Chapter 2. Methods of Integration

Basic Integrals

2.1 Note: We have the following list of **Basic Integrals**

$$\int x^p\,dx=\frac{x^{p+1}}{p+1}+c\ ,\ \text{for } p\neq -1 \qquad \int \sec^2 x\ dx=\tan x+c$$
$$\int \frac{dx}{x}=\ln|x|+c \qquad \int \sec x\tan x\ dx=\sec x+c$$
$$\int e^x\,dx=e^x+c \qquad \int \tan x\ dx=\ln|\sec x|+c$$
$$\int a^x\,dx=\frac{a^x}{\ln a}+c \qquad \int \sec x\ dx=\ln|\sec x+\tan x|+c$$
$$\int \ln x\ dx=x\ln x-x+c \qquad \int \frac{dx}{1+x^2}=\tan^{-1}x+c$$
$$\int \sin x\ dx=-\cos x+c \qquad \int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$$
$$\int \cos x\ dx=\sin x+c \qquad \int \frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+c$$

Proof: Each of these equalities is easy to verify by taking the derivative of the right side. For example, we have $\int \ln x\ dx=x\ln x-x+c$ since $\frac{d}{dx}(x\ln x-x)=1\cdot\ln x+x\cdot\frac{1}{x}-1=\ln x$, and we have $\int \tan x\ dx=\ln|\sec x|+c$ since $\frac{d}{dx}(\ln|\sec x|)=\frac{\sec x\tan x}{\sec x}=\tan x$, and we have $\int \sec x\ dx=\ln|\sec x+\tan x|+c$ since $\frac{d}{dx}(\ln|\sec x+\tan x|)=\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}=\sec x$.

2.2 Example: Find $\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx$.

Solution: By the Fundamental Theorem of Calculus and Linearity, we have

$$\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx=\int_1^4 x^{3/2}-5x^{-1/2}\,dx=\Big[\tfrac{2}{5}x^{5/2}-10x^{1/2}\Big]_1^4=\big(\tfrac{64}{5}-20\big)-\big(\tfrac{2}{5}-10\big)=\tfrac{12}{5}\,.$$

2.3 Example: Find $\int_{\pi/6}^{\sqrt{\pi}/3} \sin 2x+\cos 3x\ dx$.

Solution: We find antiderivatives for $\sin 2x$ and $\cos 3x$. Since $\frac{d}{dx}\cos 2x=-2\sin 2x$ we have $\frac{d}{dx}\big(-\frac{1}{2}\cos 2x\big)=\sin 2x$ and since $\frac{d}{dx}\sin 3x=3\cos 3x$ we have $\frac{d}{dx}\big(\frac{1}{3}\sin 3x\big)\cos 3x$, and so

$$\int_{\pi/6}^{\pi/3}\sin 2x+\cos 3x\ dx=\Big[-\tfrac{1}{2}\cos 2x+\tfrac{1}{3}\sin 3x\Big]_{\pi/6}^{\pi/3}=\big(\tfrac{1}{4}+0\big)-\big(-\tfrac{1}{4}+\tfrac{1}{3}\big)=\tfrac{1}{6}\,.$$

Substitution

2.4 Theorem: *(Substitution, or Change of Variables) Let $u = g(x)$ be differentiable on an interval and let $f(u)$ be continuous on the range of $g(x)$. Then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

and if g' is integrable then

$$\int_{x=a}^{b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{g(b)} f(u)\,du\,.$$

Proof: Let $F(u)$ be an antiderivative of $f(u)$ so $F'(u) = f(u)$ and $\int f(u)\,du = F(u) + c$. Then from the Chain Rule, we have $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$, and so

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c = F(u) + c = \int f(u)\,du$$

and

$$\begin{aligned}\int_{x=a}^{b} f(g(x))g'(x)\,dx &= \Big[F(g(x))\Big]_{x=a}^{b} = F(g(b)) - F(g(a)) \\ &= \Big[F(u)\Big]_{u=g(a)}^{g(b)} = \int_{u=g(a)}^{g(b)} f(u)\,du\,.\end{aligned}$$

2.5 Notation: For $u = g(x)$ we write $du = g'(x)\,dx$. More generally, for $f(u) = g(x)$ we write $f'(u)\,du = g'(x)\,dx$. This notation makes the above theorem easy to remember and to apply.

2.6 Example: Find $\int \sqrt{2x+3}\,dx$.

Solution: Make the substitution $u = 2x + 3$ so $du = 2dx$. Then

$$\int \sqrt{2x+3}\,dx = \int \tfrac{1}{2}u^{1/2}\,du = \tfrac{1}{3}\,u^{3/2} + c = \tfrac{1}{3}(2x+3)^{3/2} + c\,.$$

(In applying the Substitution Rule, we used $u = g(x) = 2x + 3$ and $f(u) = \sqrt{u} = u^{1/2}$, but the notation $du = g'(x)\,dx$ allows us to avoid explicit mention of the function $f(u)$ in our solution).

2.7 Example: Find $\int x\,e^{x^2}\,dx$.

Solution: Make the substitution $u = x^2$ so $du = 2x\,dx$. Then

$$\int x\,e^{x^2}\,dx = \int \tfrac{1}{2}e^u\,du = \tfrac{1}{2}e^u + c = \tfrac{1}{2}e^{x^2} + c\,.$$

2.8 Example: Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

Solution: Let $u = \ln x$ so $du = \dfrac{1}{x}\,dx$. Then

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \tfrac{1}{2}u^2 + c = \tfrac{1}{2}(\ln x)^2 + c\,.$$

2.9 Example: Find $\displaystyle\int \tan x\;dx$.

Solution: We have $\tan x = \dfrac{\sin x}{\cos x}$. Let $u = \cos x$ so $du = -\sin x\;dx$. Then

$$\int \tan x\;dx = \int \frac{\sin x\;dx}{\cos x} = \int \frac{-du}{u} = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c\,.$$

2.10 Example: Find $\displaystyle\int \frac{dx}{x+\sqrt{x}}$.

Solution: Let $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u} = \int \frac{2\,du}{u+1}\,.$$

Now let $v = u+1$ do $dv = du$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2\,du}{u+1} = \int \frac{2}{v}\,dv = 2\ln|v| + c = 2\ln|u+1| + c = 2\ln(\sqrt{x}+1) + c\,.$$

2.11 Example: Find $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{2x^2+1}}$.

Solution: Let $u = 2x^2+1$ so $du = 4x\,dx$. Then

$$\int_{x=0}^{2} \frac{x\,dx}{\sqrt{2x^2+1}} = \int_{u=1}^{9} \frac{\frac{1}{4}\,du}{\sqrt{u}} = \int_1^9 \tfrac{1}{4}\,u^{-1/2}\,du = \Big[\tfrac{1}{2}\,u^{1/2}\Big]_1^9 = \tfrac{3}{2} - \tfrac{1}{2} = 1\,.$$

2.12 Example: Find $\displaystyle\int_0^1 \frac{dx}{1+3x^2}$.

Solution: Let $u = \sqrt{3}\,x$ so $du = \sqrt{3}\,dx$. Then

$$\int_0^1 \frac{dx}{1+3x^2} = \int_0^{\sqrt{3}} \frac{\frac{1}{\sqrt{3}}\,du}{1+u^2} = \Big[\tfrac{1}{\sqrt{3}}\tan^{-1}u\Big]_0^{\sqrt{3}} = \tfrac{1}{\sqrt{3}}\,\tfrac{\pi}{3} = \tfrac{\pi}{3\sqrt{3}}\,.$$

Integration by Parts

2.13 Theorem: *(Integration by Parts) Let $f(x)$ and $g(x)$ be differentiable in an interval. Then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

and if f' and g' are integrable then

$$\int_{x=a}^{b} f(x)g'(x)\,dx = \left[f(x)g(x) - \int g(x)f'(x)\,dx\right]_{x=a}^{b}.$$

Proof: By the Product Rule, we have $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ and so

$$\int f'(x)g(x) + f(x)g'(x)\ dx = f(x)g(x) + c\,,$$

which can be rewritten as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx\,.$$

(We do not need to include the arbitrary constant c since there is now an integral on both sides of the equation).

