Chapter 7. Sequences and Series of Functions

Pointwise Convergence

7.1 Definition: Let $A \subseteq \mathbb{R}$, let $g : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n\geq p}$ **converges pointwise** to g on A, and we write $f_n \to g$ pointwise on A, when $\lim\limits_{n\to\infty} f_n(x) = g(x)$ for all $x \in A$, that is when for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers n we have

$$n \geq m \Longrightarrow |f_n(x) - g(x)| < \epsilon\,.$$

7.2 Note: By the Cauchy Criterion for convergence, the sequence $(f_n)_{n\geq p}$ converges pointwise to some function $g(x)$ on A if and only if for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers k, ℓ we have

$$k, \ell \geq m \Longrightarrow \big|f_k(x) - f_\ell(x)\big| < \epsilon\,.$$

7.3 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function g with $f_n \to g$ pointwise on $[0,1]$ such that each f_n is continuous but g is not.

Solution: Let $f_n(x) = x^n$. Then $\lim\limits_{n\to\infty} f_n(x) = \left\{ \begin{array}{l} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1 \end{array} \right\}$.

7.4 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function g with $f_n \to g$ pointwise on $[0,1]$ such that each f_n is differentiable and g is differentiable, but $\lim\limits_{n\to\infty} {f_n}' \neq g'$.

Solution: Let $f_n(x) = \frac{1}{n}\tan^{-1}(nx)$. Then $\lim\limits_{n\to\infty} f_n(x) = 0$, and ${f_n}'(x) = \dfrac{1}{1+(nx)^2}$ so $\lim\limits_{n\to\infty} {f_n}'(x) = \left\{ \begin{array}{l} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{array} \right\}$.

7.5 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function g with $f_n \to g$ pointwise on $[0,1]$ such that each f_n is integrable but g is not.

Solution: We have $\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, \cdots\}$ where

$$(a_n)_{n\geq 1} = \left(\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{0}{2}, \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{0}{4}, \cdots, \tfrac{4}{4}, \cdots\right).$$

(as an exercise, you can check that $a_n = \frac{k}{\ell}$ where $\ell = \left\lceil \frac{-3+\sqrt{9-8n}}{2} \right\rceil$ and $k = n - \frac{\ell^2+\ell}{2}$). For $x \in [0,1]$, let $f_n(x) = \left\{ \begin{array}{l} 0 \text{ if } x \notin \{a_1, a_2, \cdots, a_n\} \\ 1 \text{ if } x \in \{a_1, a_2, \cdots, a_n\} \end{array} \right\}$. Then $\lim\limits_{n\to\infty} f_n(x) = \left\{ \begin{array}{l} 0 \text{ if } x \notin \mathbb{Q} \\ 1 \text{ if } x \in \mathbb{Q} \end{array} \right\}$.

7.6 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function g with $f_n \to g$ pointwise on $[0,1]$ such that each f_n is integrable and g is integrable but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 g(x)\,dx\,.$$

Solution: Let $f_1(x) = \left\{ \begin{array}{cc} 48\left(x - \frac{1}{2}\right)\left(1-x\right) & \text{if } \frac{1}{2} \leq x \leq 1 \\ 0 & \text{otherwise} \end{array} \right\}$. For $n \geq 1$ let $f_n(x) = nf_1(nx)$.

Then each f_n is continuous with $\displaystyle\int_0^1 f_n(x)\,dx = 1$, and $\lim\limits_{n\to\infty} f_n(x) = 0$ for all x.

Uniform Convergence

7.7 Definition: Let $A \subseteq \mathbb{R}$, let $g : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n \geq p}$ **converges uniformly** to g on A, and we write $f_n \to g$ uniformly on A, when for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $n \in \mathbb{Z}_{\geq p}$ we have

$$n \geq m \Longrightarrow \left|f_n(x) - g(x)\right| < \epsilon\,.$$

7.8 Theorem: *(Cauchy Criterion for Uniform Convergence of Sequences of Functions) Let $(f_n)_{n \geq p}$ be a sequence of functions on $A \subseteq \mathbb{R}$. Then (f_n) converges uniformly (to some function g) on A if and only if for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $k, \ell \in \mathbb{Z}_{\geq p}$ we have*

$$k, \ell \geq m \Longrightarrow \left|f_k(x) - f_\ell(x)\right| < \epsilon\,.$$

Proof: Suppose that (f_n) converges uniformly to g on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ we have $n \geq m \Longrightarrow \left|f_n(x) - g(x)\right| < \frac{\epsilon}{2}$. Then for $k, \ell \geq m$ we have $\left|f_k(x) - g(x)\right| < \frac{\epsilon}{2}$ and $\left|f_\ell(x) - g(x)\right| < \frac{\epsilon}{2}$ and so

$$\left|f_k(x) - f_\ell(x)\right| \leq \left|f_k(x) - g(x)\right| + \left|f_\ell(x) - g(x)\right| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

Conversely, suppose that (f_n) satisfies the Cauchy Criterion for uniform convergence, that is for all $\epsilon > 0$ there exists m such that for all $x \in A$ and all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \left|f_n(x) - f_\ell(x)\right| < \epsilon\,.$$

For each fixed $x \in A$, $(f_n(x))$ is a Cauchy sequence, so $(f_n(x))$ converges, and we can define $g(x)$ by

$$g(x) = \lim_{n \to \infty} f_n(x)\,.$$

We know that $f_n \to g$ pointwise on A, but we must show that $f_n \to g$ uniformly on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ and for all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \left|f_n(x) - f_\ell(x)\right| < \tfrac{\epsilon}{2}\,.$$

Let $x \in A$. Since $\lim\limits_{\ell \to \infty} f_\ell(x) = g(x)$, we can choose $\ell \geq m$ so that $\left|f_\ell(x) - g(x)\right| < \frac{\epsilon}{2}$. Then for $n \geq m$ we have

$$\left|f_n(x) - g(x)\right| \leq \left|f_n(x) - f_\ell(x)\right| + \left|f_\ell(x) - g(x)\right| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

7.9 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that $f_n \to g$ uniformly on A. Let a be a limit point of A. If $\lim\limits_{x \to a} f_n(x)$ exists for each n, then*

$$\lim_{x \to a} g(x) = \lim_{x \to a} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to a} f_n(x)\,.$$

In particular, if each f_n is continuous in A, then so is g.

Proof: Suppose that $\lim\limits_{x \to a} f_n(x)$ exists for all n, and let $b_n = \lim\limits_{x \to a} f_n(x)$. We must show that $\lim\limits_{x \to a} g(x) = \lim\limits_{n \to \infty} b_n$. We claim first that (b_n) converges. Let $\epsilon > 0$. Since (f_n) converges uniformly, we can choose m so that $k, \ell \geq m \Longrightarrow \left|f_k(x) - f_\ell(x)\right| < \frac{\epsilon}{3}$ for all $x \in A$. Let $k, \ell \geq m$. Since $\lim\limits_{x \to a} f_n(x) = b_n$ for all n, we can choose $x \in A$ so that $\left|f_k(x) - b_k\right| < \frac{\epsilon}{3}$ and $\left|f_\ell(x) - b_\ell\right| < \frac{\epsilon}{3}$. Then we have

$$\left|b_k - b_\ell\right| \leq \left|b_k - f_k(x)\right| + \left|f_k(x) - f_\ell(x)\right| + \left|f_\ell(x) - b_\ell\right| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

By the Cauchy Criterion for sequences, (b_n) converges, as claimed.

Now, let $c = \lim_{n\to\infty} b_n$. We must show that $\lim_{x\to a} f(x) = c$. Let $\epsilon > 0$. Since $f_n \to g$ uniformly on A, and since $b_n \to c$, we can choose m so that when $n \geq m$ we have $\left|f_n(x) - g(x)\right| < \frac{\epsilon}{3}$ for all $x \in A$ and we have $|b_n - c| < \frac{\epsilon}{3}$. Let $n \geq m$. Since $\lim_{x\to a} f_n(x) = b_n$ we can choose $\delta > 0$ so that $0 < |x-a| < \delta \Longrightarrow \left|f_n(x) - b_n\right| < \frac{\epsilon}{3}$. Then when $0 < |x-a| < \delta$ we have

$$\left|g(x) - c\right| \leq \left|g(x) - f_n(x)\right| + \left|f_n(x) - b_n\right| + \left|b_n - c\right| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

Thus $\lim_{x\to a} f(x) = c$, as required.

In particular, if $a \in A$ and each f_n is continuous at a then we have

$$\lim_{x\to a} g(x) = \lim_{x\to a}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to a} f_n(x) = \lim_{n\to\infty} f_n(a) = g(a)$$

so g is continuous at a.

