Chapter 5: Differentiation

5.1 Definition: Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f : A \to F$ and let $a \in A$ be a limit point of A. We say that f is **differentiable** at a when the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in F. In this case we call the limit the **derivative** of f at a, and we denote to by $f'(a)$, so we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

When $a \in A$ is a limit point of A from the right, we say that f is **differentiable from the right** at a and that $f'_+(a)$ is the **derivative from the right** of f at a, when

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}.$$

Similarly, when $a \in A$ is a limit point of A from the left, we say that f is **differentiable from the left** at a and that $f'_-(a)$ is the **derivative from the left** of f at a when

$$f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}.$$

5.2 Definition: We say that f is **differentiable** (in A) when f is differentiable at every point $a \in A$. In this case, the **derivative** of f is the function $f' : A \to F$ defined by

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}.$$

When f' is differentiable at a, denote the derivative of f' at a by $f''(a)$, and we call $f''(a)$ the **second derivative** of f at a. When $f''(a)$ exists for every $a \in A$, we say that f is **twice differentiable** (in A), and the function $f'' : A \to F$ is called the **second derivative** of f. Similarly, $f'''(a)$ is the derivative of f'' at a and so on. More generally, for any function $f : A \to F$, we define its **derivative** to be the function $f' : B \to F$ where $B = \big\{a \in A \big| f \text{ is differentiable at } a\big\}$, and we define its **second derivative** to be the function $f'' : C \to F$ where $C = \big\{a \in B \big| f' \text{ is differentiable at } a\big\}$ and so on.

5.3 Remark: Note that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

To be precise, the limit on the left exists in F if and only if the limit on the right exists in F, and in this case the two limits are equal.

5.4 Theorem: *Let F be a subfield of* $\mathbf{R}$, *let $A \subseteq F$, let $f : A \to F$, and let $a \in A$ be a limit point of A. Then f is differentiable at a with derivative $f'(a)$ if and only if*

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in A \Big(|x-a| \le \delta \Longrightarrow \big| f(x) - f(a) - f'(a)(x-a) \big| \le \epsilon \Big)$$

Proof: We have

$$\begin{aligned}
&f \text{ is differentiable at } a \text{ with derivative } f'(a) \iff \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x-a| \le \delta \Longrightarrow \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| \le \epsilon \right) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x-a| \le \delta \Longrightarrow \left| \frac{f(x)-f(a)-f'(a)(x-a)}{x-a} \right| \le \epsilon \right) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \Big(0 < |x-a| \le \delta \Longrightarrow \big| f(x)-f(a)-f'(a)(x-a) \big| \le \epsilon |x-a| \Big) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \Big(|x-a| \le \delta \Longrightarrow \big| f(x)-f(a)-f'(a)(x-a) \big| \le \epsilon |x-a| \Big)
\end{aligned}$$

where on the last line, we can remove the condition that $0 < |x-a|$ because when $x = a$ we have $\big| f(x) - f(a) - f'(a)(x-a) \big| = 0$.

5.5 Definition: When $f : A \to F$ is differentiable at a with dervative $f'(a)$, the function

$$l(x) = f(a) + f'(a)(x-a)$$

is called the **linearization** of f at a. Note that the graph $y = l(x)$ of the linearization is the line through the point $(a, f(a))$ with slope $f'(a)$. This line is called the **tangent line** to the graph $y = f(x)$ at the point $(a, f(a))$.

5.6 Theorem: *(Differentiability Implies Continuity) Let F be a subfield of* $\mathbf{R}$, *let $A \subseteq F$, let $f : A \to F$ and let $a \in A$ be a limit point of A. Suppose that f is differentiable at a. Then f is continuous at a.*

Proof: We have

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a) \longrightarrow f'(a) \cdot 0 = 0 \;\text{ as } x \to a$$

and so

$$f(x) = \big(f(x) - f(a)\big) + f(a) \longrightarrow 0 + f(a) = f(a) \;\text{ as } x \to a.$$

This proves that f is continuous at a.

5.7 Theorem: *(Local Determination of the Derivative) Let F be a subfield of* $\mathbf{R}$, *let $A, B \subseteq F$, let $f : A \to F$ and $g : B \to F$, and let $a \in A \cap B$ be a limit point of both A and B. Suppose that for some $\delta > 0$ we have $\big\{x \in A \big| |x-a| \le \delta\big\} \subset \big\{x \in B \big| |x-a| \le \delta\big\}$. If g is differentiable at a then so is f and we have $f'(a) = g'(a)$.*

Proof: The proof is left as an exercise.