2.14 Notation: If we write $u = f(x)$, $du = f'(x)\,dx$, $v = g(x)$ and $dv = g'(x)\,dx$, then the top formula in the above theorem becomes

$$\int u\,dv = uv - \int v\,du\,.$$

2.15 Note: To find the integral of a polynomial multiplied by an exponential function or a trigonometric function, try Integrating by parts with u equal to the polynomial (you may need to integrate by parts repeatedly if the polynomial is of high degree).

To integrate a polynomial (or an algebraic) function times a logarithmic or inverse trigonometric function, try integrating by parts letting u be the logarithmic or inverse trigonometric function.

To integrate an exponential function times a sine or cosine function, try integrating by parts twice, letting u be the exponential function both times.

2.16 Example: Find $\int x\sin x\ dx$.

Solution: Integrate by parts using $u = x$, $du = dx$, $v = -\cos x$ and $dv = \sin x\ dx$ to get

$$\int x\sin x\ dx = \int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x\ dx = -x\cos x + \sin x + c\,.$$

2.17 Example: Find $\displaystyle\int (x^2+1)e^{2x}\,dx$.

Solution: Integrate by parts using $u = x^2+1$, $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$ and $dv = e^{2x}\,dx$ to get

$$\int (x^2+1)e^{2x}\,dx = \int u\,dv = uv - \int v\,du = \tfrac{1}{2}(x^2+1)e^{2x} - \int x\,e^{2x}\,dx\,.$$

To find $\displaystyle\int x\,e^{2x}\,dx$ we integrate by parts again, this time using $u_2 = x$, $du_2 = dx$, $v_2 = \frac{1}{2}e^{2x}$ and $dv_2 = e^{2x}\,dx$ to get

$$\begin{aligned}\int (x^2+1)e^{2x}\,dx &= \tfrac{1}{2}(x^2+1)e^{2x} - \int x\,e^{2x}\,dx \\ &= \tfrac{1}{2}(x^2+1)e^{2x} - \left(\tfrac{1}{2}\,xe^{2x} - \int \tfrac{1}{2}e^{2x}\,dx\right) \\ &= \tfrac{1}{2}(x^2+1)e^{2x} - \left(\tfrac{1}{2}\,xe^{2x} - \tfrac{1}{4}e^{2x}\right) + c \\ &= \tfrac{1}{4}\left(2x^2-2x+3\right)e^{2x} + c\end{aligned}$$

2.18 Example: Find $\displaystyle\int \ln x\;dx$.

Solution: Integrate by parts using $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $v = x$ and $dv = dx$ to get

$$\int \ln x\;dx = x\ln x - \int 1\,dx = x\ln x - x + c\,.$$

2.19 Example: Find $\displaystyle\int_1^4 \sqrt{x}\;\ln x\;dx$.

Solution: Integrate by parts using $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $v = \frac{2}{3}x^{3/2}$ and $dv = x^{1/2}\,dx$ to get

$$\begin{aligned}\int_1^4 \sqrt{x}\;\ln x\;dx &= \left[\tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx\right]_1^4 = \left[\tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2}\right]_1^4 \\ &= \left(\tfrac{16}{3}\ln 4 - \tfrac{32}{9}\right) - \left(\tfrac{2}{3}\ln 1 - \tfrac{4}{9}\right) = \tfrac{16}{3}\ln 4 - \tfrac{28}{9}\,.\end{aligned}$$

2.20 Example: Find $\displaystyle\int e^x\sin x\;dx$

Solution: Write $I = \displaystyle\int e^x\sin x\,dx$. Integrate by parts twice, first using $u_1 = e^x$, $du = e^x\,dx$, $v = -\cos x$ and $dv = \sin x\;dx$, and next using $u_2 = e^x$, $du_2 = e^x\,dx$, $v_2 = \sin x$ and $dv_2 = \cos x\;dx$ to get

$$\begin{aligned}I &= -e^x\cos x + \int e^x\cos x\;dx \\ &= -e^x\cos x + \left(e^x\sin x - \int e^x\sin x\;dx\right)\,. \\ &= -e^x\cos x + e^x\sin x - I\end{aligned}$$

Thus $2I = -e^x\cos x + e^x\sin x + c$ and so $I = \frac{1}{2}(\sin x - \cos x)e^x + d$.

2.21 Example: Let $n \geq 2$ be an integer. Find a formula for $\displaystyle\int \sin^n x\ dx$ in terms of $\displaystyle\int \sin^{n-2} x\ dx$, and hence find $\displaystyle\int \sin^2 x\ dx$ and $\displaystyle\int \sin^4 x\ dx$.

Solution: Let $I = \displaystyle\int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$. Integrate by parts using $u = \sin^{n-1} x$, $du = (n-1)\sin^{n-2} x \cos x\, dx$, $v = -\cos x$ and $dv = \sin x\, dx$ to get

$$\begin{aligned} I &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\ dx \\ &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x(1-\sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)I\,. \end{aligned}$$

Add $(n-1)I$ to both sides to get $nI = -\sin^{n-1} x \cos x + (n-1)\displaystyle\int \sin^{n-2} x\ dx$, that is

$$\int \sin^n x\ dx = -\tfrac{1}{n}\sin^{n-1} x \cos x + \tfrac{n-1}{n}\int \sin^{n-2} x\ dx\,.$$

In particular, when $n = 2$ we get

$$\int \sin^2 x\ dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int 1\, dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + c$$

and when $n = 4$ we get

$$\int \sin^4 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}\int \sin^2 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x + c\,.$$

2.22 Example: Let $n \geq 2$ be an integer. Find a formula for $\displaystyle\int \sec^n x\ dx$ in terms of $\displaystyle\int \sec^{n-2} x\ dx$, and hence find $\displaystyle\int \sec^3 x\ dx$.