7.10 Theorem: *(Uniform Convergence and Integration) Suppose that $f_n \to g$ uniformly on $[a,b]$. If each f_n is integrable on $[a,b]$ then so is g. In this case, if $F_n(x) = \int_a^x f_n(t)\,dt$ and $G(x) = \int_a^x g(t)\,dt$, then $F_n \to G$ uniformly on $[a,b]$. In particular, we have*

$$\int_a^b g(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\;dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx\,.$$

Proof: Suppose that each f_n is integrable on $[a,b]$. We claim that g is integrable on $[a,b]$. Let $\epsilon > 0$. Since $f_n \to g$ uniformly on $[a,b]$, we can choose an integer N so that $n \geq N \Longrightarrow \left|f_n(x) - g(x)\right| < \frac{\epsilon}{4(b-a)}$ for all $x \in [a,b]$. Fix $n \geq N$. Since f_n is integrable, we can choose a partition X of $[a,b]$ so that $U(f_n,X) - L(f_n,X) < \frac{\epsilon}{2}$. Note that since $\left|f_n(x) - g(x)\right| < \frac{\epsilon}{4(b-a)}$ we have $M_k(g) \leq M_k(f_n) + \frac{\epsilon}{4(b-a)}$ and $m_k(g) \geq m_k(f_n) - \frac{\epsilon}{4(b-a)}$, and so

$$\begin{aligned} U(g,X) - L(g,X) &= \sum_{k=1}^n \big(M_k(g) - m_k(g)\big)\Delta_k x \leq \sum_{k=1}^n \Big(M_k(f_n) - m_k(f_n) + \tfrac{\epsilon}{2(b-a)}\Big)\,\Delta_k x \\ &= U(f_n,X) - L(f_n,X) + \tfrac{\epsilon}{2} < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

Thus g is integrable on $[a,b]$.

Now define $F_n(x) = \int_a^x f_n(t)\,dt$ and $G(x) = \int_a^x g(t)\,dt$. We claim that $F_n \to G$ uniformly on $[a,b]$. Let $\epsilon > 0$. Since $f_n \to g$ uniformly on $[a,b]$, we can choose N so that $n \geq N \Longrightarrow \left|f_n(t) - g(t)\right| < \frac{\epsilon}{2(b-a)}$ for all $t \in [a,b]$. Let $n \geq N$. Let $x \in [a,b]$. Then we have

$$\begin{aligned} \left|F_n(x) - G(x)\right| &= \left|\int_a^x f_n(t)\,dt - \int_a^x g(t)\,dt\right| = \left|\int_a^x f_n(t) - g(t)\;dt\right| \\ &\leq \int_a^x \left|f_n(t) - g(t)\right|dt \leq \int_a^x \tfrac{\epsilon}{2(b-a)}\;dt = \tfrac{\epsilon}{2(b-a)}(x-a) \leq \tfrac{\epsilon}{2} < \epsilon\,. \end{aligned}$$

Thus $F_n \to G$ uniformly on $[a,b]$, as required.

In particular, we have $\lim_{n\to\infty} F_n(b) = G(b)$, that is

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b g(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\,.$$

7.11 Theorem: *(Uniform Convergence and Differentiation) Let (f_n) be a sequence of functions on $[a,b]$. Suppose that each f_n is differentiable on $[a,b]$, $\big({f_n}'\big)$ converges uniformly on $[a,b]$, and $\big(f_n(c)\big)$ converges for some $c \in [a,b]$. Then (f_n) converges uniformly on $[a,b]$, $\lim\limits_{n\to\infty} f_n(x)$ is differentiable, and*

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{d}{dx} f_n(x)\,.$$

Proof: We claim that (f_n) converges uniformly on $[a,b]$. Let $\epsilon > 0$. Since (f_n) converges uniformly on $[a,b]$, and since $(f_n(c))$ converges, we can choose N so that when $n,m \geq N$ we have $\left|{f_n}'(t) - {f_m}'(t)\right| < \frac{\epsilon}{2(b-a)}$ for all $t \in [a,b]$ and we have $\left|f_n(c) - f_m(c)\right| < \frac{\epsilon}{2}$. Let $n,m \geq N$. Let $x \in [a,b]$. By the Mean Value Theorem applied to the function $f_n(x) - f_m(x)$, we can choose t between c and x so that

$$\big(f_n(x) - f_m(x) - f_n(c) + f_m(c)\big) = \big({f_n}'(t) - {f_m}'(t)\big)(x-c)\,.$$

Then we have

$$\begin{aligned}\left|f_n(x) - f_m(x)\right| &\leq \left|f_n(x) - f_m(x) - f_n(c) + f_m(c)\right| + \left|f_n(c) - f_m(c)\right| \\ &= \left|{f_n}'(t) - {f_m}'(t)\right||x-c| + \left|f_n(c) - f_m(c)\right| \\ &< \tfrac{\epsilon}{2(b-a)}(b-a) + \tfrac{\epsilon}{2} = \epsilon\,.\end{aligned}$$

Thus (f_n) converges uniformly on $[a,b]$.

Let $g(x) = \lim\limits_{n\to\infty} f_n(x)$. We claim that g is differentiable with $fg(x) = \lim\limits_{n\to\infty} {f_n}'(x)$ for all $x \in [a,b]$. Fix $x \in [a,b]$. Note that

$$\begin{aligned} g'(x) = \lim_{n\to\infty} {f_n}'(x) &\iff \lim_{y\to x}\frac{g(y)-g(x)}{y-x} = \lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x} \\ &\iff \lim_{y\to x}\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x} = \lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}\end{aligned}$$

so it suffices to show that $\big(h_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, where

$$h_n(y) = \frac{f_n(y)-f_n(x)}{y-x}\,.$$

Let $\epsilon > 0$. Since (f_n') converges uniformly on $[a,b]$, we can choose an integer N so that $n,m \geq N \implies \left|{f_n}'(t) - {f_m}'(t)\right| < \epsilon$ for all $t \in [a,b]$. Let $n,m \geq N$. Let $y \in [a,b]\setminus\{x\}$. By the Mean Value Theorem, we can choose t between x and y so that

$$\big(f_n(y) - f_m(y) - f_n(x) + f_m(x)\big) = \big({f_n}'(t) - {f_m}'(t)\big)(y-x)\,.$$

Then

$$\left|h_n(y) - h_m(y)\right| = \left|\frac{f_n(y) - f_m(y) - f_n(x) + f_m(x)}{y-x}\right| = \left|{f_n}'(t) - {f_m}'(t)\right| < \epsilon\,.$$

Thus $\big(h_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, as required.

Series of Functions

7.12 Definition: Let $(f_n)_{n\geq p}$ be a sequence of functions $f_n : A \to \mathbb{R}$. The **series of functions** $\sum\limits_{n\geq p} f_n$ is defined to be the sequence $\big(S_\ell\big)_{n\geq p}$ where $S_\ell(x) = \sum\limits_{n=p}^{\ell} f_n(x)$. The function S_ℓ is called the ℓ^{th} **partial sum** of the series. We say the series $\sum\limits_{n\geq p} f_n$ converges pointwise (or uniformly) on A when the sequence $(S_\ell)_{n\geq p}$ converges, pointwise (or uniformly) on A. In this case, the **sum** of the series of functions is defined to be the function

$$g(x) = \sum_{n=p}^{\infty} f_n(x) = \lim_{\ell\to\infty} S_\ell(x)\,.$$

7.13 Theorem: *(Cauchy Criterion for the Uniform Convergence of a Series of Functions) The series $\sum\limits_{n\geq p} f_n$ converges uniformly (to some function g) on A if and only if for every $\epsilon > 0$ there exists $N \geq p$ such that for all $x \in A$ and for all $m, \ell \geq p$ we have*

$$m > \ell \geq N \Longrightarrow \Big|\sum_{n=\ell+1}^{m} f_n(x)\Big| < \epsilon\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions, since $S_m(x) - S_\ell(x) = \sum_{n=\ell+1}^{m} f_n(x)$.

7.14 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that $\sum\limits_{n\geq p} f_n(x)$ converges uniformly on A. Let a be a limit point of A. If $\lim\limits_{x\to a} f_n(x)$ exists for all $n \geq p$, then*

$$\lim_{x\to a}\sum_{n=p}^{\infty} f_n(x) = \sum_{n=p}^{\infty}\lim_{x\to a} f_n(x)\,.$$

In particular, if each f_n is continuous on A then so is $\sum_{n=p}^{\infty} f_n$.

Proof: This follows immediately from the analogous theorem for sequences of functions.