5.8 Theorem: *(Operations on Derivatives) Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f, g : A \to F$, let $a \in A$ be a limit point of A, and let $c \in F$. Suppose that f and g are differentiable at a. Then*

(1) (Linearity) the functions cf, $f+g$ and $f-g$ are differentiable at a with

$$(cf)'(a) = c\,f'(a)\ ,\ (f+g)'(a) = f'(a) + g'(a)\ ,\ (f-g)'(a) - f'(a) - g'(a),$$

(2) (Product Rule) the function fg is differentiable at a with

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a),$$

(3) (Reciprocal Rule) if $g(a) \neq 0$ then the function $1/g$ is differentiable at a with

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2},$$

(4) (Quotient Rule) if $g(a) \neq 0$ then the function f/g is differentiable at a with

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof: We prove Parts (2), (3) and (4). For $x \in A$ with $x \neq a$, we have

$$\begin{aligned}\frac{(fg)(x) - (fg)(a)}{x-a} &= \frac{f(x)g(x) - f(a)g(a)}{x-a} \\ &= \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} \\ &= f(x)\cdot\frac{g(x)-g(a)}{x-a} + g(a)\cdot\frac{f(x)-f(a)}{x-a} \\ &\longrightarrow f(a)\cdot g'(a) + g(a)\cdot f'(a)\ \text{ as } x \to a.\end{aligned}$$

Note that $f(x) \to f(a)$ as $x \to a$ because f is continuous at a since differentiability implies continuity. This proves the Product Rule.

Suppose that $g(a) \neq 0$. Since g is continuous at a (because differentiability implies continuity) we can choose $\delta > 0$ so that $|x-a| \leq \delta \implies |g(x) - g(a)| \leq \frac{|g(a)|}{2}$ and then when $|x-a| \leq \delta$ we have $|g(x)| \geq \frac{|g(a)|}{2}$ so that $g(x) \neq 0$. For $x \in A$ with $|x-a| \leq \delta$ we have

$$\frac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(a)}{x - -a} = \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} = \frac{-1}{g(x)g(a)}\cdot\frac{g(x)-g(a)}{x-a} \longrightarrow \frac{-1}{g(a)^2}\cdot g'(a)$$

as $x \to a$. This Proves the Reciprocal Rule.

Finally, note that Part (4) follows from Parts (2) and (3). Indeed when $g(a) \neq 0$, we have

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f\cdot\frac{1}{g}\right)'(a) = f'(a)\cdot\left(\frac{1}{g}\right)(a) + f(a)\cdot\left(\frac{1}{g}\right)'(a) \\ &= f'(a)\cdot\frac{1}{g(a)} + f(a)\cdot\frac{-g'(a)}{g(a)^2} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.\end{aligned}$$

5.9 Theorem: *(Chain Rule) Let F be a subfield of $\mathbf{R}$, let $A, B \subseteq F$, let $f : A \to F$, let $g : B \to F$ and let $h = g \circ f : C \to F$ where $C = A \cap f^{-1}(B)$. Let $a \in C$ be a limit point of C (hence also of A) and let $b = f(a) \in B$ be a limit point of B. Suppose that f is differentiable at a and g is differentiable at b. Then h is differentiable at a with*

$$h'(a) = g'\big(f(a)\big)\, f'(a).$$

Proof: We shall use the ϵ-δ formulation of the derivative from Theorem 5.3. Note first that for $x \in C$ and $y = f(x) \in B$ we have

$$\begin{aligned}
&\big|h(x) - h(a) - g'(f(a))f'(a)(x-a)\big| \\
&\quad = \big|g(f(x)) - g(f(a)) - g'(f(a))f'(a)(x-a)\big| \\
&\quad = \big|g(y) - g(b) - g'(b)f'(a)(x-a)\big| \\
&\quad = \big|g(y) - g(b) - g'(b)(y-b) + g'(b)(y-b) - g'(b)f'(a)(x-a)\big| \\
&\quad \le \big|g(y) - g(b) - g'(b)(y-b)\big| + |g'(b)|\,|y - b - f'(a)(x-a)| \\
&\quad = \big|g(y) - g(b) - g'(b)(y-b)\big| + |g'(b)|\,\big|f(x) - f(a) - f'(a)(x-a)\big|
\end{aligned}$$

and also

$$\begin{aligned}
|y-b| &= \big|f(x) - f(a)\big| = \big|f(x) - f(a) - f'(a)(x-a) + f'(a)(x-a)\big| \\
&\le \big|f(x) - f(a) - f'(a)(x-a)\big| + |f'(a)|\,|x-a|.
\end{aligned}$$