Solution: Let $I = \displaystyle\int \sec^n x\ dx = \int \sec^{n-2} x \sec^2 x\ dx$. Using Integrate by Parts with $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x \tan x\, dx$, $v = \tan x$ and $dv = \sec^2 x\, dx$, we obtain

$$\begin{aligned} I &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\ dx \\ &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\ dx \\ &= \sec^{n-2} x \tan x - (n-2)I + (n-2)\int \sec^{n-2} x\ dx \end{aligned}$$

Add $(n-2)I$ to both sides to get $(n-1)I = \sec^{n-2} x \tan x + (n-2)\displaystyle\int \sec^{n-2} x\ dx$, that is

$$\int \sec^n x\ dx = \tfrac{1}{n-1}\sec^{n-2} x \tan x + \tfrac{n-2}{n-1}\int \sec^{n-2} x\ dx\,.$$

In particular, when $n = 3$ we get

$$\int \sec^3 x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \sec x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln\left|\sec x + \tan x\right| + c$$

Trigonometric Integrals

2.23 Note: To find $\int f(\sin x)\cos^{2n+1} x\ dx$, write $\cos^{2n+1} x = (1-\sin^2 x)^n \cos x$ then try the substitution $u = \sin x$, $du = \cos x\,dx$.

To find $\int f(\cos x)\sin^{2n+1} x\ dx$, write $\sin^{2n+1} x = (1-\cos^2 x)^n \sin x$ then try the substitution $u = \cos x$, $du = -\sin x\ dx$.

To find $\int \sin^{2m} x \cos^{2n} x\, dx$, try using the trigonometric identities $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$. Alternatively, write $\cos^{2n} x = (1-\sin^2 x)^n$ and use the formula from Example 2.21.

To find $\int f(\tan x)\sec^{2n+2} x\ dx$, write $\sec^{2n+2} x = (1+\tan^2 x)^n \sec^2 x\, dx$ and try the substitution $u = \tan x$, $du = \sec^2 x\, dx$.

To find $\int f(\sec x)\tan^{2n+1} x\ dx$, write $\tan^{2n+1} x = \dfrac{(\sec^2 x - 1)^n}{\sec x} \sec x \tan x\ dx$ and try the substitution $u = \sec x$, $du = \sec x \tan x\ dx$.

To find $\int \sec^{2n+1} x \tan^{2n} x\ dx$, write $\tan^{2n} x = (\sec^2 x - 1)^n$ and use the formula from Example 2.22.

2.24 Example: Find $\displaystyle\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\, dx$.

Solution: Make the substitution $u = \cos x$ so $du = -\sin x\, dx$. Then

$$\begin{aligned}\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\, dx &= \int_0^{\pi/3} \frac{(1-\cos^2 x)\sin x\, dx}{\cos^2 x} = \int_1^{1/2} -\frac{(1-u^2)\, du}{u^2} = \int_1^{1/2} -\frac{1}{u^2} + 1\ du \\ &= \left[\tfrac{1}{u} + u\right]_1^{1/2} = \left(2 + \tfrac{1}{2}\right) - (1+1) = \tfrac{1}{2}\,.\end{aligned}$$

2.25 Example: Find $\int \sin^6 x\ dx$.

Solution: We could use the method of example 2.21, but we choose instead to use the half-angle formulas. We have

$$\begin{aligned}\int_0^{\pi/4} \sin^6 x\ dx &= \int_0^{\pi/4} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)^3 dx = \int_0^{\pi/4} \tfrac{1}{8} - \tfrac{3}{8}\cos 2x + \tfrac{3}{8}\cos^2 2x - \tfrac{1}{8}\cos^3 2x\ dx \\ &= \int_0^{\pi/4} \tfrac{1}{8} - \tfrac{3}{8}\cos 2x + \tfrac{3}{8}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) - \tfrac{1}{8}\left(1 - \sin^2 2x\right)\cos 2x\ dx \\ &= \int_0^{\pi/4} \tfrac{5}{16} - \tfrac{1}{2}\cos 2x + \tfrac{3}{16}\cos 4x + \tfrac{1}{8}\sin^2 2x \cos 2x\ dx \\ &= \left[\tfrac{5}{15}x - \tfrac{1}{4}\sin 2x + \tfrac{3}{64}\sin 4x + \tfrac{1}{48}\sin^3 2x\right]_0^{\pi/4} \\ &= \tfrac{5\pi}{64} - \tfrac{1}{4} + \tfrac{1}{48} = \tfrac{5\pi}{64} - \tfrac{11}{48}\,.\end{aligned}$$

2.26 Example: Find $\displaystyle\int_0^{\pi/4} \tan^4 x\ dx$.

Solution: Note first that

$$\tan^4 x = \tan^2 x(\sec^2 x - 1) = \tan^2 x \sec^2 x - \tan^2 x = \tan^2 x \sec^2 x - \sec^2 x + 1\,.$$

To find $\displaystyle\int \tan^2 x \sec^2 x\ dx$, make the substitution $u = \tan\theta$, $du = \sec^2\theta\, d\theta$ to get

$$\int \tan^2 x \sec^2 x\ dx = \int u^2\, du = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}\tan^3 x + c\,.$$

Thus we have

$$\begin{aligned}\int_0^{\pi/4} \tan^4 x\, dx &= \int_0^{\pi/4} \tan^2 x \sec^2 x - \sec^2 x + 1 \\ &= \Big[\tfrac{1}{3}\ \tan^3 x - \tan x + x\Big]_0^{\pi/4} = \tfrac{1}{3} - 1 + \tfrac{\pi}{4} = \tfrac{\pi}{4} - \tfrac{2}{3}\,.\end{aligned}$$

2.27 Example: Find $\displaystyle\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x + 1}}\, dx$.

Solution: Make the substitution $u = \tan x$ so $du = \sec^2 x\, dx$. Then

$$\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x + 1}}\, dx = \int_0^{\pi/4} \frac{(\tan^2 x + 1)\sec^2 x\, dx}{\sqrt{\tan x + 1}} = \int_0^1 \frac{(u^2+1)\, du}{\sqrt{u+1}}$$

Now make the substitution $v = u + 1$ so $u = v - 1$ and $du = dv$. Then

$$\begin{aligned}\int_0^1 \frac{u^2+1}{\sqrt{u+1}}\, du &= \int_1^2 \frac{(v-1)^2+1}{\sqrt{v}}\, dv = \int_1^2 v^{3/2} - 2\, v^{1/2} + 2\, v^{-1/2}\ dv \\ &= \Big[\tfrac{2}{5}\, v^{5/2} - \tfrac{4}{3}\, v^{3/2} + 4\, v^{1/2}\Big]_1^2 = \big(\tfrac{2\cdot 4\sqrt{2}}{5} - \tfrac{4\cdot 2\sqrt{2}}{3} + 4\sqrt{2}\big) - \big(\tfrac{2}{5} - \tfrac{4}{3} + 4\big) \\ &= \tfrac{(24-40+60)\sqrt{2}}{15)} - \tfrac{6-20+60}{15} = \tfrac{44\sqrt{2}-46}{15}\,.\end{aligned}$$

2.28 Note: To find $\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$, or $\int \sin(ax)\cos(bx)\,dx$, use one of the identities

$$\begin{aligned}\cos(A-B)-\cos(A+B)&=2\sin A\sin B\\ \cos(A-B)+\cos(A+B)&=2\cos A\cos B\\ \sin(A-B)+\sin(A+B)&=2\sin A\cos B\,.\end{aligned}$$

2.29 Example: Find $\displaystyle\int_0^{\pi/6}\cos 3x\cos 2x\,dx$.

Solution: Since $2\cos 3x\cos 2x=\cos(3x-2x)+\cos(3x+2x)=\cos x+\cos 5x$, we have

$$\int_0^{\pi/6}\cos 2x\cos 3x\,dx=\int_0^{\pi/6}\tfrac{1}{2}(\cos x+\cos 5x)\,dx=\Big[\tfrac{1}{2}\sin x+\tfrac{1}{10}\sin 5x\Big]_0^{\pi/6}=\tfrac{1}{4}+\tfrac{1}{20}=\tfrac{3}{10}\,.$$

2.30 Note: The **Weirstrass substitution** $u=\tan\frac{x}{2}$, $x=2\tan^{-1}u$, $dx=\dfrac{2du}{1+u^2}$ converts $\sin x$ and $\cos x$ into rational functions of u: indeed we have $\sin\frac{x}{2}=\frac{u}{\sqrt{1-u^2}}$ and $\cos\frac{x}{2}=\frac{1}{\sqrt{1-u^2}}$ so that $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=\frac{2u}{1+u^2}$ and $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{1-u^2}{1+u^2}$.