7.15 Theorem: *(Uniform Convergence and Integration) Suppose that $\sum\limits_{n\geq p} f_n$ converges uniformly on $[a,b]$. If each f_n is integrable on $[a,b]$, then so is $\sum\limits_{n=p}^{\infty} f_n$. In this case, if we define $F_n(x) = \displaystyle\int_a^x f_n(t)\,dt$ and $G(x) = \displaystyle\int_a^x \sum_{n=p}^{\infty} f_n(t)\,dt$, then $\sum\limits_{n\geq p} F_n$ converges uniformly to G on A. In particular, we have*

$$\int_a^b \sum_{n=p}^{\infty} f_n(x)\,dx = \sum_{n=p}^{\infty}\int_a^b f_n(x)\,dx\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

7.16 Theorem: *(Uniform Convergence and Differentiation) Suppose that each* f_n *is differentiable on* $[a,b]$*, and* $\sum\limits_{n\geq p} {f_n}'$ *converges uniformly on* $[a,b]$*, and* $\sum\limits_{n\geq p} f_n(c)$ *converges for some* $c\in[a,b]$*. Then* $\sum\limits_{n\geq p} f_n$ *converges uniformly on* $[a,b]$ *and*

$$\frac{d}{dx}\sum_{n=p}^{\infty} f_n(x) = \sum_{n=p}^{\infty} \frac{d}{dx} f_n(x)\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

7.17 Theorem: *(The Weierstrass M-Test) Suppose that each* $f_n : A \to \mathbb{R}$ *is bounded with* $|f_n(x)| \leq M_n$ *for all* $x \in A$*, and that* $\sum\limits_{n\geq p} M_n$ *converges. Then* $\sum\limits_{n\geq p} f_n(x)$ *converges uniformly on* A*.*

Proof: Let $\epsilon > 0$. Since the series $\sum M_n$ converges, we can choose an integer N so that $m > \ell \geq N \Longrightarrow \sum\limits_{n=\ell+1}^{m} M_n < \epsilon$. Let $m > \ell \geq N$ and let $x \in A$. Then

$$\Big|\sum_{n=\ell+1}^{m} f_n(x)\Big| \leq \sum_{n=\ell+1}^{m} \big|f_n(x)\big| \leq \sum_{n=\ell+1}^{m} M_n < \epsilon\,.$$

7.18 Example: Find a sequence of functions $\big(f_n(x)\big)_{n\geq 0}$, each of which is differentiable on $\mathbb{R}$, such that $\sum\limits_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$, but the sum $g(x) = \sum\limits_{n=0}^{\infty} f_n(x)$ is nowhere differentiable.

Solution: Let $f_n(x) = \frac{1}{2^n}\sin^2(8^n x)$. Since $|f_n(x)| \leq \frac{1}{2^n}$ and $\sum \frac{1}{2^n}$ converges, $\sum\limits_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-Test. Let $g(x) = \sum\limits_{n=0}^{\infty} f_n(x)$. We claim that $g(x)$ is nowhere differentiable. Let $x\in\mathbb{R}$. For each n, let m, a_n and b_n be such that $a_n = \frac{m\pi}{2\cdot 8^n}$, $b_n = \frac{(m+1)\pi}{2\cdot 8^n}$ and $x \in [a_n, b_n)$. Note that one of $f_n(a_n)$ and $f_n(b_n)$ is equal to $\frac{1}{2^n}$ and the other is equal to 0 so we have $\big|f_n(b_n) - f_n(a_n)\big| = \frac{1}{2^n}$. Note also that for $k > n$ we have $f_k(a_n) = f_k(b_n) = 0$. Also, for all k we have $f_k(x) = \frac{1}{2^k}\sin^2(8^k x)$, ${f_k}'(x) = 4^k \sin(2\cdot 8^k x)$, and $\big|{f_k}'(x)\big| \leq 4^k$, so by the Mean Value Theorem,

$$\big|f_k(b_n) - f_k(a_n)\big| \leq 4^k |b_n - a_n|\,.$$

Finally, note that if $g'(x)$ did exist, then we would have $g'(x) = \lim\limits_{n\to\infty} \dfrac{f(b_n)-f(a_n)}{b_n - a_n}$, but

$$\begin{aligned}\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}\right| &= \left|\sum_{k=0}^{\infty}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| = \left|\sum_{k=0}^{n}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| \\ &\geq \left|\frac{f_n(b_n)-f_n(a_n)}{b_n-a_n}\right| - \sum_{k=0}^{n-1}\left|\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| \\ &\geq \frac{\frac{1}{2^n}}{\frac{\pi}{2\cdot 8^n}} - \sum_{k=0}^{n-1} 4^k = \frac{2\cdot 4^n}{\pi} - \frac{4^n-1}{3} = \left(\frac{2}{\pi}-\frac{1}{3}\right)4^n + \frac{1}{3} \to \infty \text{ as } n\to\infty\end{aligned}$$

Power Series

7.19 Definition: A **power series centred at** a is a series of the form $\sum_{n\geq 0} c_n(x-a)^n$ for some real numbers c_n, where we use the convention that $(x-a)^0 = 1$.

7.20 Example: The geometric series $\sum_{n\geq 0} x^n$ is a power series centred at 0. It converges when $|x| < 1$ and for all such x the sum of the series is

$$\sum_{n=0}^{\infty} x^n = \tfrac{1}{1-x}\,.$$

7.21 Lemma: *(Abel's Formula) Let $(a_n)_{n\geq p}$ and $(b_n)_{n\geq p}$ be sequences. Let $S_\ell = \sum_{n=p}^{\ell} a_n$. Then*

$$\sum_{n=p}^{\ell} a_n b_n = S_\ell b_\ell - \sum_{j=p}^{\ell-1} S_j(b_{j+1} - b_j)\,.$$

Proof: We have

$$\begin{aligned}\sum_{j=p}^{\ell-1} S_j(b_{j+1}-b_j) &= a_p(b_{p+1}-b_p) + (a_p + a_{p+1})(b_{p+2}-b_{p+1}) \\ &\qquad + (a_p + a_{p+1} + a_{p+2})(b_{p+3}-b_{p+2}) \\ &\qquad + \cdots + (a_p + a_{p+1} + a_{p+2} + \cdots + a_{\ell-1})(b_\ell - b_{\ell-1}) \\ &= -a_p b_p - a_{p+1}b_{p+1} - \cdots - a_{\ell-1}b_{\ell-1} \\ &\quad + (a_p + a_{p+1} + \cdots a_{\ell-1})b_\ell - a_\ell b_\ell + a_\ell b_\ell \\ &= S_\ell b_\ell - \sum_{n=p}^{\ell} a_n b_n\,.\end{aligned}$$

7.22 Note: Recall that for a sequence (a_n) in $\mathbb{R}$, we define $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$ where $s_n = \sup\{a_k \mid k \geq n\}$ (with $\limsup_{n\to\infty} a_n = \infty$ when (a_n) is not bounded above).

7.23 Theorem: *(The Interval and Radius of Convergence) Let $\sum_{n\geq 0} c_n(x-a)^n$ be a power series and let $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}} \in [0,\infty]$. Then the set of $x \in \mathbb{R}$ for which the power series converges is an interval I centred at a of radius R. Indeed*

(1) If $|x-a| > R$ then $\lim_{n\to\infty} c_n(x-a) \neq 0$ so $\sum_{n\geq 0} c_n(x-a)^n$ diverges.

(2) If $|x-a| < R$ then $\sum_{n\geq 0} c_n(x-a)^n$ converges absolutely.

(3) If $0 < r < R$ then $\sum_{n\geq 0} c_n(x-a)^n$ converges uniformly in $[a-r, a+r]$.

(4) (Abel's Theorem) If $\sum_{n\geq 0} c_n R^n$ converges then $\sum_{n\geq 0} c_n(x-a)^n$ converges uniformly on $[a, a+R]$. If $\sum_{n\geq 0} c_n(-R)^n$ converges then $\sum_{n\geq 0} c_n(x-a)^n$ converges uniformly on $[a-R, a]$.

Proof: To prove Part 1, suppose that $|x-a|>R$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = |x-a| \limsup_{n\to\infty} \sqrt[n]{|c_n|} > R\cdot\tfrac{1}{R} = 1\,,$$

and so $\lim\limits_{n\to\infty} c_n(x-a)^n \neq 0$ and $\sum c_n(x-a)^n$ diverges, by the Root Test.

To prove Part 2, suppose that $|x-a|<R$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = |x-a| \limsup_{n\to\infty} \sqrt[n]{|c_n|} < R\cdot\tfrac{1}{R} = 1\,,$$

and so $\sum \big|c_n(x-a)^n\big|$ converges, by the Root Test.

To prove Part 3, fix $0<r<R$. By part 2, $\sum\big|c_n(x-a)^n\big|$ converges when $x=a+r$, that is $\sum|c_nr^n|$ converges. Let $x\in[a-r,a+r]$. Then $|c_n(x-a)^n|\leq|c_nr^n|$ and $\sum|c_nr^n|$ converges, and so $\sum|c_n(x-a)^n|$ converges uniformly by the Weierstrass M-Test.

