap in a smaller triangle with base 2 and height $\frac{1}{2}\sqrt{5}$, so the area of this smaller triangle is $\frac{1}{2} \cdot 2 \cdot \frac{1}{2}\sqrt{5} = \frac{1}{2}\sqrt{5}$. Thus the area of each of the front and back faces is $A_4 = 2\sqrt{5} + 2\sqrt{5} - \frac{1}{2}\sqrt{5} = \frac{7\sqrt{5}}{2}$. Finally, the total surface area of the solid (including the base) is $A_1 + 2A_2 + 2A_3 + 2A_4 = 8 + 2 \cdot 4\sqrt{5} + 2 \cdot \frac{9\sqrt{2}}{2} + 2 \cdot \frac{7\sqrt{5}}{2} = 8 + 9\sqrt{2} + 15\sqrt{5}$

Now, let us find the volume with the help of some pictures of the intersection $A \cap B$.

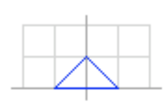
$A \cap B$



Top View



Front View



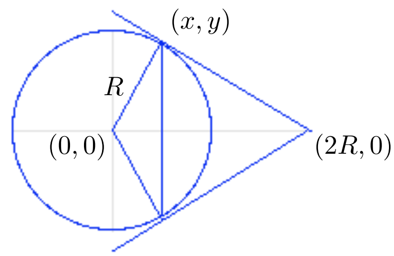
Side View

The two rectangle-based cones have base area 8 and height 4 and so they each have volume $V_1 = \frac{1}{3} \cdot 8 \cdot 4 = \frac{32}{3}$. The intersection $A \cap B$ is in the form of a roof as shown. We can cut $A \cap B$ into three pieces, along the green lines, as shown. The centre piece is a triangle-based prism with base area $\frac{1}{2} \cdot 2 \cdot 1 = 1$ and height 3, so its volume is $1 \cdot 3 = 3$. The other two pieces can be put together to form a rectangle-based pyramid with base area $1 \cdot 2 = 2$ and height 1, hence with volume $\frac{1}{3} \cdot 2 \cdot 1 = \frac{2}{3}$. Thus the volume of $A \cap B$ is $V_2 = 3 + \frac{2}{3} = \frac{11}{3}$. The total volume of $A \cup B$ is $V = 2V_1 - V_2 = \frac{64}{3} - \frac{11}{3} = \frac{53}{3}$.

4: Let R be the radius of the Earth ($R \cong 6,000$ km).

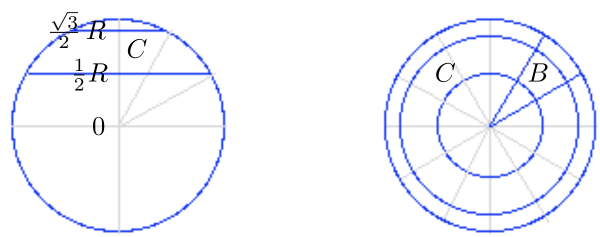
(a) A satellite orbits the Earth at a distance $2R$ from the Earth's center. Let A be the set of points on the Earth's surface from which the satellite is visible (at one instant in time). Find the area of A.

Solution: In the diagram below, the satellite is represented by the point $(2R, 0)$. From similar triangles, we see that $x = \frac{1}{2}R$, so the portion of the Earth's surface from which the satellite is visible is a spherical cap of thickness $l = \frac{1}{2}R$, and so its surface area is $A = 2\pi R\, l = \pi R^2$.



(b) Let B be the portion of the Earth's surface which lies between 30° and 60° latitude and between 30° and 60° longitude. Find the area of B.

Solution: Let C be the portion of the Earth's surface which lies between 30° and 60° latitude. Then C is a slice of the sphere of thickness $l = \frac{\sqrt{3}}{2}R - \frac{1}{2}R = \frac{\sqrt{3}-1}{2}R$, and so its area is $2\pi R l = 2\pi R \cdot \frac{\sqrt{3}-1}{2}\, R = (\sqrt{3}-1)\pi R^2$. The lines of latitude at 0°, 30°, 60°, 90°, 120° and 150° cut C into 12 equal parts, one of which is the region B, and so the area of B is $\frac{1}{12}(\sqrt{3}-1)\pi\, R^2$.

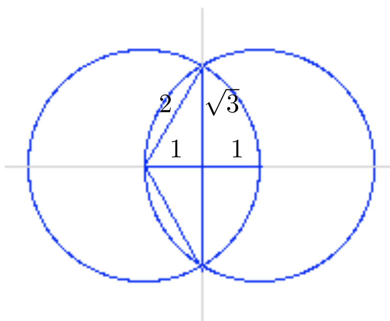


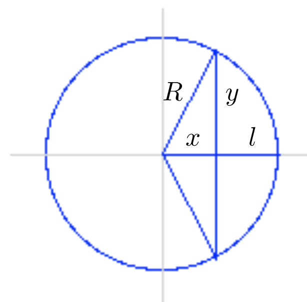
5: (a) Let A be the ball of radius 2 centered at $(1,0,0)$ and let B be the ball of radius 2 centered at $(-1,0,0)$. Find the volume of the solid $A \cap B$.

