Chapter 6. Sequences and Series of Real Numbers

Sequences (Review)

6.1 Definition: A **sequence** in a set A is a function $a:\{k,k+1,k+2,\cdots\}\to A$ for some integer k. For a sequence $a:\{k,k+1,\cdots\}\to A$, we write $a_n=a(n)$ for $n\geq k$, we refer to the function a as the sequence (a_n) or the sequence $(a_n)_{n\geq k}$, and we write

$$(a_n)_{n\geq k}=\big(a_k,a_{k+1},a_{k+2},\cdots\big).$$

When $(a_n)_{n\geq k}$ is a sequence in $\mathbb{R}$, we say the sequence $(a_n)_{n\geq k}$ **converges** to the real number $b\in\mathbb{R}$, or that the **limit** of the sequence $(a_n)_{n\geq k}$ is equal to b, and we write $\lim\limits_{n\to\infty}a_n=b$ or we write $a_n\to b$ (as $n\to\infty$), when for every $\epsilon>0$ there exists an integer $m\geq k$ such that for every integer n we have

$$n\geq m\Longrightarrow |a_n-b|<\epsilon\,.$$

We say the sequence (a_n) **converges** (in $\mathbb{R}$) if it converges to some real number $b\in\mathbb{R}$.

We say the sequence (a_n) **diverges to infinity**, or that the **limit** of (a_n) is equal to infinity, and write $\lim\limits_{n\to\infty}a_n=\infty$ or $a_n\to\infty$, when for every $r\in\mathbb{R}$ there exists an integer $m\geq k$ such that for every integer n we have

$$n\geq m\Longrightarrow a_n>r\,.$$

We say that (a_n) **diverges to negative infinity**, or that the limit of (a_n) is equal to negative infinity, and write $\lim\limits_{n\to\infty}a_n=-\infty$ or $a_n\to-\infty$, when for every $r\in\mathbb{R}$ there exists an integer $m\geq k$ such that for every integer n we have

$$n\geq m\Longrightarrow a_n<r\,.$$

6.2 Theorem: *(First Finitely Many Terms do Not Affect Convergence) Let $(a_n)_{n\geq k}$ be a sequence in $\mathbb{R}$ and let $\ell\in\mathbb{Z}^+$. Then $\lim\limits_{n\to\infty}a_n$ exists if and only if $\lim\limits_{n\to\infty}a_{n+\ell}$ exists, and in this case the limits are equal.*

6.3 Note: Because of the above theorem, we often omit the starting value k from our notation and write the sequence $(a_n)_{n\geq k}$ simply as (a_n). Also, we often choose a specific starting value k (often $k=1$) in the statements or the proofs of various theorems with the understanding that the theorem holds for any any integer k.

6.4 Theorem: *(Linearity, Products and Quotients) If (a_n) and (b_n) are convergent sequences in $\mathbb{R}$ then*

(1) for any real number c, the sequence $(c\,a_n)$ converges with $\lim\limits_{n\to\infty}c\,a_n=c\lim\limits_{n\to\infty}a_n$,

(2) the sequence (a_n+b_n) converges with $\lim\limits_{n\to\infty}(a_n+b_n)=\lim\limits_{n\to\infty}a_n+\lim\limits_{n\to\infty}b_n$,

(3) the sequence (a_nb_n) converges with $\lim\limits_{n\to\infty}(a_nb_n)=\left(\lim\limits_{n\to\infty}a_n\right)\left(\lim\limits_{n\to\infty}b_n\right)$, and

(4) if $\lim\limits_{n\to\infty}b_n\neq 0$ then the sequence $\left(\frac{a_n}{b_n}\right)$ converges with $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\dfrac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n}$.

6.5 Note: By defining algebraic operations in the **extended real numbers** $\mathbb{R}\cup\{\pm\infty\}$, the above theorem can be extended to include many cases in which $\lim\limits_{n\to\infty}a_n=\pm\infty$ or $\lim\limits_{n\to\infty}b_n=\pm\infty$, but care is needed for the **indeterminate forms** $\infty-\infty$, $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$.

6.6 Theorem: *(Comparison and Squeeze) Let* (a_n), (b_n) *and* (c_n) *be sequences in* $\mathbb{R}$.

(1) If $a_n \le b_n$ *for all* n *and* $\lim\limits_{n\to\infty} a_n$ *and* $\lim\limits_{n\to\infty} b_n$ *both exist, then* $\lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$.

(2) If $a_n \le b_n \le c_n$ *for all* n *and if* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n$ *then* $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n$.

6.7 Theorem: *(Sequences and Absolute Values) Let* (a_n) *be a sequence in* $\mathbb{R}$.

(1) If $\lim\limits_{n\to\infty} a_n$ *exists then* $\lim\limits_{n\to\infty} |a_n| = \left|\lim\limits_{n\to\infty} a_n\right|$.

(2) If $\lim\limits_{n\to\infty} |a_n| = 0$ *then* $\lim\limits_{n\to\infty} a_n = 0$.

(3) If $|a_n| \le b_n$ *for all* $n \ge k$ *and* $\lim\limits_{n\to\infty} b_n = 0$ *then* $\lim\limits_{n\to\infty} a_n = 0$.

6.8 Definition: Let $(a_n)_{n\ge k}$ be a sequence in $\mathbb{R}$. We say that $(a_n)_{n\ge k}$ is **increasing** (or **non-decreasing**) when $a_n \le a_{n+1}$ for all $n \ge k$, or equivalently when $n \le m \Longrightarrow a_n \le a_m$ for all integers $n, m \ge k$. We say that $(a_n)_{n\ge k}$ is **strictly increasing** when $a_n < a_{n+1}$ for all $n \ge k$. We say that $(a_n)_{n\ge k}$ is **bounded above** by the real number b when $a_n \le b$ for all $n \ge k$, and in this case b is called an **upper bound** for the sequence. We say that $(a_n)_{n\ge k}$ is **bounded above** when it is bounded above by some real number b. We have similar definitions for the terms **decreasing** (or **nonincreasing**), **strictly decreasing**, **bounded below** and **lower bound**.

6.9 Theorem: *(The Monotone Convergence Theorem) Let* (a_n) *be a sequence in* $\mathbb{R}$.

(1) If (a_n) *is increasing and bounded above by* b*, then* (a_n) *converges and* $\lim\limits_{n\to\infty} a_n \le b$.