2.31 Example: Find $\displaystyle\int\frac{dx}{1-\cos x}$.

Solution: We use the Weirstrass substitution $u=\tan\frac{x}{2}$, $dx=\frac{2}{1+u^2}\,du$, and $\cos x=\frac{1-u^2}{1+u^2}$ to get

$$\int\frac{dx}{1-\cos x}=\int\frac{\frac{2}{1+u^2}\,du}{1-\frac{1-u^2}{1+u^2}}=\int\frac{2\,du}{(1+u^2)-(1-u^2)}=\int\frac{du}{u^2}=-\frac{1}{u}+c=-\cot\frac{x}{2}+c\,.$$

Inverse Trigonometric Substitution

2.32 Note: To solve an integral involving $\sqrt{a^2+b^2(x+c)^2}$ or $1/(a^2+b^2(x+c)^2)$, try the substitution $\theta=\tan^{-1}\frac{b(x+c)}{a}$ so that $a\tan\theta=b(x+c)$, $a\sec\theta=\sqrt{a^2+b^2(x+c)^2}$ and $a\sec^2\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{a^2-b^2(x+c)^2}$, try the substitution $\theta=\sin^{-1}\frac{b(x+c)}{a}$ so that $a\sin\theta=b(x+c)$, $a\cos\theta=\sqrt{a^2-b^2(x+c)^2}$ and $a\cos\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{b^2(x+c)^2-a^2}$, try the substitution $\theta=\sec^{-1}\frac{b(x+c)}{a}$ so that $a\sec\theta=b(x+c)$, $a\tan\theta=\sqrt{b^2(x+c)^2-a^2}$ and $a\sec\theta\tan\theta\,d\theta=b\,dx$.

2.33 Example: Find $\displaystyle\int_0^1\frac{dx}{(4-3x^2)^{3/2}}$.

Solution: Let $2\sin\theta=\sqrt{3}\,x$ so $2\cos\theta=\sqrt{4-3x^2}$ and $2\cos\theta\,d\theta=\sqrt{3}\,dx$. Then

$$\int_0^1\frac{dx}{(4-3x^2)^{3/2}}=\int_0^{\pi/3}\frac{\frac{2}{\sqrt{3}}\cos\theta\,d\theta}{(2\cos\theta)^3}=\int_0^{\pi/3}\tfrac{1}{4\sqrt{3}}\sec^2\theta\,d\theta=\Big[\tfrac{1}{4\sqrt{3}}\tan\theta\Big]_0^{\pi/3}=\tfrac{1}{4}\,.$$

2.34 Example: Find $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}$.

Solution: Let $\sqrt{3}\tan\theta=x$ so $\sqrt{3}\sec\theta=\sqrt{x^2+3}$ and $\sqrt{3}\sec^2\theta\,d\theta=dx$, and also let $u=\sin\theta$ so $du=\cos\theta\,d\theta$. Then

$$\begin{aligned}\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}&=\int_{\pi/6}^{\pi/4}\frac{\sqrt{3}\sec^2\theta\,d\theta}{3\tan^2\theta\sqrt{3}\sec\theta}=\int_{\pi/6}^{\pi/4}\frac{1}{3}\,\frac{\sec\theta}{\tan^2\theta}\,d\theta=\int_{\pi/6}^{\pi/4}\frac{1}{3}\,\frac{\cos\theta\,d\theta}{\sin^2\theta}\\&=\int_{1/2}^{1/\sqrt{2}}\frac{1}{3\,u^2}\,du=\Big[-\frac{1}{3u}\Big]_{1/2}^{1/\sqrt{2}}=-\tfrac{\sqrt{2}}{3}+\tfrac{2}{3}=\tfrac{2-\sqrt{2}}{3}\,.\end{aligned}$$

2.35 Example: Find $\displaystyle\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx$.

Solution: Let $2\sec\theta=x$ so $2\tan\theta=\sqrt{x^2-4}$ and $2\sec\theta\tan\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx&=\int_0^{\pi/3}\frac{\tan^2\theta\sec\theta\,d\theta}{\sec^2\theta}=\int_0^{\pi/3}\frac{\tan^2\theta}{\sec\theta}\,d\theta=\int_0^{\pi/3}\frac{\sec^2\theta-1}{\sec\theta}\,d\theta\\&=\int_0^{\pi/3}\sec\theta-\cos\theta\;d\theta=\Big[\ln|\sec\theta+\tan\theta|-\sin\theta\Big]_0^{\pi/3}=\ln(2+\sqrt{3})-\tfrac{\sqrt{3}}{2}\,.\end{aligned}$$

2.36 Example: Find $\displaystyle\int_2^3(4x-x^2)^{3/2}\,dx$.

Solution: Let $2\sin\theta=x-2$ so $2\cos\theta=\sqrt{4x-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^3(4x-x^2)^{3/2}\,dx&=\int_0^{\pi/6}16\cos^4\theta\,d\theta=\int_0^{\pi/6}4\,(1+\cos2\theta)^2\,d\theta\\&=\int 4+8\cos2\theta+4\cos^22\theta\;d\theta=\int 4+8\cos2\theta+2+2\cos4\theta\;d\theta\\&=\Big[6\theta+4\sin2\theta+\tfrac{1}{2}\sin4\theta\Big]_0^{\pi/6}=\pi+2\sqrt{3}+\tfrac{\sqrt{3}}{4}=\pi+\tfrac{9\sqrt{3}}{4}\,.\end{aligned}$$

Partial Fractions

2.37 Note: We can find the integral of a rational function $\dfrac{f(x)}{g(x)}$ as follows:

Step 1: use long division to find polynomials $q(x)$ and $r(x)$ with $\deg r(x) < \deg g(x)$ such that $f(x) = g(x)q(x) + r(x)$ for all x, and note that $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r(x)}{g(x)}$ so

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x) + \frac{r(x)}{g(x)}\,dx\,.$$

(If $\deg f(x) < \deg g(x)$ then $q(x) = 0$ and $r(x) = f(x)$).

Step 2: factor $g(x)$ into linear and irreducible quadratic factors.

Step 3: write $\dfrac{r(x)}{g(x)}$ as a sum of terms so that for each linear factor $(ax+b)^k$ we have the k terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

and for each irreducible quadratic factor $(ax^2+bx+c)^k$ we have the k terms

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}\,.$$

Writing $\dfrac{r(x)}{g(x)}$ in this form is called splitting $\dfrac{r(x)}{g(x)}$ into its **partial fractions** decomposition.