Now let us prove the first statement in Part 4 (the proof of the second statement is similar). Suppose that $\sum c_nR^n$ converges. Let $\epsilon>0$. Choose an integer N so that $m>\ell\geq N \Longrightarrow \left|\sum\limits_{n=\ell+1}^{m} c_nR^n\right| < \epsilon$. Let $x\in[a,a+R]$. By Abel's Lemma, using $a_n=c_nR^n$, $b_n=\left(\frac{x-a}{R}\right)^n$, $S_m=\sum\limits_{\ell+1}^{m} a_n$, and noting that $0\leq\frac{x-a}{R}\leq 1$ so that $0\leq b_{j+1}\leq b_j\leq 1$,

$$\begin{aligned}
\left|\sum_{n=\ell+1}^{m} c_n(x-a)^n\right| &= \left|\sum_{n=\ell+1}^{m} c_nR^n\left(\tfrac{x-a}{R}\right)^n\right| = \left|\sum_{n=\ell+1}^{m} a_nb_n\right| \\
&= \left|S_m\,b_m - \sum_{j=\ell+1}^{m-1} S_j(b_{j+1}-b_j)\right| \;\leq\; |S_m||b_m| + \sum_{j=\ell+1}^{m-1}|S_j||b_{j+1}-b_j| \\
&= |S_m|\,b_m + \sum_{j=\ell+1}^{m-1}|S_j|(b_j-b_{j+1}) \;<\; \epsilon\,b_m + \epsilon\sum_{j=\ell+1}^{m-1}(b_j-b_{j+1}) \\
&= \epsilon\,b_m + \epsilon\,(b_{\ell+1}-b_m) = \epsilon\,b_{\ell+1} \leq \epsilon.
\end{aligned}$$

7.24 Definition: The number R in the above theorem is called the **radius of convergence** of the power series, and the interval I is called the **interval of convergence** of the power series.

7.25 Example: Find the interval of convergence of the power series $\displaystyle\sum_{n\geq1}\frac{(3-2x)^n}{\sqrt{n}}$.

Solution: First note that this is in fact a power series, since $\dfrac{(3-2x)^n}{\sqrt{n}} = \frac{(-2)^n}{\sqrt{n}}\left(x-\frac{3}{2}\right)^n$, and so $\displaystyle\sum_{n\geq1}\frac{(3-2x)^n}{\sqrt{n}} = \sum_{n\geq0} c_n(x-a)^n$, where $c_0=0$, $c_n=\frac{(-2)^n}{\sqrt{n}}$ for $n\geq1$ and $a=\frac{3}{2}$.

Now, let $a_n=\dfrac{(3-2x)^n}{\sqrt{n}}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(3-2x)^{n+1}}{\sqrt{n+1}}\,\dfrac{\sqrt{n}}{(3-2x)^n}\right| = \sqrt{\frac{n}{n+1}}\,|3-2x|$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |3-2x|$. By the Ratio Test, $\sum a_n$ converges when $|3-2x|<1$ and diverges when $|3-2x|>1$. Equivalently, it converges when $x\in(1,2)$ and diverges when $x\notin[1,2]$. When $x=1$ so $(3-2x)=1$, we have $\sum a_n=\sum\frac{1}{\sqrt{n}}$, which diverges (its a p-series), and when $x=2$ so $(3-2x)=-1$, we have $\sum a_n=\sum\frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $I=(1,2\,]$.

Operations on Power Series

7.26 Theorem: *(Continuity of Power Series) Suppose that the power series* $\sum c_n(x-a)^n$ *converges in an interval* I*. Then the sum* $f(x)=\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ *is continuous in* I*.*

Proof: This follows from uniform convergence of $\sum c_n(x-a)^n$ on closed subintervals of I. Indeed, given $x\in I$ we can choose a closed interval $[b,c]\subseteq I$ with $x\in[b,c]$, and then since $\sum c_n(x-a)^n$ converges uniformly on $[b,c]$, the sum is continuous on $[b,c]$, and hence at x.

7.27 Theorem: *(Addition and Subtraction of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in the interval* I*. Then* $\sum(a_n+b_n)(x-a)^n$ *and* $\sum(a_n-b_n)(x-a)^n$ *both converge in* I*, and for all* $x\in I$ *we have*

$$\sum_{n=0}^{\infty}(a_n\pm b_n)(x-a)^n=\Big(\sum_{n=0}^{\infty}a_n(x-a)^n\Big)\pm\Big(\sum_{n=0}^{\infty}b_n(x-a)^n\Big).$$

Proof: This follows from Linearity.

7.28 Theorem: *(Multiplication of Power Series) Suppose the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a\in I$*. Let* $c_n=\sum\limits_{k=0}^{n}a_kb_{n-k}$*. Then* $\sum c_n(x-a)^n$ *converges in* I *and for all* $x\in I$ *we have*

$$\sum_{n=0}^{\infty}c_n(x-a)^n=\Big(\sum_{n=0}^{\infty}a_n(x-a)^n\Big)\Big(\sum_{n=0}^{\infty}b_n(x-a)^n\Big).$$

Proof: This follows from the Multiplication of Series Theorem, since the power series converge absolutely in I.

7.29 Theorem: *(Integration of Power Series) Suppose that* $\sum c_n(x-a)^n$ *converges in the interval* I*. Then for all* $x\in I$*, the sum* $f(x)=\sum\limits_{n=0}^{\infty}c_n(x-a)^n$ *is integrable on the closed interval between* a *and* x *(that is on* $[a,x]$ *or* $[x,a]$*) and*

$$\int_a^x\sum_{n=0}^{\infty}c_n(t-a)^n\,dt=\sum_{n=0}^{\infty}\int_a^x c_n(t-a)^n\,dt=\sum_{n=0}^{\infty}\tfrac{1}{n+1}\,c_n(x-a)^{n+1}.$$

Proof: This follows from uniform convergence on closed subintervals of I.

7.30 Theorem: *(Differentiation of Power Series) Suppose that* $\sum c_n(x-a)^n$ *converges in the open interval* I*. Then the sum* $f(x)=\sum\limits_{n=0}^{\infty}c_n(x-a)^n$ *is differentiable in* I *and*

$$f'(x)=\tfrac{d}{dx}\sum c_n(x-a)^n=\sum_{n=0}^{\infty}\tfrac{d}{dx}c_n(x-a)^n=\sum_{n=1}^{\infty}n\,c_n(x-a)^{n-1}.$$

Proof: Note that the two power series $\sum c_n(x-a)^n$ and $\sum n\,c_n(x-a)^{n-1}$ have the same radius of convergence R, because $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ so that $\limsup\sqrt[n]{|nc_n|}=\limsup\sqrt[n]{|c_n|}$.

The theorem now follows from the convergence of $\sum c_n(x-a)^n$ and the uniform convergence of $\sum n\,c_n(x-a)^{n-1}$ on closed subintervals of I.

7.31 Theorem: *(Division of Power Series) Suppose that* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* J *with* $a \in J$*, and that* $b_0 \neq 0$*. Define* c_n *by*

$$c_0 = \tfrac{a_0}{b_0}\ \text{, and for } n > 0,\ c_n = \tfrac{a_n}{b_0} - \tfrac{b_n c_0}{b_0} - \tfrac{b_{n-1}c_1}{b_0} - \cdots - \tfrac{b_1 c_{n-1}}{b_0}\,.$$

Then there is an open interval I *with* $a \in I \subseteq J$ *such that* $\sum c_n(x-a)^n$ *converges in* I *with*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \frac{\sum\limits_{n=0}^{\infty} a_n(x-a)^n}{\sum\limits_{n=0}^{\infty} b_n(x-a)^n}\,.$$

Proof: For all $x \in J$, let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$. Since g is continuous with $g(a) = b_0 \neq 0$, we can choose $r > 0$ small enough so that $[a-r, a+r] \subseteq J$ and $g(x) \neq 0$ for all $x \in [a-r, a+r]$. Note that $\sum |a_n r^n|$ and $\sum |b_n r^n|$ both converge. Since $|a_n r^n| \to 0$ and $|b_n r^n| \to 0$ and $b_0 \neq 0$, we can choose M so that $M \geq \left|\frac{a_n r^n}{b_0}\right|$ and $M \geq \left|\frac{b_n r^n}{b_0}\right|$ for all n. Note that $|c_0| = \left|\frac{a_0}{b_0}\right| \leq M$ and since $c_1 = \frac{a_1}{b_0} + \frac{b_1 c_0}{b_0}$ we have

$$|c_1 r| \leq \left|\tfrac{a_1 r}{b_0}\right| + \left|\tfrac{b_1 r}{b_0}\right| |c_0| \leq M + M^2 = M(1+M)\,.$$

Suppose, inductively, that $|c_k r^k| \leq M(1+M)^k$ for all $k < n$. Then since

$$a_n = b_n c_0 + b_{n-1}c_1 + \cdots + b_1 c_{n-1} + b_0 c_n\,,$$

we have

$$\begin{aligned}
|c_n r^n| &\leq \left|\tfrac{a_n r^n}{b_0}\right| + \left|\tfrac{b_n r^n}{b_0}\right| |c_0| + \left|\tfrac{b_{n-1}r^{n-1}}{b_0}\right| |c_1 r| + \cdots + \left|\tfrac{b_1 r}{b_0}\right| |c_{n-1}r^{n-1}| \\
&\leq M + M^2 + M^2(1+M) + M^2(1+M)^2 + M^2(1+M)^3 + \cdots + M^2(1+M)^{n-1} \\
&= M + M^2\left(\tfrac{(1+M)^n - 1}{M}\right) = M(1+M)^n\,.
\end{aligned}$$

By induction, we have $|c_n r^n| \leq M(1+M)^n$ for all $n \geq 0$. Let $I = (a-s, a+s)$ with $s = \frac{r}{1+M}$. Note that $I \subseteq (a-r, a+r) \subseteq J$. When $x \in I$, for all n we have

$$|c_n(x-a)^n| = |c_n r^n| \cdot \frac{1}{(1+M)^n} \cdot \left|\frac{x-a}{r/(1+M)}\right|^n \leq M \left|\frac{x-a}{r/(1+M)}\right|^n$$

and so $\sum |c_n(x-a)^n|$ converges in I by the Comparison Test.