(2) If (a_n) *is increasing and is not bounded above, then* $\lim\limits_{n\to\infty} a_n = \infty$.

(3) If (a_n) *is decreasing and bounded below by* c*, then* (a_n) *converges and* $\lim\limits_{n\to\infty} a_n \ge c$.

(4) If (a_n) *is decreasing and is not bounded below, then* $\lim\limits_{n\to\infty} a_n = -\infty$.

6.10 Definition: A sequence $(a_n)_{n\ge k}$ is said to be **Cauchy** when for every $\epsilon > 0$ there exists an integer $N \ge k$ such that for all integers $n, m \ge k$ we have

$$n, m \ge N \Longrightarrow |a_n - a_m| < \epsilon\,.$$

6.11 Theorem: *(The Cauchy Criterion for Sequences) Let* (a_n) *be a sequence in* $\mathbb{R}$*. Then* (a_n) *converges if and only if* (a_n) *is Cauchy.*

6.12 Theorem: *(The Sequential Characterization of Limits of Functions) Let* $A \subseteq \mathbb{R}$*, let* $f : A \to \mathbb{R}$*, and let* $a, b \in \mathbb{R} \cup \{\pm\infty\}$ *where either* $a \in A$ *or* a *is a limit point of* A*. Then* $\lim\limits_{x\to a} f(x) = b$ *if and only if* $\lim\limits_{n\to\infty} f(x_n) = b$ *for every sequence* (x_n) *in* $A \setminus \{a\}$ *with* $\lim\limits_{n\to\infty} x_n = a$.

6.13 Definition: Let $(a_n)_{n\ge k}$ be a sequence in $\mathbb{R}$. We define the **limit supremum** and the **limit infemum** of $(a_n)_{n\ge k}$ to be to be the the extended real numbers

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\left\{a_k \,\middle|\, k \ge n\right\} \quad , \quad \liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\left\{a_k \,\middle|\, k \ge n\right\}.$$

The above two limits do exist as extended real numbers because for $b_n = \sup\{a_k | k \ge n\}$ the sequence (b_n) is decreasing, and for $c_n = \inf\{a_k | k \ge n\}$ the sequence (c_n) is increasing.

6.14 Theorem: *Let* $(a_n)_{n\ge k}$ *be a sequence in* $\mathbb{R}$ *and let* $b \in \mathbb{R}$.

(2) (a_n) *is bounded above if and only if* $\limsup\limits_{n\to\infty} a_n < \infty$,

(2) (a_n) *is bounded below if and only if* $\liminf\limits_{n\to\infty} a_n > -\infty$,

(3) $\lim\limits_{n\to\infty} a_n = b$ *if and only if* $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$.

Series

6.15 Definition: Let $(a_n)_{n\geq k}$ be a sequence. The **series** $\sum\limits_{n\geq k} a_n$ is defined to be the sequence $(S_\ell)_{\ell\geq k}$ where

$$S_\ell = \sum_{n=k}^{\ell} a_n = a_k + a_{k+1} + \cdots + a_\ell\,.$$

The term S_ℓ is called the ℓ^{th} **partial sum** of the series $\sum\limits_{n\geq k} a_n$. The **sum** of the series, denoted by

$$S = \sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots\,,$$

is the limit of the sequence of partial sums, if it exists, and we say the series **converges** when the sum exists and is finite. We remark that it is quite common to write $\sum\limits_{n=k}^{\infty} a_n$ (somewhat abusively) both to denote the sequence of partial sums (which may or may not converge) and to denote its limit (when it does converge).

6.16 Example: (Geometric Series) Show that for $a \neq 0$, the series $\sum\limits_{n=1}^{\infty} a\,r^n$ converges if and only if $|r| < 1$, and that in this case

$$\sum_{n=k}^{\infty} ar^n = \tfrac{ar^k}{1-r}\,.$$

Solution: The ℓ^{th} partial sum is

$$S_\ell = \sum_{n=k}^{\infty} ar^n = ar^k + ar^{k+1} + ar^{k+2} + \cdots + ar^l\,.$$

When $r = 1$ we have $S_\ell = a(l-k+1)$ and so $\lim\limits_{\ell\to\infty} S_\ell = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \neq 1$ we have $rS_\ell = ar^{k+1} + ar^{k+2} + \cdots + ar^\ell + ar^{\ell+1}$, so $S_\ell - rS_\ell = ar^k - ar^{\ell+1} = ar^k\big(1 - r^{\ell-k+1}\big)$ and so

$$S_\ell = \tfrac{ar^k(1-r^{\ell-k+1})}{1-r}\,.$$

When $r > 1$, $\lim\limits_{\ell\to\infty} r^{\ell-k+1} = \infty$ and so $\lim\limits_{\ell\to\infty} S_\ell = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \leq -1$, $\lim\limits_{\ell\to\infty} r^{\ell-k+1}$ does not exist, and so neither does $\lim\limits_{\ell\to\infty} S_\ell$. When $|r| < 1$, we have $\lim\limits_{\ell\to\infty} r^{\ell-k+1} = 0$ and so $\lim\limits_{\ell\to\infty} S_\ell = \frac{ar^k}{1-r}$, as required.

6.17 Example: Find $\displaystyle\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}}$.

Solution: This is a geometric series. By the formula in the previous example, we have

$$\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}} = \sum_{n=-1}^{\infty} \tfrac{9}{2}\left(\tfrac{3}{4}\right)^n = \frac{\frac{9}{2}\left(\frac{3}{4}\right)^{-1}}{1-\frac{3}{4}} = \tfrac{9}{2}\cdot\tfrac{4}{3}\cdot\tfrac{4}{1} = 24\,.$$

6.18 Example: (Telescoping Series) Find $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

Solution: We use a partial fractions decomposition. The ℓ^{th} partial sum is

$$\begin{aligned} S_\ell &= \sum_{n=1}^{\ell} \frac{1}{n(n+2)} = \sum_{n=1}^{\ell} \left(\frac{\frac12}{n} - \frac{\frac12}{n+2} \right) = \frac12 \sum_{n=1}^{\ell} \left(\tfrac1n - \tfrac{1}{n+2} \right) \\ &= \tfrac12 \Big(\big(1 - \tfrac13\big) + \big(\tfrac12 - \tfrac14\big) + \big(\tfrac13 - \tfrac15\big) + \cdots + \Big(\tfrac{1}{\ell-2} - \tfrac1\ell\Big) + \Big(\tfrac{1}{\ell-1} - \tfrac{1}{\ell+1}\Big) + \Big(\tfrac1\ell - \tfrac{1}{\ell+2}\Big) \Big) \\ &= \tfrac12 \Big(1 + \tfrac12 - \tfrac{1}{\ell+1} - \tfrac{1}{\ell+2} \Big) , \end{aligned}$$

since all the other terms cancel. Thus the sum of the series is

$$S = \lim_{\ell\to\infty} S_\ell = \tfrac12 \big(1 + \tfrac12\big) = \tfrac34 \, .$$