Step 4: solve the integral.

2.38 Example: If $g(x) = x(x-1)^3(x^2+2x+3)^2$ then in step 3 we would write

$$\frac{r(x)}{g(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+2x+3} + \frac{Gx+H}{(x^2+2x+3)^2}\,.$$

and then solve for the various constants.

2.39 Example: Find $\displaystyle\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$.

Solution: In order to get $\dfrac{x-7}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ we need

$$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = x-7\,.$$

Equating coefficients gives $A+C=0$, $A+B-2C=1$ and $-2A+2B+C=-7$. Solving these three equations gives $A=1$, $B=-2$ and $C=-1$, and so we have

$$\begin{aligned}\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx &= \int_2^3 \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}\\ &= \int_2^3 \frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2}\,dx = \Big[\ln(x-1) + \tfrac{2}{x-1} - \ln(x+2)\Big]_2^3\\ &= (\ln 2 + 1 - \ln 5) - (2 - \ln 4) = \ln\frac{8}{5} - 1\,.\end{aligned}$$

2.40 Example: Find $\displaystyle\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx$.

Solution: Use long division of polynomials to show that $\dfrac{x^4-x^3+1}{x^3+x} = x-1+\dfrac{-x^2+x+1}{x^3+x}$. Next, note that to get $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1} = \dfrac{-x^2+x+1}{x^3+x}$ we need $A(x^2+1)+(Bx+C)(x) = -x^2+x+1$. Equating coefficients gives $A+B=-1$, $C=1$ and $A=1$. Solving these three equations gives $A=1$, $B=-2$ and $C=1$. Thus

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx &= \int_1^{\sqrt{3}} x-1+\frac{1}{x}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}\,dx \\
&= \Big[\tfrac{1}{2}\,x^2 - x + \ln x - \ln(x^2+1) + \tan^{-1} x\Big]_1^{\sqrt{3}} \\
&= \left(\tfrac{3}{2} - \sqrt{3} + \ln\sqrt{3} - \ln 4 + \tfrac{\pi}{3}\right) - \left(\tfrac{1}{2} - 1 - \ln 2 + \tfrac{\pi}{4}\right) \\
&= 2 - \sqrt{3} + \ln\tfrac{\sqrt{3}}{2} + \tfrac{\pi}{12}\,.
\end{aligned}$$

2.41 Example: Find $I = \displaystyle\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$.

Solution: To get

$$\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-2x+5}+\frac{Ex+F}{(x^2-2x+5)^2} = \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}$$

we need $Ax(x^2-2x+5)^2+B(x^2-2x+5)^2+(Cx+D)(x^2)(x^2-2x+5)+(Ex+F)(x^2) = x^5+x^4-2x^3-2x^2-5x-25$. Expanding the left hand side then equating coefficients gives the 5 equations

$$\begin{aligned}
&A+C=1\ ,\ -4A+B-2C+D=1\ ,\ 14A-4B+5C-2D+E=-2 \\
&-20A+14B+5D+F=-2\ ,\ 25A-20B=-5\ ,\ 25B=-25
\end{aligned}$$

Solving these equations gives $A=-1$, $B=-1$, $C=2$, $D=2$, $E=2$ and $F=-18$, so

$$\begin{aligned}
I &= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x+2}{x^2-2x+5}+\frac{2x-18}{(x^2-2x+5)^2}\,dx \\
&= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x-2+4}{x^2-2x+5}+\frac{2x-2-16}{(x^2-2x+5)^2}\,dx \\
&= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x-2}{x^2-2x+5}+\frac{4}{x^2-2x+5}+\frac{2x-2}{(x^2-2x+5)^2}-\frac{16}{(x^2-2x+5)^2}\,dx
\end{aligned}$$

We have $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$ and $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}+c$. Make the substitution $u = x^2-2x+5$, $du = (2x-2)\,dx$ to get

$$\int \frac{(2x-2)\,dx}{x^2-2x+5} = \int \frac{du}{u} = \ln u + c = \ln(x^2-2x+5)+c$$

and

$$\int \frac{(2x-2)\,dx}{(x^2-2x+5)^2} = \int \frac{du}{u^2} = \frac{-1}{u}+c = \frac{-1}{x^2-2x+5}+c.$$

Make the substitution $2\tan\theta = x-1$, $2\sec\theta = \sqrt{x^2-2x+5}$, $2\sec^2\theta\,d\theta = dx$ to get

$$\int \frac{4\,dx}{x^2-2x+5} = \int \frac{4\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^2} = \int 2\,d\theta = 2\theta + c = 2\tan^{-1}\left(\tfrac{x-1}{2}\right)+c$$

and

$$\int \frac{16\,dx}{(x^2-2x+5)^2} = \int \frac{16\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^4}\,d\theta = \int \frac{2\,d\theta}{\sec^2\theta} = \int 2\cos^2\theta\,d\theta = \int 1+\cos 2\theta\;d\theta$$

$$= \theta + \tfrac{1}{2}\,\sin 2\theta + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\tfrac{x-1}{2}\right) + \tfrac{2(x-1)}{x^2-2x+5} + c\,.$$

Thus we have

$$\begin{aligned} I &= \Bigg[-\ln x + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\frac{x-1}{2} \\ &\qquad - \frac{1}{x^2-2x+5} - \tan^{-1}\frac{x-1}{2} - \frac{2(x-1)}{x^2-2x+5}\Bigg]_1^2 \\ &= \left[\ln\frac{x^2-2x+5}{x} + \frac{1}{x} - \frac{2x-1}{x^2-2x+5} + \tan^{-1}\frac{x-1}{2}\right]_1^2 \\ &= \left(\ln\tfrac{5}{2} + \tfrac{1}{2} - \tfrac{3}{5} + \tan^{-1}\tfrac{1}{2}\right) - \left(\ln 4 + 1 - \tfrac{1}{4}\right) \\ &= \ln\tfrac{5}{8} - \tfrac{17}{20} + \tan^{-1}\tfrac{1}{2}\,. \end{aligned}$$

2.42 Example: Find $\displaystyle\int \frac{\sec^3 x\,dx}{\sec x - 1}$.

Solution: Multiply the numerator and denominator by $\sec x + 1$ to get

$$\int \frac{\sec^3 x\,dx}{\sec x - 1} = \int \frac{\sec^3 x(\sec x + 1)}{(\sec^2 x - 1)}\,dx = \int \frac{\sec^4 x + \sec^3 x}{\tan^2 x}\,dx = \int \frac{\sec^4 x}{\tan^2 x}\,dx + \int \frac{\sec^3 x}{\tan^2 x}\,dx\,.$$

Make the substitution $u = \tan x$, $du = \sec^2 x\,dx$ to get

$$\begin{aligned} \int \frac{\sec^4 x}{\tan^2 x}\,dx &= \int \frac{(\tan^2 x + 1)\sec^2 x\,dx}{\tan^2 x} = \int \frac{u^2+1}{u^2}\,du \\ &= \int 1 + \frac{1}{u^2}\,du = u - \frac{1}{u} + c = \tan x - \cot x + c\,. \end{aligned}$$