Note that from the definition of c_n we have $a_n = \sum\limits_{k=0}^{n} c_k b_{n-k}$, and so by multiplying power series, we have

$$\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in I$.

7.32 Theorem: *(Composition of Power Series). Let* $f(x)=\sum\limits_{n=0}^{\infty}a_n(x-a)^n$ *in an open interval* J *with* $a\in J$*, and let* $g(y)=\sum\limits_{m=0}^{\infty}b_m(y-b)^m$ *in an open interval* K *with* $b=f(a)\in K$*. Let* $c_{m,k}$ *be the coefficient of* $(x-a)^k$ *in the product* $\big(\sum\limits_{n=1}^{k}a_n(x-a)^n\big)^m$*, and let* $d_k=\sum\limits_{m=0}^{k}b_mc_{m,k}$*. Then there exists an open interval* $I\subseteq J$ *with* $a\in I$ *such that* $f(I)\subseteq K$ *and the series* $\sum\limits_{k\geq 0}d_k(x-a)^k$ *converges in* I *with* $g(f(x))=\sum\limits_{k=0}^{\infty}d_k(x-a)^k$*.*

Proof: When $x\in J$ and $f(x)\in K$, we have

$$g\big(f(x)\big)=\sum_{m=0}^{\infty}b_m\Big(\sum_{n=1}^{\infty}a_n(x-a)^n\Big)^m=\sum_{m=0}^{\infty}b_m\sum_{k=0}^{\infty}c_{m,k}(x-a)^k=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}b_mc_{m,k}(x-a)^k.$$

If we can apply Fubini's Theorem to interchange the order of summation, then we obtain

$$g\big(f(x)\big)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}b_mc_{m,k}(x-a)^k=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}b_mc_{m,k}(x-a)^k=\sum_{m=0}^{\infty}d_k(x-a)^k.$$

We need to verify that the hypotheses of Fubini's Theorem are satisfied, so we must show that $\sum_{k\geq 0}|b_mc_{m,k}(x-a)^k|$ converges for each m, and that $\sum_{m\geq 0}\sum_{k=0}^{\infty}|b_mc_{m,k}(x-a)^k|$ converges (at least in some subinterval $I\subseteq J$ with $a\in I$). We know that $\sum a_n(x-a)^n$ converges in J, so by multiplication of power series, the series $\sum c_{m,k}(x-a)^k$ also converges in J with $\sum_{k=0}^{\infty}c_{m,k}(x-a)^k=\big(\sum_{n=1}^{\infty}a_n(x-a)^n\big)^m$, and hence by multiplying by b_m, the series $\sum_{k\geq 0}b_mc_{m,k}(x-a)^k$ also converges in J. Since power series converge absolutely in their open interval of convergence, $\sum_{k\geq 0}|b_mc_{m,k}(x-a)^k|$ converges in J.

It remains to show that $\sum_{m\geq 0}\sum_{k=0}^{\infty}|b_mc_{m,k}(x-a)^k|$ converges in an open interval I with $a\in I$. Let R and S be the radii of convergence of the series $\sum a_n(x-a)^n$ and $\sum b_m(y-b)^m$. Note that the series $\sum|a_n|(x-a)^n$ and $\sum|b_m|(y-|b|)^m$ have the same radii of convergence, R and S, so we can define functions $\overline{f}(x)=\sum_{n=0}^{\infty}|a_n|(x-a)^n$ for $x\in(a-R,a+R)$ and $\overline{g}(y)=\sum_{m=0}^{\infty}|b_m|(y-|b|)^m$ for $y\in(|b|-S,|b|+S)$. Since $\overline{f}$ is continuous with $\overline{f}(a)=|a_0|=|f(a)|=|b|$, we can choose r with $0<r\leq R$ such that $|x-a|<r\Longrightarrow\big|\overline{f}(a)-|b|\big|<S$. Let $I=(a-r,a+r)\cap J$, and note that $\overline{g}\big(\overline{f}(x)\big)$ is defined for all $x\in I$. Let $\overline{c}_{m,k}$ be the coefficient of $(x-a)^k$ in the product $\big(\sum_{n=1}^{k}|a_n|(x-a)^n\big)^m$. Note that, by the triangle inequality, $|c_{m,k}|\leq\overline{c}_{m,k}$ for all m,k. For all $x\in I$, we have

$$\overline{g}\big(\overline{f}(x)\big)=\sum_{m=0}^{\infty}|b_m|\Big(\sum_{n=1}^{\infty}|a_n|(x-a)^n\Big)^m=\sum_{m=0}^{\infty}|b_m|\sum_{k=0}^{\infty}\overline{c}_{m,k}(x-a)^k=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}|b_m|\,\overline{c}_{m,k}(x-a)^k.$$

Since power series converge absolutely, $\sum_{m\geq 0}\sum_{k=0}^{\infty}|b_m\overline{c}_{m,k}(x-a)^k|$ converges in I. Since $|c_{m,k}|\leq\overline{c}_{m,k}$, the series $\sum_{m\geq 0}\sum_{k=0}^{\infty}|b_mc_{m,k}(x-a)^k|$ also converges in I, by comparison.

7.33 Example: We have $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x|<1$. By Integration of Power Series, $\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^n$ for $|x|<1$. In particular, we can take $x=\frac{1}{2}$ to get $\ln\frac{3}{2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^n}$ and we can take $x=-\frac{1}{2}$ to get $\ln\frac{1}{2} = \sum_{n=1}^{\infty}\frac{-1}{n\cdot 2^n}$, that is $\ln 2 = \sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$.

Let us also argue that we can also take $x=1$. Note when $x=-1$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ becomes the harmonic series, which diverges, and when $x=1$ it becomes the alternating harmonic series, which converges, so the interval of convergence is $(-1,1]$. Thus the sum $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ is defined for $-1<x\leq 1$. We know already that $f(x)=\ln(1+x)$ for $-1<x<1$. By Abel's Theorem, the series converges uniformly on $[0,1]$, so by the Continuity of Power Series Theorem, the sum $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ is continuous on $[0,1]$ and in particular $f(x)$ is continuous at $x=1$. Since $f(x)=\ln(1+x)$ for $|x|<1$ and and since both $f(x)$ and $\ln(1+x)$ are continuous at 1 it follows that $f(1)=\ln 2$. Thus we have $\ln 2 = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.

7.34 Example: Let $f(x)=\frac{1}{x^2+3x+2}$. Find a power series centred at 0 whose sum is $f(x)$ in its interval of convergence, and find a power series centred at -4 whose sum is $f(x)$ in its interval of convergence.

Solution: Let us find a series centred at 0. For all $x\neq -1,-2$ we have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{1+x} - \frac{\frac{1}{2}}{1+\frac{x}{2}} \\ &= \sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}\tfrac{1}{2}\left(-\tfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n x^n - \sum_{n=0}^{\infty}\tfrac{(-1)^n}{2^{n+1}}x^n \\ &= \sum_{n=0}^{\infty}(-1)^n\left(1-\tfrac{1}{2^{n+1}}\right)x^n\,. \end{aligned}$$

We remark that since $\sum_{n\geq 0}(-x)^n$ converges if and only if $|x|<1$ and $\sum_{n\geq 0}\frac{1}{2}\left(-\frac{x}{2}\right)^n$ converges when $|x|<2$, it follows from Linearity the sum of these two series converges if and only if $|x|<1$.

Solution: Now let us find a series centred at -4. For all $x\neq -1,-2$, we have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+4)-3} - \frac{1}{(x+4)-2} \\ &= \frac{-\frac{1}{3}}{1-\frac{x+4}{3}} + \frac{\frac{1}{2}}{1-\frac{x+4}{2}} = \sum_{n=0}^{\infty}-\tfrac{1}{3}\left(\tfrac{x+4}{3}\right)^n + \sum_{n=0}^{\infty}\tfrac{1}{2}\left(\tfrac{x+4}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\left(\tfrac{1}{2^{n+1}} - \tfrac{1}{3^{n+1}}\right)(x+4)^n\,. \end{aligned}$$

Since $\sum_{n\geq 0}-\frac{1}{3}\left(\frac{x+4}{3}\right)^n$ converges if and only if $|x+4|<3$ and $\sum_{n\geq 0}\frac{1}{2}\left(\frac{x+4}{2}\right)^n$ converges if and only if $|x+4|<2$, it follows that their sum converges if and only if $|x+4|<2$.