6.19 Theorem: *(First Finitely Many Terms do Not Affect Convergence) Let $(a_n)_{n\geq k}$ be a sequence in $\mathbb{R}$. Then for any integer $m \geq k$, the series $\sum_{n\geq k} a_n$ converges if and only if the series $\sum_{n\geq m} a_n$ converges, and in this case*

$$\sum_{n=k}^{\infty} a_n = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \sum_{n=m}^{\infty} a_n \, .$$

Proof: Let $S_\ell = \sum_{n=k}^{\ell} a_n$ and let $T_\ell = \sum_{n=m}^{\ell} a_n$. Then for all $\ell \geq m$ we have

$$S_\ell = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + T_\ell \, ,$$

and so (S_ℓ) converges if and only if (T_ℓ) converges, and in this case

$$\lim_{\ell\to\infty} S_\ell = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \lim_{\ell\to\infty} T_\ell \, .$$

6.20 Note: Since the first finitely many terms do not affect the convergence of a series, we often omit the subscript $n \geq k$ in the expression $\sum_{n\geq k} a_n$, and simply write $\sum a_n$, when we are interested in whether or not the series converges. On the other hand, we cannot omit the subscript $n = k$ when we are interested in the value of the sum $\sum_{n=k}^{\infty} a_n$.

6.21 Definition: When we approximate a value x by the value y, the **error** in our approximation is $|x - y|$.

6.22 Note: If $\sum_{n\geq k} a_n$ converges and $\ell \geq k$ then, by the above theorem, so does $\sum_{n\geq \ell+1}^{\infty} a_n$. If we approximate the sum $S = \sum_{n=k}^{\infty} a_n$ by the ℓ^{th} partial sum $S_\ell = \sum_{n=k}^{\ell} a_n$, then the **error** in our approximation is

$$\big|S - S_\ell\big| = \left| \sum_{n=\ell+1}^{\infty} a_n \right| .$$

6.23 Theorem: *(Linearity) If* $\sum a_n$ *and* $\sum b_n$ *are convergent series then*
(1) for any real number c, $\sum c a_n$ *converges and* $\sum\limits_{n=k}^{\infty} c a_n = c \sum\limits_{n=k}^{\infty} a_n$, *and*
(2) the series $\sum (a_n + b_n)$ *converges and* $\sum\limits_{n=k}^{\infty} (a_n + b_n) = \sum\limits_{n=k}^{\infty} a_n + \sum\limits_{n=k}^{\infty} b_n$.

Proof: This follows immediately from the Linearity Theorem for sequences.

6.24 Theorem: *(Series of Positive Terms) Let* $\sum a_n$ *be a series.*
(1) If $a_n \geq 0$ *for all* n *then either* $\sum a_n$ *converges or* $\sum\limits_{n=k}^{\infty} a_n = \infty$.
(2) If $a_n \leq 0$ *for all* n *then either* $\sum a_n$ *converges or* $\sum\limits_{n=k}^{\infty} a_n = -\infty$.

Proof: This follows from the Monotone Convergence Theorem for sequences. Indeed if $a_n \geq 0$ for all $n \geq k$, then $(S_\ell)_{\ell \geq k}$ is increasing (since $S_{\ell+1} = S_\ell + a_{\ell+1} \geq S_\ell$ for all ℓ). Either (S_ℓ) is bounded above, in which case (S_ℓ) converges hence $\sum a_n$ converges, or the sequence (S_ℓ) is unbounded, in which case $\lim\limits_{\ell \to \infty} S_\ell = \infty$ hence $\sum\limits_{n=k}^{\infty} a_n = \infty$.

6.25 Theorem: *(Cauchy Criterion for Series) Let* $\sum a_n$ *be a series. Then* $\sum a_n$ *converges if and only if for all* $\epsilon > 0$ *there exists* N *such that for all* $\ell, m \in \mathbb{Z}$,

$$m > \ell \geq N \Longrightarrow \left| \sum_{n=\ell+1}^{m} a_n \right| < \epsilon \, .$$

Proof: This follows from the Cauchy Criterion for Sequences, applied to the sequence of partial sums. Indeed (S_ℓ) converges if and only if for all $\epsilon > 0$ there exists N such that $m > \ell \geq N \Longrightarrow |S_m - S_\ell| < \epsilon$, and we have

$$|S_m - S_\ell| = \left| \sum_{n=k}^{m} a_n - \sum_{n=k}^{\ell} a_n \right| = \left| \sum_{n=\ell+1}^{m} a_n \right| .$$

Convergence Tests

6.26 Theorem: *(The Divergence Test) If* $\sum a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$*. Equivalently, if* $\lim_{n\to\infty} a_n$ *either does not exist, or exists but is not equal to 0, then* $\sum a_n$ *diverges.*

Proof: Suppose that $\sum a_n$ converges, and say $\sum_{n=k}^{\infty} a_n = S$. Let S_ℓ be the ℓ^{th} partial sum. Then we have $\lim_{\ell\to\infty} S_\ell = S = \lim_{\ell\to\infty} S_{\ell-1}$, and we have $a_\ell = S_\ell - S_{\ell-1}$, and so

$$\lim_{\ell\to\infty} a_\ell = \lim_{\ell\to\infty} S_\ell - \lim_{\ell\to\infty} S_{\ell-1} = S - S = 0\,.$$

6.27 Example: Determine whether $\sum e^{1/n}$ converges.

Solution: Since $\lim_{n\to\infty} e^{1/n} = e^0 = 1$, $\sum e^{1/n}$ diverges by the Divergence Test.