Make the substitution $v = \sin x$, $dv = \cos x\,dx$ and integrate by parts to get

$$\begin{aligned} \int \frac{\sec^3 x}{\tan^2 x}\,dx &= \int \frac{dx}{\cos x\sin^2 x} = \int \frac{\cos x\,dx}{(1-\sin^2 x)\sin^2 x} = \int \frac{dv}{(1-v^2)\,v^2} \\ &= \int \frac{1}{1-v^2} + \frac{1}{v^2}\;dv = \int \frac{\frac{1}{2}}{1-v} + \frac{\frac{1}{2}}{1+v} + \frac{1}{v^2}\;dv \\ &= -\tfrac{1}{2}\ln|1-v| + \tfrac{1}{2}\ln|1+v| - \tfrac{1}{v} + c = \tfrac{1}{2}\ln\left|\tfrac{1+v}{1-v}\right| - \tfrac{1}{v} + c \\ &= \tfrac{1}{2}\ln\tfrac{1+\sin x}{1-\sin x} - \csc x + c = \tfrac{1}{2}\ln\tfrac{(1+\sin x)^2}{(\cos x)^2} - \csc x + c = \ln\left|\tfrac{1+\sin x}{\cos x}\right| - \csc x + c\,. \end{aligned}$$

Thus $\displaystyle\int \frac{\sec^3 x}{\sec x - 1}\,dx = \tan x - \cot x + \ln|\sec x + \tan x| - \csc x + c.$

Approximate Integration

2.43 Definition: Let f be integrable on $[a,b]$. We can approximate the integral of f on $[a,b]$ by any Riemann sum

$$I = \int_a^b f(x)\,dx \cong \sum_{k=1}^n f(c_k)\Delta_k x$$

where $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta_k x = x_k - x_{k-1}$ and $c_k \in [x_{k-1}, x_k]$. The n^{th} **Left Endpoint Approximation** L_n, the n^{th} **Right Endpoint Approximation** R_n, and the n^{th} **Midpoint Approximation** M_n, for the integral $I = \displaystyle\int_a^b f(x)\,dx$ are the Riemann sums for f obtained by using the partition of $[a,b]$ into n equal sized subintervals and by choosing c_k to be the left endpoint, the right endpoint, or the midpoint of the k^{th} subinterval $[x_{k-1}, x_k]$. We have

$$\begin{aligned}
L_n &= \sum_{k=1}^n f(x_{k-1})\Delta x = \frac{b-a}{n}\Big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big) \\
R_n &= \sum_{k=1}^n f(x_k)\Delta x = \frac{b-a}{n}\Big(f(x_1) + f(x_2) + \cdots + f(x_n)\Big) \\
M_n &= \sum_{k=1}^n f\Big(\frac{x_{k-1}+x_k}{2}\Big)\Delta x = \frac{b-a}{n}\Big(f\left(\tfrac{x_0+x_1}{2}\right) + f\left(\tfrac{x_1+x_2}{2}\right) + \cdots + f\left(\tfrac{x_{n-1}+x_n}{2}\right)\Big)
\end{aligned}$$

where $x_k = a + \frac{b-a}{n}\,k$ and $\Delta x = \frac{b-a}{n}$.

2.44 Definition: Let f be integrable on $[a,b]$. The **Trapezoidal Approximation** T_n for the integral $I = \displaystyle\int_a^b f(x)\,dx$ is defined as follows. We use the partition of $[a,b]$ into n equal-sized subintervals, so we let $x_k = a + \frac{b-a}{n}\,k$ and $\Delta x = \frac{b-a}{n}$. Let g_k be the linear polynomial with $g_k(x_{k-1}) = f(x_{k-1})$ and $g_k(x_k) = f(x_k)$. Let g be the piecewise-linear function defined by $g(x) = g_k(x)$ for $x \in [x_{k-1}, x_k]$. We define

$$T_n = \int_a^b g(x)\,dx\,.$$

Note that

$$\int_{x_{k-1}}^{x_k} g(x)\,dx = \int_{x_{k-1}}^{x_k} g_k(x)\,dx = \frac{f(x_{k-1}) + f(x_k)}{2}\,\Delta x$$

(indeed, the integral measures the area of a trapezoid) so we have

$$\begin{aligned}
T_n &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} g(x)\,dx = \sum_{k=1}^n \frac{f(x_{k-1}) + f(x_k)}{2}\,\Delta x = \frac{L_n + R_n}{2} \\
&= \frac{b-a}{2n}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big)\,.
\end{aligned}$$

2.45 Definition: Let f be integrable on $[a,b]$. For an even positive integer n, we define the **Simpson Approximation** S_n for the integral $I=\int_a^b f(x)\,dx$ as follows. We partition $[a,b]$ into n equal-sized subintervals. Let $x_k=a+\frac{b-a}{n}\,k$ and $\Delta x=\frac{b-a}{n}$. For $k=1,2,\cdots,\frac{n}{2}$, let g_k be the quadratic polynomial with $g(x_{2k-2})=f(x_{2k-2})$, $g(x_{2k-1})=f(x_{2k-1})$ and $g(x_{2k})=f(x_{2k})$. Let g be the piecewise-quadratic function given by $g(x)=g_k(x)$ for $x\in[x_{2k-2},x_{2k}]$. We define

$$S_n=\int_a^b g(x)\,dx\,.$$

Note that if $h(x)=Ax^2+Bx+C$ is the quadratic polynomial with $h(-1)=u$, $h(0)=v$ and $h(1)=w$, then we must have $u=h(-1)=A-B+C$, $v=h(0)=C$ and $w=h(1)=A+B+C$. Solving these three equations gives $A=\frac{u-2v+w}{2}$, $B=\frac{w-u}{2}$ and Cv so we have

$$\begin{aligned}\int_{-1}^1 h(x)\,dx&=\int_{-1}^1 \tfrac{u-2v+w}{2}\,x^2+\tfrac{w-u}{2}\,x+v\;dx\\&=\Big[\tfrac{u-2v+w}{6}\,x^3+\tfrac{w-u}{4}\,x^2+v\,x\Big]_{-1}^1\\&=\tfrac{u-2v+w}{3}+2v=\tfrac{u+4v+w}{3}\,.\end{aligned}$$

It follows, by shifting and scaling, that

$$\int_{x_{2k-2}}^{x_{2k}} g_k(x)\,dx=\frac{f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k})}{3}\,\Delta x\,.$$

Thus

$$\begin{aligned}S_n&=\sum_{k=1}^{n/2}\int_{x_{2k-2}}^{x_{2k}} g(x)\,dx=\sum_{k=1}^{n/2}\frac{f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k})}{3}\,\Delta x\\&=\frac{b-a}{3n}\Big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big)\,.\end{aligned}$$

2.46 Theorem: *(Error Bounds for Approximate Integration) Suppose that the higher order derivatives of f exist and are continuous on $[a,b]$. Let $I=\int_a^b f(x)\,dx$. and let L_n, R_n, T_n, M_n and S_n be the left endpoint, right endpoint, midpoint, trapezoidal and Simpson approximation of I. Then*

$$\big|L_n-I\big|\le\frac{(b-a)^2}{2\,n}\max_{a\le x\le b}\big|f'(x)\big|$$

$$\big|R_n-I\big|\le\frac{(b-a)^2}{2\,n}\max_{a\le x\le b}\big|f'(x)\big|$$

$$\big|T_n-I\big|\le\frac{(b-a)^3}{12\,n^2}\max_{a\le x\le b}\big|f''(x)\big|$$

$$\big|M_n-I\big|\le\frac{(b-a)^3}{24\,n^2}\max_{a\le x\le b}\big|f''(x)\big|$$

$$\big|S_n-I\big|\le\frac{(b-a)^5}{180\,n^4}\max_{a\le x\le b}\big|f''''(x)\big|$$

Proof: We may assign some proofs as exercises and we may provide some proofs later.