7.35 Example: Find a power series centered at 0 whose sum is $f(x) = \dfrac{1}{(1-x)^2}$.

Solution: We provide three solutions. For the first solution, we multiply two power series. For $|x| < 1$ we have

$$\begin{aligned} f(x) &= \frac{1}{1-x} \cdot \frac{1}{1-x} \\ &= \big(1 + x + x^2 + x^3 + \cdots\big)\big(1 + x + x^2 + x^3 + \cdots\big) \\ &= 1 + (1+1)x + (1+1+1)x^2 + (1+1+1+1)x^3 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)x^n \,. \end{aligned}$$

For the second solution, we note that $f(x) = \dfrac{1}{1 - 2x + x^2}$ and we use long division.

$$\begin{array}{rl} & \quad 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \\ 1 - 2x + x^2 & \big) \; 1 + 0x + 0x^2 + 0x^3 + 0x^4 - \cdots \\ & \quad \underline{1 - 2x + x^2} \\ & \qquad 2x - x^2 \\ & \qquad \underline{2x - 4x^2 + 2x^3} \\ & \qquad\qquad 3x^2 - 2x^3 \\ & \qquad\qquad \underline{3x^2 - 6x^3 + 3x^4} \\ & \qquad\qquad\qquad 4x^3 - 8x^4 + \cdots \\ & \qquad\qquad\qquad \underline{4x^3 - 8x^4 + \cdots} \\ & \qquad\qquad\qquad\qquad 5x^4 + \cdots \end{array}$$

For the third solution, we note that $\displaystyle\int \frac{1}{(1-x)^2} = \frac{1}{1-x}$ and we use differentiation.

$$\begin{aligned} \frac{1}{1-x} &= 1 + x^2 + x^3 + x^4 + x^5 + \cdots \\ \frac{d}{dx}\Big(\frac{1}{1-x}\Big) &= \tfrac{d}{dx}\big(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\big) \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \,. \end{aligned}$$

Taylor Series

7.36 Theorem: *Suppose that* $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ *in an open interval* I *centred at* a. *Then* f *is infinitely differentiable at* a *and for all* $n\geq 0$ *we have*

$$a_n=\frac{f^{(n)}(a)}{n!}\ ,$$

where $f^{(n)}(a)$ *denotes the* n^{th} *derivative of* f *at* a.

Proof: By repeated application of the Differentiation of Power Series Theorem, for all $x\in I$, we have $f'(x)=\sum_{n=1}^{\infty}na_n(x-a)^{n-1}$, $f''(x)=\sum_{n=2}^{\infty}n(n-1)a_n(x-a)^{n-2}$ and $f'''(x)=\sum_{n=3}^{\infty}n(n-1)(n-2)a_n(x-a)^{n-3}$, and in general

$$f^{(k)}(x)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_n(x-a)^{n-k}$$

and so $f(a)=a_0$, $f'(a)=a_1$, $f''(a)=2\cdot 1\,a_2$ and $f'''(a)=3\cdot 2\cdot 1\,a_3$, and in general

$$f^{(n)}(a)=n!\ a_n$$

7.37 Definition: Given a function $f(x)$ which is infinitely differentiable at $x=a$, we define the **Taylor series** of $f(x)$ centred at a to be the power series

$$T(x)=\sum_{n\geq 0}a_n(x-a)^n \quad \text{where } a_n=\frac{f^{(n)}(a)}{n!}$$

and we define the l^{th} **Taylor Polynomial** of $f(x)$ centred at a to be the l^{th} partial sum

$$T_l(x)=\sum_{n=0}^{l}a_n(x-a)^n \quad \text{where } a_n=\frac{f^{(n)}(a)}{n!}$$

7.38 Example: Find the Taylor series centred at 0 for $f(x)=e^x$.

Solution: We have $f^{(n)}(x)=e^x$ for all n, so $f^{(n)}(0)=1$ and $a_n=\frac{1}{n!}$ for all $n\geq 0$. Thus the Taylor series is

$$T(x)=\sum_{n=0}^{\infty}\tfrac{1}{n!}\,x^n=1+x+\tfrac{1}{2!}x^2+\tfrac{1}{3!}x^3+\tfrac{1}{4!}x^4+\cdots\ .$$

7.39 Example: Find the Taylor series centred at 0 for $f(x)=\sin x$.

Solution: We have $f'(x)=\cos x$, $f''(x)=-\sin x$, $f'''(x)=-\cos x$, $f''''(x)=\sin x$ and so on, so that in general $f^{(2n)}(x)=(-1)^n\sin x$ and $f^{(2n+1)}(x)=(-1)^n\cos x$. It follows that $f^{(2n)}(0)=0$ and $f^{(2n+1)}(0)=(-1)^n$, so we have $a_{2n}=0$ and $a_{2n+1}=\frac{(-1)^n}{(2n+1)!}$. Thus

$$T(x)=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\tfrac{1}{3!}x^3+\tfrac{1}{5!}x^5-\tfrac{1}{7!}x^7+\cdots.$$

7.40 Example: Find the Taylor series centred at 0 for $f(x) = (1+x)^p$ where $p \in \mathbb{R}$. This series is called the **binomial series**

Solution: $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}$, $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$, and in general

$$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n}\,,$$

so $f(0) = 1$, $f'(0) = p$, $f''(0) = p(p-1)$, and in general $f^{(n)}(0) = p(p-1)(p-2)\cdots(p-n+1)$, and so we have $a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \tfrac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

where we use the notation

$$\binom{p}{0} = 1\ ,\text{ and for } n \geq 1,\ \binom{p}{n} = \tfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

7.41 Theorem: *(Taylor) Let $f(x)$ be infinitely differentiable in an open interval I with $a \in I$. Let $T_l(x)$ be the l^{th} Taylor polynomial for $f(x)$ centered at a. Then for all $x \in I$ there exists a number c between a and x such that*

$$f(x) - T_l(x) = \frac{f^{(l+1)}(c)}{(l+1)!}(x-a)^{l+1}\,.$$

Proof: When $x = a$ both sides of the above equation are 0. Suppose that $x > a$ (the case that $x < a$ is similar). Since $f^{(l+1)}$ is differentiable and hence continuous, by the Extreme Value Theorem it attains its maximum and minimum values, say M and m, on the interval $[a, x]$. Since $m \leq f^{(l+1)}(t) \leq M$ for all $t \in [a, x]$, we have

$$\int_a^{t_1} m\,dt \leq \int_a^{t_1} f^{(l+1)}(t)\,dt \leq \int_a^{t_1} M\,dt$$

that is

$$m(t_1 - a) \leq f^{(l)}(t_1) - f^{(l)}(a) \leq M(t_1 - a)$$

for all $t_1 \in [a, x]$. Integrating each term with respect to t_1 from a to t_2, we get

$$\tfrac{1}{2}m(t_2-a)^2 \leq f^{(l-1)}(t_2) - f^{(l-1)}(a) - f^{(l)}(a)(t_2-a) \leq \tfrac{1}{2}M(t_t-a)^2$$

for all $t_2 \in [a, x]$. Integrating with respect to t_2 from a to t_3 gives

$$\tfrac{1}{3!}m(t_3-a)^3 \leq f^{(l-2)}(t_3) - f^{(l-2)}(a) - f^{(l-1)}(a)(t_3-a) - \tfrac{1}{2}f^{(l)}(a)(t_3-a)^2 \leq \tfrac{1}{3!}M(t_3-a)^3$$

for all $t_3 \in [a, x]$. Repeating this procedure eventually gives

$$\tfrac{1}{(l+1)!}m(t_{l+1}-a)^{l+1} \leq f(t_{l+1}) - T_l(t_{l+1}) \leq \tfrac{1}{(l+1)!}M(t_{l+1}-a)^{l+1}$$

for all $t_{l+1} \in [a, x]$. In particular $\frac{1}{(l+1)!}m(x-a)^{l+1} \leq f(x) - T_l(x) \leq \frac{1}{(l+1)!}M(x-a)^{l+1}$, so

$$m \leq \big(f(x) - T_l(x)\big)\tfrac{(l+1)!}{(x-a)^{l+1}} \leq M\,.$$

By the Intermediate Value Theorem, there is a number $c \in [a, x]$ such that

$$f^{(l+1)}(c) = \big(f(x) - T_l(x)\big)\frac{(l+1)!}{(x-a)^{l+1}}$$

.

7.42 Theorem: *The functions e^x and $\sin x$ are equal to the sum of their Taylor series centred at 0 for all $x \in \mathbb{R}$. For $p \in \mathbb{R}$, the function $(1+x)^p$ is equal to the sum of its Taylor series, centred at 0, for all $|x| < 1$*

Proof: First let $f(x) = e^x$ and let $x \in \mathbb{R}$. By Taylor's Theorem, $f(x) - T_l(x) = \dfrac{e^c x^{l+1}}{(l+1)!}$ for some c between 0 and x, and so

$$\left|f(x) - T_l(x)\right| \leq \frac{e^{|x|}|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum \dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, we have $\lim\limits_{l\to\infty} \dfrac{e^{|x|}|x|^{l+1}}{(l+1)!} = 0$ by the Divergence Test, so $\lim\limits_{l\to\infty}\big(f(x) - T_l(x)\big) = 0$, and so $f(x) = \lim\limits_{l\to\infty} T_l(x) = T(x)$.