6.28 Note: The converse of the Divergence Test is false. For example, as we shall see below, $\sum \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

6.29 Theorem: *(Integral Test) Let* $f(x)$ *be positive and decreasing for* $x \ge k$*, and let* $a_n = f(n)$ *for all integers* $n \ge k$*. Then* $\sum a_n$ *converges if and only if* $\int_k^\infty f(x)\,dx$ *converges, and in this case, for any* $\ell \ge k$ *we have*

$$\int_{\ell+1}^\infty f(x)\,dx \le \sum_{n=\ell+1}^{\infty} a_n \le \int_\ell^\infty f(x)\,dx\,.$$

Proof: Let T_m be the m^{th} partial sum for $\sum_{n\ge\ell+1} a_n$, so $T_m = \sum_{n=\ell+1}^{m} a_n$. Note that since $f(x)$ is decreasing, it is integrable on any closed interval. Also, for each $n \ge \ell$ we have $a_n = f(n) \le f(x)$ for all $x \in [n-1, n]$, so $\int_{n-1}^{n} f(x)\,dx \ge \int_{n-1}^{n} a_n\,dx = a_n$ and so

$$T_m = \sum_{n=\ell+1}^{m} a_n \le \sum_{n=\ell+1}^{m} \int_{n-1}^{n} f(x)\,dx = \int_\ell^m f(x)\,dx \le \int_\ell^\infty f(x)\,dx\,.$$

Since $f(n) = a_n$ is positive, the sequence (T_m) is increasing. If $\int_k^\infty f(x)\,dx$ converges, then (T_n) is bounded above by $\int_\ell^\infty f(x)\,dx$, and so it converges with $\lim_{m\to\infty} T_m \le \int_\ell^\infty f(x)\,dx$.

Similarly, for each $n \ge \ell$ we have $a_n = f(n) \ge f(x)$ for all $x \in [n, n+1]$ so that $\int_n^{n+1} f(x)\,dx \le \int_n^{n+1} a_n dx = a_n$ and so

$$T_m = \sum_{n=\ell+1}^{m} a_n \ge \sum_{n=\ell+1}^{m} \int_n^{n+1} f(x)\,dx = \int_{\ell+1}^{m+1} f(x)\,dx\,.$$

If $\int_k^\infty f(x)\,dx$ converges, then $\lim_{m\to\infty} T_m \ge \lim_{m\to\infty} \int_{\ell+1}^{m+1} f(x)\,dx = \int_{\ell+1}^\infty f(x)\,dx$. If $\int_k^\infty f(x)\,dx$ diverges, then $\lim_{m\to\infty} \int_{\ell+1}^{m+1} f(x)\,dx = \infty$, and so $\lim_{m\to\infty} T_m = \infty$ too, by Comparison.

6.30 Example: (p-Series) Show that the series $\sum\limits_{n\geq 1} \frac{1}{n^p}$ converges if and only if $p>1$. In particular, the **harmonic series** $\sum \frac{1}{n}$ diverges.

Solution: If $p<0$ then $\lim\limits_{n\to\infty} \frac{1}{n^p}=\infty$ and if $p=0$ then $\lim\limits_{n\to\infty}\frac{1}{n^p}=1$, so in either case $\sum\frac{1}{n^p}$ diverges by the Divergence Test. Suppose that $p>0$. Let $a_n=\frac{1}{n^p}$ for integers $n\geq 1$, and let $f(x)=\frac{1}{x^p}$ for real numbers $x\geq 1$. Note that $f(x)$ is positive and decreasing for $x\geq 1$ and $a_n=f(n)$ for all $n\geq 1$. Since we know that $\int_1^\infty f(x)\,dx$ converges if and only if $p>1$, it follows from the Integral Test that $\sum a_n$ converges if and only if $p>1$.

6.31 Example: Approximate $S=\sum\limits_{n=1}^{\infty}\frac{1}{2n^2}$ so that the error is at most $\frac{1}{100}$.

Solution: We let $a_n=\frac{1}{2n^2}$ and $f(x)=\frac{1}{2x^2}$ so that we can apply the Integral Test. If we choose to approximate the sum S by the ℓ^{th} partial sum S_ℓ, then the error is

$$E=S-S_\ell=\sum_{n=\ell+1}^{\infty}a_n\leq\int_\ell^\infty \frac{1}{2x^2}\,dx=\Big[-\frac{1}{2x}\Big]_\ell^\infty=\frac{1}{2\ell}\,,$$

and so to insure that $E\leq\frac{1}{100}$ we can choose ℓ so that $\frac{1}{2\ell}\leq\frac{1}{100}$, that is $\ell\geq 50$. Since it would be tedious to add up the first 50 terms of the series, we take an alternate approach. The Integral Test gives us upper and lower bounds: we have

$$\int_{\ell+1}^\infty f(x)\,dx\leq S-S_\ell\leq\int_\ell^\infty f(x)\,dx$$
$$\frac{1}{2(l+1)}\leq S-S_\ell\leq\frac{1}{2\ell}$$
$$S_\ell+\frac{1}{2(\ell+1)}\ \leq S\leq\ S_\ell+\frac{1}{2\ell}\,.$$

If approximate S using the midpoint of the upper and lower bounds, that is if we make the approximation $S\cong S_\ell+\frac{1}{2}\left(\frac{1}{2\ell}+\frac{1}{2(\ell+1)}\right)$, then the error E will be at most half of the difference of the bounds:

$$E\leq\frac{1}{2}\left(\frac{1}{2\ell}-\frac{1}{2(\ell+1)}\right)=\frac{1}{4\ell(\ell+1)}\,.$$

To get $E\leq\frac{1}{100}$ we want $\frac{1}{4\,l\,(l+1)}\leq\frac{1}{100}$, that is $\ell\,(\ell+1)\geq 25$, and so we can take $\ell=5$. Thus we estimate

$$S\cong S_5+\tfrac{1}{2}\left(\tfrac{1}{10}+\tfrac{1}{12}\right)=\tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{18}+\tfrac{1}{32}+\tfrac{1}{50}+\tfrac{1}{20}+\tfrac{1}{24}=\tfrac{5929}{7200}\,.$$

(Incidentally, the exact value of this sum is $\frac{\pi^2}{12}$).