2.47 Example: Let $f(x)=\sin^2 x$. Find the exact value $I=\int_0^{4\pi/3} f(x)\,dx$, find the approximations L_8, R_8, M_8, T_8 and S_8, and find a bound on the error for each of these approximations.

Solution: The exact value of the integral is

$$I=\int_0^{4\pi/3}\sin^2 x\;dx=\int_0^{4\pi/3}\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\;dx=\Big[\tfrac{1}{2}\,x-\tfrac{1}{4}\sin 2x\Big]_0^{4\pi/3}=\tfrac{4\pi}{3}-\tfrac{\sqrt{3}}{8}\,.$$

When we divide the interval $\big[0,4\pi/3\big]$ into 8 equal subintervals, the size each of the subintervals is $\Delta x=\frac{\pi}{6}$ and the endpoints of the subintervals are $0,\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\frac{5\pi}{6},\pi,\frac{7\pi}{6},\frac{4\pi}{3}$. Thus the approximations are

$$\begin{aligned}
L_8&=\tfrac{\pi}{6}\Big(f(0)+f\big(\tfrac{\pi}{6}\big)+f\big(\tfrac{\pi}{3}\big)+f\big(\tfrac{\pi}{2}\big)+f\big(\tfrac{2\pi}{3}\big)+f\big(\tfrac{5\pi}{6}\big)+f(\pi)+f\big(\tfrac{7\pi}{6}\big)\Big)\\
&=\tfrac{\pi}{6}\big(0+\tfrac{1}{4}+\tfrac{3}{4}+1+\tfrac{3}{4}+\tfrac{1}{4}+0+\tfrac{1}{4}\big)=\tfrac{13\pi}{24}\,,\\
R_8&=\tfrac{\pi}{6}\Big(f\big(\tfrac{\pi}{6}\big)+f\big(\tfrac{\pi}{3}\big)+f\big(\tfrac{\pi}{2}\big)+f\big(\tfrac{2\pi}{3}\big)+f\big(\tfrac{5\pi}{6}\big)+f(\pi)+f\big(\tfrac{7\pi}{6}\big)+f\big(\tfrac{4\pi}{3}\big)\Big)\\
&=\tfrac{\pi}{6}\big(\tfrac{1}{4}+\tfrac{3}{4}+1+\tfrac{3}{4}+\tfrac{1}{4}+0+\tfrac{1}{4}+\tfrac{3}{4}\big)=\tfrac{2\pi}{3}\,,\\
T_8&=\tfrac{1}{2}\big(L_8+R_8\big)=\tfrac{29\pi}{48}\,,\\
M_8&=\tfrac{\pi}{6}\Big(f\big(\tfrac{\pi}{12}\big)+f\big(\tfrac{\pi}{4}\big)+f\big(\tfrac{5\pi}{12}\big)+f\big(\tfrac{7\pi}{12}\big)+f\big(\tfrac{3\pi}{4}\big)+f\big(\tfrac{11\pi}{12}\big)+f\big(\tfrac{13\pi}{12}\big)+f\big(\tfrac{15\pi}{12}\big)\Big)\\
&=\tfrac{\pi}{6}\big(\tfrac{2-\sqrt{3}}{4}+\tfrac{1}{2}+\tfrac{2+\sqrt{3}}{4}+\tfrac{2+\sqrt{3}}{4}+\tfrac{1}{2}+\tfrac{2-\sqrt{3}}{4}+\tfrac{2-\sqrt{3}}{4}+\tfrac{1}{2}\Big)=\tfrac{\pi}{6}\Big(4-\tfrac{\sqrt{3}}{4}\Big)\,,\\
S_8&=\tfrac{\pi}{18}\Big(f(0)+4f\big(\tfrac{\pi}{6}\big)+2f\big(\tfrac{\pi}{3}\big)+4f\big(\tfrac{\pi}{2}\big)+2f\big(\tfrac{2\pi}{3}\big)+4f\big(\tfrac{5\pi}{6}\big)+2f(\pi)+4f\big(\tfrac{7\pi}{6}\big)+f\big(\tfrac{4\pi}{3}\big)\Big)\\
&=\tfrac{\pi}{18}\big(0+1+\tfrac{3}{2}+4+\tfrac{3}{2}+1+0+1+\tfrac{3}{4}\big)=\tfrac{43\pi}{72}\,.
\end{aligned}$$

Note that to find the values of f needed for the midpoint approximation M_8, we used the identity $f(x)=\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$. From this same identity, we obtain $f'(x)=\sin 2x$ and then $f''(x)=2\cos 2x$, $f'''(x)=-4\sin 2x$ and $f''''(x)=-8\cos 2x$. Thus we find that

$$\max_{0\le x\le 4\pi/3}\big|f'(x)\big|=1\;,\quad\max_{0\le x\le 4\pi/3}\big|f''(x)\big|=2\;\text{ and }\;\max_{0\le x\le 4\pi/3}\big|f''''(x)\big|=8\,.$$

The above theorem gives the following error bounds.

$$\begin{aligned}
\big|L_8-I\big|&\le\frac{(4\pi/3)^2}{16}\cdot 1=\tfrac{\pi^2}{9}\\
\big|R_r-I\big|&\le\frac{(4\pi/3)^2}{16}\cdot 1=\tfrac{\pi^2}{9}\\
\big|T_n-I\big|&\le\frac{(4\pi/3)^3}{12\cdot 6^2}\cdot 2=\tfrac{8\pi^3}{3^6}\\
\big|M_n-I\big|&\le\frac{(4\pi/3)^3}{24\cdot 6^2}\cdot 2=\tfrac{4\pi^3}{3^6}\\
\big|S_n-I\big|&\le\frac{(4\pi/3)^5}{180\cdot 6^4}\cdot 8=\tfrac{2^7\pi^5}{5\cdot 3^{11}}
\end{aligned}$$

Improper Integration

2.48 Definition: Suppose that $f:[a,b)\to\mathbb{R}$ is integrable on every closed interval contained in $[a,b)$. Then we define the **improper integral** of f on $[a,b)$ to be

$$\int_a^b f=\lim_{t\to b^-}\int_a^t f$$

provided the limit exists and, when the improper integral exists and is finite, we say that f is **improperly integrable** on $[a,b)$, (or that the improper integral of f on $[a,b)$ **converges**). In this definition we also allow the case that $b=\infty$, and then we have

$$\int_a^\infty f=\lim_{t\to\infty}\int_a^t f\,.$$

Similarly, if $f:(a,b]\to\mathbb{R}$ is integrable on every closed interval in $(a,b]$ then we define the **improper integral** of f on $(a,b]$ to be

$$\int_a^b f=\lim_{t\to a^+}\int_t^b f$$

provided the limit exists, and we say that f is **improperly integrable** on $(a,b]$ when the improper integral is finite. In this definition we also allow the case that $a=-\infty$. For a function $f:(a,b)\to\mathbb{R}$, which is integrable on every closed interval in (a,b), we choose a point $c\in(a,b)$, then we define the **improper integral** of f on (a,b) to be

$$\int_a^b f=\int_a^c f+\int_c^b f$$

provided that both of the improper integrals on the right exist and can be added, and we say that f is **improperly integrable** on (a,b) when both of the improper integrals on the right are finite. As an exercise, you should verify that the value of this integral does not depend on the choice of c.