Solution: As seen with the help of the diagram below on the left, $A \cap B$ is the union of two spherical caps of thickness 1. The volume of $A \cap B$ is twice the volume of a spherical cap of thickness 1 on a sphere of radius 2. Let us make a general formula for the volume of a spherical cap of thickness l on a sphere of radius R. Notice that a sphere-based cone whose base is a spherical circle, is the union of a flat-based cone with a spherical cap. Let x and y be as shown in the diagram on the right, and note that $x = R - l$ and $x^2 + y^2 = R^2$. The surface area of the spherical cap of thickness l is $A_1 = 2\pi\, Rl$, and the sphere-based cone with base area A_1 has volume $V_1 = \frac{1}{3}\, A_1 R = \frac{2}{3}\, \pi R^2\, l$. The area of the flat disc of radius $r = y$ is $A_2 = \pi\, r^2 = \pi\, y^2$, and the flat-based cone with base area A_2 and height $h = x$ has volume $V_2 = \frac{1}{3}\, A_2 h = \frac{1}{3}\pi y^2 x$. Thus the volume of the spherical cap is $V = V_1 - V_2 = \frac{2}{3}\pi R^2 l - \frac{1}{3}\pi y^2 x$. Put in $x = R - l$ and $y^2 = R^2 - x^2$ and simplify to get

$$V = \pi\, l^2 \left(R - \tfrac{1}{3} l\right) .$$

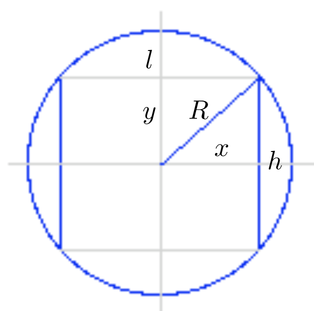
In particular, when $R = 2$ and $l = 1$ we get $V = \frac{5}{3}\pi$, and so the volume of $A \cap B$ is $\frac{10}{3}\pi$.





(b) A cylindrical hole is bored through the centre of a solid spherical ball. Let A be the portion of the ball which remains. Let h be the height of the cylindrical face of A. Find the volume of A in terms of h (somewhat surprisingly, the final answer involves neither the radius of the sphere, nor the radius of the hole).

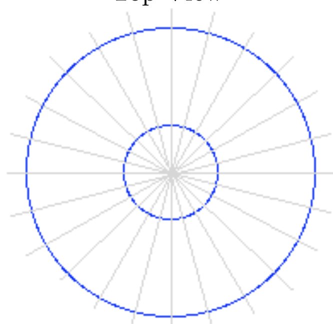
Solution: Let R be the radius of the ball, let x be the radius of the hole, let $y = \frac{1}{2} h$ and let $l = R - y$ (see the diagram below). Then the portion of the sphere that is removed when the hole is bored consists of a cylinder of radius x and height h, which has volume $V_1 = \pi\, x^2\, h$, and two spherical caps of thickness $l = R - y$, which each have volume $V_2 = \pi\, l^2 \left(R - \frac{1}{3}\, l\right)$ by our work in question 4. Thus the volume of the portion of the ball that remains is $V = \frac{4}{3}\, \pi\, R^3 - V_1 - 2V_2 = \frac{4}{3}\pi R^3 - \pi x^2 h - 2\pi\, l^2 \left(R - \frac{1}{3}\right)$. Put in $l = R - y = R - \frac{1}{2}\, h$ and $x = R^2 - y^2 = R^2 - \frac{1}{4}\, h^2$ and simplify to get $V = \frac{1}{6}\, \pi\, h^3$.



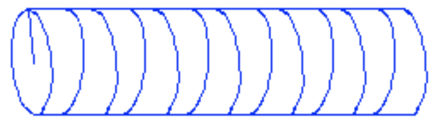
6: a) Let A be the the solid torus obtained by revolving the disc $(x-R)^2+y^2 \le r^2$ about the y-axis. Find the volume and the surface area of A. (Hint: slice A into pieces which can be reassembled to form a cylinder).

Solution: As shown below, the torus can be sliced into pieces which can be reassembled to form a cylinder of radius r and height (or length) $h = 2\pi R$. Thus the volume is $V = \pi r^2 h = \pi r^2 \cdot 2\pi R = 2\pi^2 r^2 R$ and the surface area is $A = 2\pi r h = 2\pi r \cdot 2\pi R = 4\pi^2 r R$.

Top View



Reassembled



(b) Let B be the paraboloidal solid which is obtained by revolving the region given by $0 \le x \le 1$ and $x^2 \le y \le 1$ about the y-axis. Find the volume of B. (Hint: slice B horizontally into n thin discs each of thickness $\frac{1}{n}$, find the approximate volume of each disc by treating it as a cylinder, add these volumes and take the limit as $n \to \infty$).

Solution: Slice the solid into n thin horizontal slices with each one being approximately in the form of a disc of thickness $h = \frac{1}{n}$. The i^{th} disc from the bottom will be at height $y_i = \sqrt{\frac{i}{n}}$, so its radius is $r_i = \sqrt{y_i} = \sqrt{\frac{i}{n}}$ and its volume will be $V_i = \pi {r_i}^2 h = \pi \cdot \frac{i}{n} \cdot \frac{1}{n} = \frac{i\,\pi}{n^2}$. The total volume of all the slices is approximately $V \cong \sum_{i=1}^{n} V_i = \frac{\pi}{n^2} + \frac{2\pi}{n^2} + \frac{3\pi}{n^2} + \cdots + \frac{n\pi}{n^2} = \frac{\pi}{2}\big(1+2+3+\cdots+n\big) = \frac{\pi}{2} \cdot \frac{n(n+1)}{2} = \frac{\pi}{2} \cdot \frac{n^2+n}{n^2} = \frac{\pi}{2}\big(1+\frac{1}{n}\big)$. Take the limit as $n \to \infty$ to get $V = \frac{\pi}{2}$