6.32 Theorem: *(Comparison Test) Let $0\leq a_n\leq b_n$ for all $n\geq k$. Then if $\sum b_n$ converges then so does $\sum a_n$ and in this case,*

$$\sum_{n=k}^{\infty}a_n\leq\sum_{n=k}^{\infty}b_n\,.$$

Proof: Let $S_\ell=\sum\limits_{n=k}^{\ell}a_n$ and let $T_\ell=\sum\limits_{n=k}^{\ell}b_n$. Since $0\leq a_n,b_n$ for all n, the sequences (S_ℓ) and (T_ℓ) are increasing. Since $a_n\leq b_n$ for all n we have $S_\ell\leq T_\ell$ for all ℓ. Suppose that $\sum b_n$ converges with say $\sum\limits_{n=k}^{\infty}b_n=T$ so that $\lim\limits_{\ell\to\infty}T_\ell=T$. Then $S_\ell\leq T_\ell\leq T$ for all ℓ, so (S_ℓ) is increasing and bounded above, hence convergent, and $\lim\limits_{\ell\to\infty}S_\ell\leq\lim\limits_{\ell\to\infty}T_\ell$.

6.33 Example: Determine whether $\sum\limits_{n\geq 0} \frac{1}{\sqrt{n^3+1}}$ converges.

Solution: Note that $0 \leq \frac{1}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$ for all $n \geq 1$, and $\sum \frac{1}{n^{3/2}}$ converges since it is a p-series with $p = \frac{3}{2}$, and so $\sum \frac{1}{\sqrt{n^3+1}}$ also converges, by the Comparison Test.

6.34 Example: Determine whether $\sum\limits_{n\geq 1} \tan \frac{1}{n}$ converges.

Solution: For $0 < x < \frac{\pi}{2}$ we have $x < \tan x$, so for $n \geq 1$ we have $0 < \frac{1}{n} < \tan \frac{1}{n}$. Since the harmonic series $\sum \frac{1}{n}$ diverges, the series $\sum \tan \frac{1}{n}$ also diverges by the Comparison Test.

6.35 Example: Approximate $S = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$ so that the error is at most $\frac{1}{100}$.

Solution: If we make the approximation $S \cong S_\ell = \sum\limits_{n=0}^{l} \frac{1}{n!}$ then the error is

$$\begin{aligned} E = S - S_\ell &= \sum_{n=\ell+1}^{\infty} \frac{1}{n!} \\ &= \frac{1}{(\ell+1)!} + \frac{1}{(\ell+2)!} + \frac{1}{(\ell+3)!} + \frac{1}{(\ell+4)!} + \cdots \\ &= \frac{1}{(\ell+1)!}\left(1 + \frac{1}{\ell+2} + \frac{1}{(\ell+2)(\ell+3)} + \frac{1}{(\ell+2)(\ell+3)(\ell+4)} + \cdots\right) \\ &\leq \frac{1}{(\ell+1)!}\left(1 + \frac{1}{\ell+2} + \frac{1}{(\ell+2)^2} + \frac{1}{(\ell+2)^3} + \cdots\right) \\ &= \frac{1}{(\ell+1)!}\,\frac{1}{1 - \frac{1}{\ell+2}} \\ &= \frac{\ell+2}{(\ell+1)(\ell+1)!} \end{aligned}$$

where we used the Comparison Test and the formula for the sum of a geometric series. To get $E \leq \frac{1}{100}$ we can choose l so that $\frac{l+2}{(l+1)(l+1)!} \leq \frac{1}{100}$. By trial and error, we find that we can take $l = 4$, so we make the approximation

$$S \cong S_4 = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} = \tfrac{65}{24}\,.$$

(Incidentally, the exact value of this sum is e, so we have approximated the value of e).

6.36 Theorem: *(Limit Comparison Test) Let $a_n \geq 0$ and let $b_n > 0$ for all $n \geq k$. Suppose that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = r$. Then*
(1) if $r = \infty$ and $\sum a_n$ converges then so does $\sum b_n$,
(2) if $r = 0$ and $\sum b_n$ converges then so does $\sum a_n$, and
(3) if $0 < r < \infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof: If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$, then for large n we have $\frac{a_n}{b_n} > 1$ so that $a_n > b_n$, and so if $\sum a_n$ converges, then so does $\sum b_n$ by the Comparison Test. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ then for large n we have $\frac{a_n}{b_n} < 1$ so $a_n < b_n$, and so if $\sum b_n$ converges then so does $\sum a_n$ by the Comparison Test. Suppose that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = r$ with $0 < r < \infty$. Choose m so that when $n \geq m$ we have $\left|\frac{a_n}{b_n} - r\right| < \frac{r}{2}$ so that $\frac{r}{2} < \frac{a_n}{b_n} < \frac{3r}{2}$ and hence

$$0 < \tfrac{r}{2} b_n \leq a_n \leq \tfrac{3r}{2} b_n\,.$$

If $\sum a_n$ converges, then $\sum \frac{r}{2} b_n$ converges by the Comparison Test, and hence $\sum b_n$ converges by linearity. If $\sum b_n$ converges, then $\sum \frac{3r}{2} b_n$ converges by linearity, and hence so does $\sum a_n$ by the Comparison Test.

6.37 Example: Determine whether $\sum \frac{1}{\sqrt{n^3-1}}$ converges.

Solution: Note that we cannot use the same argument that we used earlier to show that $\sum \frac{1}{\sqrt{n^3+1}}$ converges, because $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$ but $\frac{1}{\sqrt{n^3-1}} > \frac{1}{n^{3/2}}$. We use a different approach. Let $a_n = \frac{1}{\sqrt{n^3-1}}$ and let $b_n = \frac{1}{n^{3/2}}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^3}}} = 1$, and $\sum b_n = \sum \frac{1}{n^{3/2}}$ converges (its a p-series with $p = \frac{3}{2}$), and so $\sum a_n$ converges too, by the Limit Comparison Test.

6.38 Theorem: *(Ratio Test) Let $a_n > 0$ for all $n \geq k$. Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$. Then*
(1) if $r < 1$ then $\sum a_n$ converges, and
(2) if $r > 1$ then $\lim_{n\to\infty} a_n = \infty$ so $\sum a_n = \infty$.

Proof: Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. Choose s with $r < s < 1$, and then choose m so that when $n \geq m$ we have $\frac{a_{n+1}}{a_n} < s$ and hence $a_{n+1} < s\,a_n$. Fix $k \geq m$. Then $a_{k+1} < sa_k$, $a_{k+2} < sa_{k+1} < s^2 a_k$, $a_{k+3} < sa_{k+2} < s^3 a_k$, and so on, so we have $a_n < b_n = s^{n-k} a_k$ for all $n \geq k$. Since $\sum b_n$ is geometric with ratio $s < 1$, it converges, and hence so does $\sum a_n$ by the Comparison Test.