2.49 Notation: For a function $F:(a,b)\to\mathbb{R}$ write

$$\Big[F(x)\Big]_{a^+}^{b^-}=\lim_{x\to b^-}F(x)-\lim_{x\to a^+}F(x)\,.$$

We use similar notation when $F:[a,b)\to\mathbb{R}$ and when $F:(a,b]\to\mathbb{R}$.

2.50 Note: Suppose that $f:(a,b)\to\mathbb{R}$ is integrable on every closed interval contained in (a,b) and that F is differentiable with $F'=f$ on (a,b). Then

$$\int_a^b f=\Big[F(x)\Big]_{a^+}^{b^-}.$$

A similar result holds for functions defined on half-open intervals $[a,b)$ and $(a,b]$.

Proof: Choose $c\in(a,b)$. By the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_a^b f=\int_a^c f+\int_c^b f&=\lim_{s\to a^+}\int_s^c f+\lim_{t\to b^-}\int_c^t f\\&=\lim_{s\to a^+}\big(F(c)-F(s)\big)+\lim_{t\to b^-}\big(F(t)-F(c)\big)\\&=\lim_{t\to b^-}F(t)-\lim_{s\to a^+}F(s)=\Big[F(x)\Big]_{a^+}^{b^-}.\end{aligned}$$

2.51 Example: Find $\displaystyle\int_0^1 \frac{dx}{x}$ and find $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

Solution: We have
$$\int_0^1 \frac{dx}{x} = \Big[\ln x\Big]_{0^+}^1 = 0 - (-\infty) = \infty$$
and
$$\int_0^1 \frac{dx}{\sqrt{x}} = \Big[2\sqrt{x}\Big]_{0^+}^1 = 2 - 0 = 2\,.$$

2.52 Example: Show that $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges if and only if $p<1$.

Solution: The case that $p=1$ was dealt with in the previous example. If $p>1$ so that $p-1>0$ then we have
$$\int_0^1 \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_{0^+}^1 = \left(-\tfrac{1}{p-1}\right) - \big(-\infty\big) = \infty$$
and if $p<1$ so that $1-p>0$ then we have
$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_{0^+}^1 = \left(\tfrac{1}{1-p}\right) - (0) = \tfrac{1}{1-p}\,.$$

2.53 Example: Show that $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p>1$.

Solution: When $p=1$ we have
$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{1}{x} = \Big[\ln x\Big]_1^\infty = \infty - 0 = \infty\,.$$
When $p>1$ so that $p-1>0$ we have
$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^\infty = (0) - \left(-\tfrac{1}{p-1}\right) = \tfrac{1}{p-1}$$
and if $p<1$ so that $1-p>0$ then we have
$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = (\infty) - \left(\tfrac{1}{1-p}\right) = \infty\,.$$

2.54 Example: Find $\displaystyle\int_0^\infty e^{-x}\,dx$.

Solution: We have
$$\int_0^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_0^\infty = 0-(-1) = 1\,.$$

2.55 Example: Find $\displaystyle\int_0^1 \ln x\,dx$.

Solution: We have
$$\int_0^1 \ln x\;dx = \Big[x\ln x - x\Big]_{0^+}^1 = (-1)-(0) = -1\,,$$
since l'Hôpital's Rule gives $\displaystyle\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$

2.56 Theorem: *(Comparison) Let f and g be integrable on closed subintervals of (a,b), and suppose that $0\le f(x)\le g(x)$ for all $x\in(a,b)$. If g is improperly integrable on (a,b) then so is f and then we have*
$$\int_a^b f \le \int_a^b g\,.$$
On the other hand, if $\displaystyle\int_a^b f$ diverges then $\displaystyle\int_a^b g$ diverges, too. A similar result holds for functions f and g defined on half-open intervals.

Proof: The proof is left as an exercise.

2.57 Example: Determine whether $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges.

Solution: For $0\le x<\frac{\pi}{2}$ we have $\cos x\ge 1-\frac{2}{\pi}\,x$ so $\sec x\le\frac{1}{1-\frac{2}{\pi}\,x}$ hence $\sqrt{\sec x}\le\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}$. Let $u=1-\frac{2}{\pi}\,x$ so that $du=-\frac{2}{\pi}\,dx$. Then
$$\int_{x=0}^{\pi/2}\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}\,dx = \int_{u=1}^0 -\tfrac{\pi}{2}\,u^{-1/2} = \Big[-\pi\,u^{1/2}\Big]_1^0 = \pi$$
which is finite. It follows that $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges, by comparison.

2.58 Example: Determine whether $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges.

Solution: For $0\le u$ we have $e^u\ge 1+u$, so for $0\le x$ we have $e^{x^2}\ge 1+x^2$, so $e^{-x^2}\le\dfrac{1}{1+x^2}$. Since
$$\int_0^\infty\frac{dx}{1+x^2} = \Big[\tan^{-1}x\Big]_0^\infty = \tfrac{\pi}{2}\,,$$
which is finite, we see that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges, by comparison.

2.59 Theorem: *(Estimation) Let f be integrable on closed subintervals of (a,b). If $|f|$ is improperly integrable on (a,b) then so is f, and then we have*

$$\left|\int_a^b f\right| \leq \int_a^b |f| \, .$$

A similar result holds for functions defined on half-open intervals.

Proof: The proof is left as an exercise.

2.60 Example: Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$ converges.

Solution: We shall show that both of the integrals $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$ and $\displaystyle\int_1^\infty \frac{\sin x}{x}\, dx$ converge.

Since $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, the function f defined by $f(0) = 1$ and $f(x) = \dfrac{\sin x}{x}$ for $x > 0$ is continuous (hence integrable) on $[0,1]$. By part 1 of the Fundamental Theorem of Calculus, the function $\displaystyle\int_r^1 f(x)\, dx$ is a continuous function of r for $r \in [0,1]$ and so we have

$$\int_0^1 \frac{\sin x}{x}\, dx = \lim_{r\to 0^+} \int_r^1 \frac{\sin x}{x}\, dx = \lim_{r\to 0^+} \int_r^1 f(x)\, dx = \int_0^1 f(x)\, dx \, ,$$

which is finite, so $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$ converges.

Integrate by parts using $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\, dx$, $v = -\sin x$ and $dv = \cos x\, dx$ to get

$$\int_1^\infty \frac{\sin x}{x}\, dx = \left[-\frac{\cos x}{x} \right]_1^\infty - \int_1^\infty \frac{\cos x}{x^2}\, dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}\, dx \, .$$

Since $\left|\dfrac{\cos x}{x^2}\right| \leq \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, we see that $\displaystyle\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx$ converges too, by comparison. Thus $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\, dx$ also converges by the Estimation Theorem.