Now let $f(x) = \sin x$ and let $x \in \mathbb{R}$. By Taylor's Theorem, $f(x) - T_\ell(x) = \dfrac{f^{(l+1)}(c)\, x^{l+1}}{(l+1)!}$ for some c between 0 and x. Since $f^{(l+1)}(x)$ is one of the functions $\pm \sin x$ or $\pm \cos x$, we have $\left|f^{(l+1)}(c)\right| \leq 1$ for all c and so

$$\left|f(x) - T_\ell(x)\right| \leq \frac{|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum \dfrac{|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, $\lim\limits_{l\to\infty} \dfrac{|x|^{l+1}}{(l+1)!} = 0$ by the Divergence Test, and so we have and $f(x) = T(x)$ as above.

Finally, let $f(x) = (1+x)^p$. The Taylor series centred at 0 is

$$T(x) = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \tfrac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

and it converges for $|x| < 1$. Differentiating the power series gives

$$T'(x) = p + \tfrac{p(p-1)}{1!}x + \tfrac{p(p-1)(p-2)}{2!}x^2 + \tfrac{p(p-1)(p-2)(p-3)}{3!}x^3 + \cdots$$

and so

$$\begin{aligned}(1+x)T'(x) &= p + \left(p + \tfrac{p(p-1)}{1!}\right)x + \left(\tfrac{p(p-1)}{1!} + \tfrac{p(p-1)(p-2)}{2!}\right)x^2 \\ &\quad + \left(\tfrac{p(p-1)(p-2)}{2!} - \tfrac{p(p-1)(p-2)(p-3)}{3!}\right)x^3 + \cdots \\ &= p + \tfrac{p\cdot p}{1!}x + \tfrac{p\cdot p(p-1)}{2!}x^2 + \tfrac{p\cdot p(p-1)(p-2)}{3!}x^3 + \cdots \\ &= p\,T(x)\,.\end{aligned}$$

Thus we have $(1+x)T'(x) = pT(x)$, that is $(1+x)T'(x) - pT(x) = 0$, for all $|x| < 1$. Multiply both sides by $(1+x)^{-p-1}$ to get $(1+x)^{-p}\,T'(x) - p(1+x)^{-p-1} = 0$, that is $\frac{d}{dx}\big((1+x)^{-p}\,T(x)\big) = 0$ for all $|x| < 1$. It follows that $(1+x)^{-p}\,T(x) = c$ for some constant $c \in \mathbb{R}$. Taking $x = 0$ shows that $c = 1$, so we have $(1+x)^{-p}\,T(x) = 1$, and hence $T(x) = (1+x)^p$, for all $|x| < 1$.

7.43 Note: It is *not* the case that every infinitely differentiable function is equal to the sum of its Taylor series in the open interval of convergence. For example, for the function given by $f(x) = e^{-1/x^2}$ when $x \neq 0$ with $f(0) = 0$, you can verify, as an exercise, that $f^{(n)}(0) = 0$ for all $0 \leq n \in \mathbb{Z}$, so the Taylor series of f, centred at 0, is the zero function $T(x) = 0$ for all x. A function which is equal, in an open interval, to the sum of its Taylor series centred at every point a in its domain, is called **analytic**.

Applications

7.44 Example: Let $f(x)=\sin\left(\frac{1}{2}x^2\right)$. Find the 10^{th} derivative $f^{(10)}(0)$.

Solution: Say $f(x)=\sin\left(\frac{1}{2}x^2\right)=\sum_{n=0}^{\infty}c_nx^n$. For all $x\in\mathbb{R}$ we have

$$\begin{aligned}\sin x &= x-\tfrac{1}{3!}\,x^3+\tfrac{1}{5!}\,x^5-\cdots\\ \sin\left(\tfrac{1}{2}\,x^2\right) &= \tfrac{1}{2}\,x^2-\tfrac{1}{2^3\cdot 3!}\,x^6+\tfrac{1}{2^5\cdot 5!}\,x^{10}-\cdots\end{aligned}$$

and so $f^{(10)}(0)=10!\,c_{10}=10!\,\frac{1}{2^5\,5!}=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2^5}=5\cdot 9\cdot 7\cdot 3=945$.

7.45 Example: Find $\displaystyle\lim_{x\to 0}\frac{e^{-2x^2}-\cos 2x}{\left(\tan^{-1}x-\ln(1+x)\right)^2}$

Solution: For all x in an open neighbourhood of 0, and for some $a,b,c,d\in\mathbb{R}$ we have

$$\begin{aligned}\frac{e^{-2x^2}-\cos 2x}{\left(\tan^{-1}x-\ln(1+x)\right)^2} &= \frac{\left(1-2x^2+\frac{1}{2}(2x^2)^2-\cdots\right)-\left(1-\frac{1}{2}(2x)^2+\frac{1}{24}(2x)^4-\cdots\right)}{\left(\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots\right)-\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots\right)\right)^2}\\ &= \frac{\frac{4}{3}x^4+ax^6+\cdots}{\left(\frac{1}{2}x^2+bx^4+\cdots\right)^2}=\frac{\frac{4}{3}x^4+ax^6+\cdots}{\frac{1}{4}x^4+cx^6+\cdots}=\frac{16}{3}+dx^2+\cdots\\ &\longrightarrow \frac{16}{3}\quad\text{as}\quad x\to 0.\end{aligned}$$

7.46 Remark: The next few examples illustrate how one could design some of the buttons on a calculator. In particular, they show how one make an algorithm to calculate e^x, $\ln x$, $x^{2/3}$ (or $x^{1/n}$) and π (assuming one has an algorithm to calculate addition, subtraction, multiplication and division).

7.47 Example: Approximate the value of $\frac{1}{\sqrt{e}}$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned}\tfrac{1}{\sqrt{e}} &= e^{-1/2}=1-\tfrac{1}{2}+\tfrac{1}{2^2\cdot 2!}-\tfrac{1}{2^3\cdot 3!}+\tfrac{1}{2^4\cdot 4!}-\cdots\\ &\cong 1-\tfrac{1}{2}+\tfrac{1}{2^2\cdot 2!}-\tfrac{1}{2^3\cdot 3!}=1-\tfrac{1}{2}+\tfrac{1}{8}-\tfrac{1}{48}=\tfrac{29}{48}\end{aligned}$$

with error $E\leq\frac{1}{2^4\cdot 4!}=\frac{1}{384}$ by the AST (since the sequence $\frac{1}{2^n\cdot n!}$ decreases with limit zero).

7.48 Example: Approximate the value of $\sqrt{e}$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned}\sqrt{e} &= e^{1/2}=1+\tfrac{1}{2}+\tfrac{1}{2^2\cdot 2!}+\tfrac{1}{2^3\cdot 3!}+\tfrac{1}{2^4\cdot 4!}+\cdots\\ &\cong 1+\tfrac{1}{2}+\tfrac{1}{2^2\cdot 2!}+\tfrac{1}{2^3\cdot 3!}=1+\tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{24}=\tfrac{79}{48}\end{aligned}$$

with error

$$\begin{aligned}E &= \tfrac{1}{2^4\cdot 4!}+\tfrac{1}{2^5\cdot 5!}+\tfrac{1}{2^6\cdot 6!}+\tfrac{1}{2^7\cdot 7!}+\cdots\\ &= \tfrac{1}{2^4\cdot 4!}\left(1+\tfrac{1}{2\cdot 5}+\tfrac{1}{2^2\cdot 5\cdot 6}+\tfrac{1}{2^3\cdot 5\cdot 6\cdot 7}+\cdots\right)\\ &< \tfrac{1}{2^4\cdot 4!}\left(1+\tfrac{1}{10}+\left(\tfrac{1}{10}\right)^2+\left(\tfrac{1}{10}\right)^3+\cdots\right)\\ &= \tfrac{1}{2^4\cdot 4!}\cdot\tfrac{1}{1-\frac{1}{5}}=\tfrac{1}{2^4\cdot 4!}\cdot\tfrac{5}{4}=\tfrac{5}{1536}.\end{aligned}$$

7.49 Example: Approximate the value of $\ln 2$ so the error is at most $\frac{1}{50}$

Solution: For $|x|<1$ we have $\frac{1}{1-x}=1+x+x^2+\cdots$ so that $-\ln\left(1-\frac{1}{2}\right)=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$, so

$$\begin{aligned}\ln 2 &= -\ln\tfrac{1}{2} = -\ln\left(1-\tfrac{1}{2}\right) = \tfrac{1}{2}+\tfrac{1}{2\cdot 2^2}+\tfrac{1}{3\cdot 2^3}+\tfrac{1}{4\cdot 2^4}+\tfrac{1}{5\cdot 2^5}+\cdots \\ &\cong \tfrac{1}{2}+\tfrac{1}{2\cdot 2^2}+\tfrac{1}{3\cdot 2^3}+\tfrac{1}{4\cdot 2^4} = \tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{24}+\tfrac{1}{64} = \tfrac{131}{192}\end{aligned}$$

with error

$$E=\tfrac{1}{5\cdot 2^5}+\tfrac{1}{6\cdot 2^6}+\tfrac{1}{7\cdot 2^7}+\cdots < \tfrac{1}{5\cdot 2^5}\left(1+\tfrac{1}{2}+\tfrac{1}{2^2}+\tfrac{1}{2^3}+\cdots\right)=\tfrac{1}{5\cdot 2^5}\cdot 2=\tfrac{1}{80}.$$