Now suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$. Choose s with $1 < s < r$, then choose m so that when $n \geq m$ we have $\frac{a_{n+1}}{a_n} > s$ and hence $a_{n+1} > sa_n$. Fix $k \geq m$. Then as above $a_n > b_n = s^{n-k} a_k$ for all $n \geq k$, and $\lim_{n\to\infty} b_n = \infty$, so $\lim_{n\to\infty} a_n = \infty$ too.

6.39 Example: Determine whether $\sum \frac{5^n}{n!}$ converges.

Solution: Let $a_n = \frac{5^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \to 0$ as $n \to \infty$, and so $\sum a_n$ converges by the Ratio Test.

6.40 Note: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then $\sum a_n$ could converge or diverge. For example, if $a_n = \frac{1}{n}$ then $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$ as $n \to \infty$ and $\sum a_n$ diverges, but if $b_n = \frac{1}{n^2}$ then $\frac{b_{n+1}}{b_n} = \frac{n^2}{(n+1)^2} \to 1$ as $n \to \infty$ and $\sum b_n$ converges.

6.41 Theorem: *(Root Test) Let $a_n \geq 0$ for all $n \geq k$. Let $r = \limsup_{n\to\infty} \sqrt[n]{a_n}$. Then*
(1) if $r < 1$ then $\sum a_n$ converges, and
(2) if $r > 1$ then $\lim_{n\to\infty} a_n = \infty$ so $\sum a_n = \infty$.

Proof: The proof is left as an exercise. It is similar to the proof of the Ratio Test.

6.42 Example: Determine whether $\sum \left(\frac{n}{n+1}\right)^{n^2}$ converges.

Solution: Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then $\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = e^{n\ln\left(\frac{n}{n+1}\right)}$, and by l'Hôpital's Rule we have $\lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-x^2}{(x+1)^2} = -1$, and so $\lim_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1$. Thus $\sum a_n$ converges by the Root Test.

6.43 Definition: A sequence $(a_n)_{n\geq k}$ is said to be **alternating** when either we have $a_n = (-1)^n|a_n|$ for all $n \geq k$ or we have $a_n = (-1)^{n+1}|a_n|$ for all $n \geq k$.

6.44 Theorem: *(Aternating Series Test) Let $(a_n)_{n\geq k}$ be an alternating series. If the sequence $(|a_n|)$ is decreasing with $\lim\limits_{n\to\infty}|a_n| = 0$ then $\sum\limits_{n\geq k} a_n$ converges, and in this case*

$$\Big|\sum_{n=k}^{\infty} a_n\Big| \leq |a_k|\,.$$

Proof: To simplify notation, we give the proof in the case that $k = 0$ and $a_n = (-1)^n|a_n|$. Suppose the sequence $\big(|a_n|\big)$ is decreasing with $|a_n| \to 0$. Let $S_\ell = \sum\limits_{n=0}^{\ell} a_n$. We consider the sequences $(S_{2\ell})$ and $(S_{2\ell-1})$ of even and odd partial sums. Note that since $\big(|a_n|\big)$ is decreasing, we have

$$S_{2\ell} - S_{2(\ell-1)} = |a_{2\ell}| - |a_{2\ell-1}| \leq 0$$

so $(S_{2\ell})$ is decreasing, and we have

$$\begin{aligned} S_{2\ell} &= |a_0| - |a_1| + |a_2| - |a_3| + \cdots + |a_{2\ell-2}| - |a_{2\ell-1}| + |a_{2\ell}| \\ &= \big(|a_0| - |a_1|\big) + \big(|a_2| - |a_3|\big) + \cdots + \big(|a_{2\ell-2}| - |a_{2\ell-1}|\big) + |a_{2\ell}| \geq 0 \end{aligned}$$

and so $(S_{2\ell})$ is bounded below by 0. Thus $(S_{2\ell})$ converges by the Monotone Convergence Theorem. Similarly, $(S_{2\ell-1})$ is increasing and bounded above by $|a_0|$, so it also converges, and we have $\lim\limits_{\ell\to\infty} S_{2\ell-1} \leq |a_0|$.

Finally we note that since $|a_n| \to 0$, taking the limit on both sides of the equality $|a_{2\ell}| = S_{2\ell} - S_{2\ell-1}$ gives $0 = \lim\limits_{\ell\to\infty} S_{2\ell} - \lim\limits_{\ell\to\infty} S_{2\ell-1}$, and so we have $\lim\limits_{\ell\to\infty} S_{2\ell} = \lim\limits_{\ell\to\infty} S_{2\ell-1}$. It follows that (S_ℓ) converges with $\lim\limits_{\ell\to\infty} S_\ell = \lim\limits_{\ell\to\infty} S_{2\ell} = \lim\limits_{\ell\to\infty} S_{2\ell-1} \leq |a_0|$.

6.45 Example: Determine whether $\sum\limits_{n\geq 2} \frac{(-1)^n \ln n}{\sqrt{n}}$ converges.

Solution: Let $a_n = \frac{(-1)^n \ln n}{\sqrt{n}}$. Let $f(x) = \frac{\ln x}{\sqrt{x}}$ so that $|a_n| = f(n)$. Note that

$$f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{3/2}}\,,$$

so we have $f'(x) < 0$ for $x > e^2$. Thus $f(x)$ is decreasing for $x > e^2$, and so $\big(|a_n|\big)$ is decreasing for $n \geq 8$. Also, by l'Hôpital's Rule, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

and so $|a_n| \to 0$ as $n \to \infty$. Thus $\sum a_n$ converges by the Alternating Series Test.

6.46 Example: Approximate the sum $S=\sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}$ so that the error is at most $\frac{1}{2000}$.

Solution: Let $a_n=\frac{(-2)^n}{(2n)!}$. Note that

$$\frac{|a_{n+1}|}{|a_n|}=\frac{2^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{2^n}=\frac{2}{(2n+2)(2n+1)}=\frac{1}{(n+1)(2n+1)}\,.$$

Since $\frac{|a_{n+1}|}{|a_n|}\leq 1$ for all $n\geq 0$, we know that $\big(|a_n|\big)$ is decreasing. Since $\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=0$, we know that $\sum|a_n|$ converges by the Ratio Test, and so $|a_n|\to 0$ by the Divergence Test. This shows that we can apply the Alternating Series Test.