7.50 Example: Approximate the value of $10^{2/3}$ so the error is at most $\frac{1}{100}$.

Solution: Using the binomial series, we have

$$\begin{aligned}10^{2/3} &= (8+2)^{2/3} = 4\left(1+\tfrac{1}{4}\right)^{2/3} \\ &= 4\left(1+\tfrac{2}{3}\cdot\tfrac{1}{4}+\tfrac{(\frac{2}{3})(-\frac{1}{3})}{2!}\cdot\tfrac{1}{4^2}+\tfrac{(\frac{2}{3})\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}\cdot\tfrac{1}{4^3}+\cdots\right) \\ &= 4+\tfrac{2}{3}-\tfrac{2\cdot 1}{3^2\cdot 2!\cdot 4}+\tfrac{2\cdot 1\cdot 4}{3^3\cdot 3!\cdot 4^2}-\cdots+(-1)^{n+1}\tfrac{2\cdot 1\cdot 4\cdot 7\cdots(3n-5)}{3^n\cdot n!\cdot 4^{n-1}}+\cdots \\ &\cong 4+\tfrac{2}{3}-\tfrac{2\cdot 1}{3^2\cdot 2!\cdot 4} = 4+\tfrac{2}{3}-\tfrac{1}{36}=\tfrac{167}{36}\end{aligned}$$

with error $E\le\frac{2\cdot 1\cdot 4}{3^3\cdot 3!\cdot 4^2}=\frac{1}{3^3\cdot 3!\cdot 2}=\frac{1}{324}$ by the AST, which we can apply because because for $a_n=\frac{2\cdot 1\cdot 4\cdot 7\cdots(3n-5)}{3^n\cdot n!\cdot 4^{n-1}}$ we have $\frac{a_{n+1}}{a_n}=\frac{3n-2}{3(n+1)\cdot 4}<\frac{1}{4}$ so that the sequence (a_n) is decreasing, and we have $a_n=\frac{2\cdot 1\cdot 4\cdot 7\cdots(3n-5)}{3^n\cdot n!\cdot 4^{n-1}}<\frac{3\cdot 6\cdot 9\cdots(3n)}{3^n\cdot n!\cdot 4^{n-1}}=\frac{1}{4^{n-1}}\longrightarrow 0$ as $n\to\infty$.

7.51 Example: Approximate the value of π so the error is at most $\frac{1}{50}$.

Solution: For $|x|<1$ we have $\frac{1}{1+x^2}=1-x^2+x^4-\cdots$ so $\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots$. Put in $x=\frac{1}{\sqrt{3}}$ to get

$$\begin{aligned}\tfrac{\pi}{6} &= \tfrac{1}{\sqrt{3}}-\tfrac{1}{3\cdot 3\sqrt{3}}+\tfrac{1}{5\cdot 3^2\sqrt{3}}-\tfrac{1}{7\cdot 3^3\sqrt{3}}+\cdots \\ \pi &= 2\sqrt{3}\left(1-\tfrac{1}{3\cdot 3}+\tfrac{1}{5\cdot 3^2}-\tfrac{1}{7\cdot 3^3}+\cdots\right)=\sum_{n=0}^{\infty}(-1)^n\tfrac{2\sqrt{3}}{(2n+1)\cdot 3^n} \\ &\cong 2\sqrt{3}\left(1-\tfrac{1}{3\cdot 3}+\tfrac{1}{5\cdot 3^2}\right)=\tfrac{82\sqrt{3}}{45}\end{aligned}$$

with error $E\le\frac{2\sqrt{3}}{7\cdot 3^3}=\frac{2\sqrt{3}}{189}$ by the AST.

7.52 Example: Approximate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \int_0^1 1-x^2+\tfrac{1}{2!}x^4-\tfrac{1}{3!}x^6+\tfrac{1}{4!}x^8-\cdots\;dx \\ &= \left[x-\tfrac{1}{3}x^3+\tfrac{1}{5\cdot 2!}x^5-\tfrac{1}{7\cdot 3!}x^7+\tfrac{1}{9\cdot 4!}x^9-\cdots\right]_0^1 \\ &= 1-\tfrac{1}{3}+\tfrac{1}{5\cdot 2!}-\tfrac{1}{7\cdot 3!}+\tfrac{1}{9\cdot 4!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{(2n+1)\,n!} \\ &\cong 1-\tfrac{1}{3}+\tfrac{1}{5\cdot 2!}-\tfrac{1}{7\cdot 3!}=1-\tfrac{1}{3}+\tfrac{1}{10}-\tfrac{1}{42}=\tfrac{26}{35}\end{aligned}$$

with error $E\le\frac{1}{9\cdot 4!}=\frac{1}{216}$ by the AST.

7.53 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}$.

Solution: Since $\cos x = \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ for all x, we have $\sum\limits_{n=0}^{\infty}\frac{(-2)^n}{(2n)!} = \cos(\sqrt{2})$.

7.54 Example: Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty}\frac{n-2}{(-3)^n}$.

Solution: For $|x|<1$ we have $\frac{1}{1+x} = 1-x+x^2-x^3+\cdots$ and $\frac{1}{(1+x)^2} = 1-2x+3x^2-4x^3-\cdots$. Put in $x=\frac{1}{3}$ to get $\frac{9}{16} = 1-\frac{2}{3}+\frac{3}{3^3}-\frac{4}{3^3}+\frac{5}{3^4}-\cdots$ so that

$$\sum_{n=1}^{\infty}\frac{n-2}{(-3)^n} = \frac{1}{3}+\frac{0}{3^2}-\frac{1}{3^3}+\frac{2}{3^4}-\frac{3}{3^5}+\cdots = \frac{1}{3}-\frac{1}{3^3}\left(1-\frac{2}{3}+\frac{3}{3^2}-\frac{4}{3^3}+\cdots\right) = \frac{1}{3}-\frac{1}{27}\cdot\frac{9}{16} = \frac{5}{16}.$$

7.55 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty}\frac{2\cdot5\cdot8\cdot\cdots\cdot(3n+2)}{5^n\, n!}$.

Solution: For $|x|<1$ we have

$$\begin{aligned}2(1-x)^{-5/3} &= 2\left(1+\left(-\tfrac{5}{3}\right)(-x)+\frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{2!}x^2+\frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{3!}x^3+\cdots\right)\\ &= 2+\tfrac{2\cdot5}{3\cdot1!}\,x+\tfrac{2\cdot5\cdot8}{3^2\cdot2!}\,x^2+\tfrac{2\cdot5\cdot8\cdot11}{3^3\cdot3!}\,x^3+\cdots.\end{aligned}$$

Put in $x=\frac{3}{5}$ to get

$$\sum_{n=0}^{\infty}\frac{2\cdot5\cdot8\cdot\cdots\cdot(3n+2)}{5^n\,n!} = 2+\frac{2\cdot5}{5\cdot1!}+\frac{2\cdot5\cdot8}{5^2\cdot2!}+\frac{2\cdot5\cdot8\cdot11}{5^3\cdot3!}+\cdots = 2\left(\tfrac{2}{5}\right)^{-5/3} = 2\left(\tfrac{5}{2}\right)^{5/3}.$$

7.56 Example: Find the solution of the IVP $y''-2x\,y'-2y=0$ with $y(0)=1$, $y'(0)=0$. First find a power series solution, then convert the power series to closed form.

Solution: Let $y=\sum\limits_{n=0}^{\infty}a_nx^n$ so $y'=\sum\limits_{n=1}^{\infty}na_nx^{n-1}$ and $y''=\sum\limits_{n=2}^{\infty}n(n-1)a_nx^{n-2}$. Put these in the DE to get

$$\begin{aligned}0=y''-2xy'-2y &= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}2na_nx^n-\textstyle\sum_{n=0}^{\infty}2a_nx^n\\ &= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m-\sum_{m=1}^{\infty}2ma_mx^m-\textstyle\sum_{m=0}^{\infty}2a_mx^m\\ &= \big(2a_2-2a_0\big)+\sum_{m=1}^{\infty}\big((m+2)(m+1)a_{m+2}-2(m+1)a_m\big)x^m\,.\end{aligned}$$

The coefficients all vanish, so $a_2=a_0$ and for $m\geq1$, $a_{m+2}=\frac{2(m+1)a_m}{(m+2)(m+1)}=\frac{2a_m}{m+2}$. To get $y(0)=1$ and $y'(0)=0$, we need $a_0=1$ and $a_1=0$, and then the recurrence formula gives $a_k=0$ for k odd and $a_2=1$, $a_4=\frac{2}{4}$, $a_6=\frac{2^2}{2\cdot6}$, and in general $a_{2n}=\frac{2^n}{2\cdot4\cdot6\cdots(2n)}=\frac{1}{n!}$. Thus the solution is $y=\sum\limits_{n=0}^{\infty}\frac{x^{2n}}{n!}=e^{x^2}$.