If we approximate S by the ℓ^{th} partial sum $S_\ell=\sum_{n=0}^{\ell}a_n$, then by the Alternating Series Test, the error is

$$E=\big|S-S_\ell\big|=\Big|\sum_{n=\ell+1}^{\infty}a_n\Big|\leq\big|a_{\ell+1}\big|=\frac{2^{\ell+1}}{(2\ell+2)!}\,.$$

To get $E\leq\frac{1}{2000}$ we can choose ℓ so that $\frac{2^{\ell+1}}{(2\ell+2)!}\leq\frac{1}{2000}$. By trial and error we find that we can take $\ell=3$. Thus we make the approximation

$$S\cong S_3=1-\frac{2}{2!}+\frac{2^2}{4!}-\frac{2^3}{6!}=1-1+\frac{1}{6}-\frac{1}{90}=\frac{7}{45}\,.$$

$\big($We shall see later that the exact value of this sum is $\cos\sqrt{2}\big)$.

6.47 Definition: A series $\sum\limits_{n\geq k}a_n$ is said to **converge absolutely** when $\sum\limits_{n\geq k}|a_n|$ converges. The series is said to **converge conditionally** if $\sum\limits_{n\geq k}a_n$ converges but $\sum\limits_{n\geq k}|a_n|$ diverges.

6.48 Example: For $0<p\leq 1$, the p-series $\sum\frac{1}{n^p}$ diverges, but since $\left(\frac{1}{n^p}\right)$ is decreasing towards 0, $\sum\frac{(-1)^n}{n^p}$ converges by the Alternating Series Test. Thus for $0<p\leq 1$, the alternating p-series $\sum\frac{(-1)^n}{n^p}$ converges conditionally.

6.49 Theorem: *(Absolute Convergence Implies Convergence) If $\sum|a_n|$ converges then so does $\sum a_n$.*

Proof: Suppose that $\sum|a_n|$ converges. Note that $-|a_n|\leq a_n\leq|a_n|$ so that

$$0\leq a_n+|a_n|\leq 2|a_n|\text{ for all }n\,.$$

Since $\sum|a_n|$ converges, $\sum 2|a_n|$ converges by linearity, and so $\sum\big(a_n+|a_n|\big)$ converges by the Comparison Test. Since $\sum|a_n|$ and $\sum\big(a_n+|a_n|\big)$ both converge, $\sum a_n$ converges by linearity.

6.50 Example: Determine whether $\sum\frac{\sin n}{n^2}$ converges.

Solution: Let $a_n=\frac{\sin n}{n^2}$. Then $|a_n|=\frac{|\sin n|}{n^2}\leq\frac{1}{n^2}$. Since $\sum\frac{1}{n^2}$ converges (its a p-series with $p=2$), $\sum|a_n|$ converges by the Comparison Test, and hence $\sum a_n$ converges too, since absolute convergence implies convergence.

6.51 Theorem: *(Multiplication of Series) Suppose that* $\sum\limits_{n\geq 0} |a_n|$ *converges and* $\sum\limits_{n\geq 0} b_n$ *converges and define* $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$. *Then* $\sum\limits_{n\geq 0} c_n$ *converges and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof: Let $A_\ell = \sum\limits_{n=0}^{\ell} a_n$, $B_\ell = \sum\limits_{n=0}^{\ell} b_n$, $C_\ell = \sum\limits_{n=0}^{\ell} c_n$, $A = \sum\limits_{n=0}^{\infty} a_n$, $B = \sum\limits_{n=0}^{\infty} b_n$, $K = \sum\limits_{n=0}^{\infty} |a_n|$ and $E_\ell = B - B_\ell$. Then we have

$$\begin{aligned} C_\ell &= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots + (a_0b_\ell + \cdots + a_\ell b_0) \\ &= a_0B_\ell + a_1B_{\ell-1} + a_2B_{\ell-2} + \cdots + a_\ell B_0 \\ &= a_0(B - E_\ell) + a_1(B - E_{\ell-1}) + \cdots + a_\ell(B - E_0) \\ &= A_\ell B - \big(a_0E_\ell + a_1E_{\ell-1} + \cdots + a_\ell E_0\big) \end{aligned}$$

and so

$$\big|C_\ell - AB\big| \leq \big|(A_\ell - A)B\big| + \big|a_0E_\ell + a_1E_{\ell-1} + \cdots + a_\ell E_0\big|.$$

Let $\epsilon > 0$. Choose m so that $j > m \Longrightarrow E_j < \frac{\epsilon}{3K}$. Let $E = \max\big\{|E_0|, \cdots, |E_m|\big\}$. Choose $L > m$ so that when $\ell > L$ we have $\sum\limits_{n=\ell-m}^{l} |a_n| < \frac{\epsilon}{3E}$ and we have $|A_\ell - A||B| < \frac{\epsilon}{3}$. Then for $\ell > L$,

$$\begin{aligned} \big|C_\ell - AB\big| &< \big|(A_\ell - A)B\big| + \big|a_0E_\ell + \cdots + a_{\ell-m-1}E_{m+1}\big| + \big|a_{\ell-m}E_m + \cdots + a_\ell E_0\big| \\ &\leq \tfrac{\epsilon}{3} + \Big(\sum_{n=0}^{\ell-m-1} |a_n|\Big)\tfrac{\epsilon}{3K} + \Big(\sum_{n=\ell-m}^{\ell} |a_n|\Big)E \\ &< \tfrac{\epsilon}{3} + K\tfrac{\epsilon}{3K} + \tfrac{\epsilon}{3E}E = \epsilon\,. \end{aligned}$$

6.52 Example: Find an example of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that $\sum\limits_{n\geq 0} a_n$ and $\sum\limits_{n\geq 0} b_n$ both converge, but $\sum\limits_{n\geq 0} c_n$ diverges where $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$.

Solution: Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$ for $n \geq 0$, and let

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}\,.$$

Recall that for $p, q \geq 0$ we have $\sqrt{pq} \leq \frac{1}{2}(p+q)$ (indeed $(p+q)^2 - 4pq = p^2 - 2pq + q^2 = (p-q)^2 \geq 0$, so $(p+q)^2 \geq 4pq$). In particular $\sqrt{(k+1)(n-k+1)} \leq \frac{1}{2}(n+2)$ and so $|c_n| \geq \sum\limits_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$. Thus $\lim\limits_{n\to\infty} |c_n| \neq 0$ so $\sum c_n$ diverges by the Divergence Test.

6.53 Theorem: *(Fubini's Theorem for Series) Let $a_{n,m} \in \mathbb{R}$ for all integers $n, m \geq 0$. Suppose that $\sum\limits_{m\geq 0} |a_{n,m}|$ converges for each $n \geq 0$ and that $\sum\limits_{n\geq 0}\left(\sum\limits_{m=0}^{\infty} |a_{n,m}|\right)$ converges. Then $\sum\limits_{m\geq 0} a_{n,m}$ converges for all $n \geq 0$, $\sum\limits_{n\geq 0}\left(\sum\limits_{m=0}^{\infty} a_{n,m}\right)$ converges, $\sum\limits_{n\geq 0} a_{n,m}$ converges for all $m \geq 0$, $\sum\limits_{m\geq 0}\left(\sum\limits_{n=0}^{\infty} a_{n,m}\right)$ converges, and*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right).$$

Proof: First we claim that $\sum\limits_{n\geq 0} |a_{n,m}|$ converges for all $m \geq 0$, $\sum\limits_{m\geq 0}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right)$ converges, and $\sum\limits_{n=0}^{\infty}\left(\sum\limits_{m=0}^{\infty} |a_{n,m}|\right) = \sum\limits_{m=0}^{\infty}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right)$. For all n, m we have $|a_{n,m}| \leq \sum\limits_{k=0}^{\infty} |a_{n,k}|$, and $\sum\limits_{n\geq 0}\left(\sum\limits_{k=0}^{\infty} |a_{n,k}|\right)$ converges, so we know that $\sum\limits_{n\geq 0} |a_{n,m}|$ converges for all $m \geq 0$, by the Comparison Test. Let $k \geq 0$ and let $\epsilon > 0$. Since each sum $\sum\limits_{n\geq 0} |a_{n,m}|$ converges, we can choose L so that when $l > L$ we have $\sum\limits_{n=l+1}^{\infty} |a_{n,m}| < \frac{\epsilon}{k+1}$ for all $m = 0, 1, \cdots, k$. Then for $l > L$ we have

$$\begin{aligned}
\sum_{m=0}^{k}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) &= \sum_{m=0}^{k}\left(\sum_{n=0}^{l} |a_{n,m}| + \sum_{n=l+1}^{\infty} |a_{n,m}|\right) < \sum_{m=0}^{k}\left(\sum_{n=0}^{l} |a_{n,m}| + \tfrac{\epsilon}{k+1}\right) \\
&= \sum_{m=0}^{k}\left(\sum_{n=0}^{l} |a_{n,m}|\right) + \epsilon = \sum_{n=0}^{l}\left(\sum_{m=0}^{k} |a_{m,n}|\right) + \epsilon \\
&\leq \sum_{n=0}^{l}\left(\sum_{m=0}^{\infty} |a_{m,n}|\right) + \epsilon \leq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} |a_{m,n}|\right) + \epsilon
\end{aligned}$$

Since ϵ was arbitrary, we have $\sum\limits_{m=0}^{k}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right) \leq \sum\limits_{n=0}^{\infty}\left(\sum\limits_{m=0}^{\infty} |a_{m,n}|\right)$. Since the sequence of partial sums $\sum\limits_{m=0}^{k}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right)$ is increasing and bounded above by $\sum\limits_{n=0}^{\infty}\left(\sum\limits_{m=0}^{\infty} |a_{m,n}|\right)$, it converges and we have $\sum\limits_{m=0}^{\infty}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right) \leq \sum\limits_{m=0}^{\infty}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right)$. By symmetry, we obtain the opposite inequality, and the claim is proved

For all $n \geq 0$, since $\sum\limits_{m\geq 0} |a_{n,m}|$ converges we know that $\sum\limits_{m\geq 0} a_{n,m}$ converges and that $\left|\sum\limits_{m=0}^{\infty} a_{n,m}\right| \leq \sum\limits_{m=0}^{\infty} |a_{n,m}|$ by the Absolute Convergence Theorem. Since $\sum\limits_{n\geq 0}\left(\sum\limits_{m=0}^{\infty} |a_{n,m}|\right)$ converges, $\sum\limits_{n\geq 0}\left|\sum\limits_{m=0}^{\infty} a_{n,m}\right|$ converges by the Comparison Test, and so $\sum\limits_{n\geq 0}\left(\sum\limits_{m=0}^{\infty} a_{n,m}\right)$ also converges by the Absolute Convergence Theorem. Similarly, $\sum_{n\geq 0} a_{n,m}$ converges for all $m \geq 0$ and $\sum_{m\geq 0}\left(\sum_{n=0}^{\infty} a_{n,m}\right)$ converges.

Let $\epsilon > 0$. Since $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right)$ and $\sum_{m\geq 0}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)$ both converge, we can choose k and l so that $\sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right) < \frac{\epsilon}{4}$ and $\sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) < \frac{\epsilon}{4}$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{k} a_{n,m} + \sum_{m=k+1}^{\infty} a_{n,m}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right) \\
&= \sum_{n=0}^{l}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right)
\end{aligned}$$

and so we have

$$\begin{aligned}
\left|\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) - \sum_{n=0}^{l}\left(\sum_{m=0}^{k} a_{n,m}\right)\right| &\leq \left|\sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right)\right| + \left|\sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right)\right| \\
&\leq \sum_{n=l+1}^{\infty}\left|\sum_{m=0}^{k} a_{n,m}\right| + \sum_{n=0}^{\infty}\left|\sum_{m=k+1}^{\infty} a_{n,m}\right| \\
&\leq \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} |a_{n,m}|\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} |a_{n,m}|\right) \\
&= \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} |a_{n,m}|\right) + \sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) \\
&\leq \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right) + \sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}\,.
\end{aligned}$$

Similarly we have $\left|\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right) - \sum_{m=0}^{l}\left(\sum_{n=0}^{k} a_{n,m}\right)\right| < \frac{\epsilon}{2}$, and so

$$\left|\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) - \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right)\right| < \epsilon\,.$$

Since ϵ was arbitrary, we have $\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right)$ as required.