Let $\epsilon > 0$. Since g is differentiable at b, we can choose $\delta_0 > 0$ so that

$$|y-b| \le \delta_0 \Longrightarrow \big|g(y) - g(b) - g'(b)(y-b)\big| \le \tfrac{\epsilon}{2(1+|f'(a)|)}\,|y-b|.$$

Since f is continuous at a, we can choose δ_1 so that

$$|x-a| \le \delta_1 \Longrightarrow |f(x) - f(a)| \le \delta_0 \Longrightarrow |y-b| \le \delta_0.$$

Since f is differentiable at a we can choose $\delta_2 > 0$ and $\delta_3 > 0$ so that

$$\begin{aligned}
&|x-a| \le \delta_2 \Longrightarrow \big|f(x) - f(a) - f'(a)(x-a)\big| \le |x-a| \text{ and} \\
&|x-a| \le \delta_3 \Longrightarrow \big|f(x) - f(a) - f'(a)(x-a)\big| \le \tfrac{\epsilon}{2(1+g'(b)|)}.
\end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Let $x \in C$ and let $y = f(x) \in B$. Then when $|x-a| \le \delta$ we have

$$\begin{aligned}
&\big|h(x) - h(a) - g'(f(a))f'(a)(x-a)\big| \\
&\quad \le \big|g(y) - g(b) - g'(b)(y-b)\big| + |g'(b)|\,\big|f(x) - f(a) - f'(a)(x-a)\big| \\
&\quad \le \tfrac{\epsilon}{2(1+|f'(a)|)}\,|y-b| + (1 + |g'(b)|) \cdot \tfrac{\epsilon}{2(1+|g'(b)|)}\,|x-a| \\
&\quad \le \tfrac{\epsilon}{2(1+|f'(a)|)}\Big(\big|f(x) - f(a) - f'(a)(x-a)\big| + |f'(a)|\,|x-a|\Big) + \tfrac{\epsilon}{2}\,|x-a| \\
&\quad \le \tfrac{\epsilon}{2(1+|f'(a)|)}\Big(|x-a| + |f'(a)|\,|x-a|\Big) + \tfrac{\epsilon}{2}\,|x-a| \\
&\quad = \tfrac{\epsilon}{2}\,|x-a| + \tfrac{\epsilon}{2}\,|x-a| = \epsilon|x-a|.
\end{aligned}$$

Thus h is differentiable at a with $h'(a) = g'(f(a))f'(a)$, as required.

5.10 Theorem: *Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$ and let $f : A \to F$. Then f is monotonic if and only if f has the property that for all $a, b, c \in A$, if b lies between a and c then $f(b)$ lies between $f(a)$ and $f(c)$.*

Proof: The proof is left as an exercise.

5.11 Theorem: *(The Inverse Function Theorem) Let I be an interval in $\mathbf{R}$, let $f : I \to \mathbf{R}$ and let $J = f(I)$.*

(1) If f is continuous then its range $J = f(I)$ is an interval in $\mathbf{R}$.
(2) If f is injective and continuous then f is strictly monotonic.
(3) If $f : I \to J$ is strictly monotonic, then so is its inverse $g : J \to I$.
(4) If f is bijective and continuous then its inverse g is continuous.

(5) If f is bijective and continuous, and f is differentiable at a with $f'(a) \neq 0$, then its inverse g is differentiable at $b = f(a)$ with $g'(b) = \frac{1}{f'(a)}$.

Proof: Suppose that $f : I \to \mathbf{R}$ is continuous. If f is the empty function or if f is constant, then J is a degenerate interval. Suppose that J contains at least two points. Let $u, v \in J$ and let $y \in \mathbf{R}$ with $u < y < v$. Since $J = f(I)$ we can choose $a, b \in I$ with $f(a) = u$ and $f(b) = v$. Since $f(a) = u \neq v = f(b)$ we have $a \neq b$. Since y lies between $f(a) = u$ and $f(b) = v$, and since f is continuous, it follows from the Intermediate Value Theorem that we can choose x between a and b with $f(x) = y$. Since I is an interval in $\mathbf{R}$, it has the intermediate value property, and so we have $x \in I$. Since $x \in I$ and $y = f(x)$ we have $y \in f(I) = J$. This proves that J has the intermediate value property, and so J is an interval, as required. This proves Part (1).

Suppose that f is injective and continuous. Let $a, b, c \in I$ with $a < b < c$. Since f is injective and $a \neq c$, we have $f(a) \neq f(c)$. We claim that $f(b)$ lies between $f(a)$ and $f(c)$. Consider the case that $f(a) < f(c)$ (the case that $f(a) > f(c)$ is similar). Suppose, for a contradiction, that $f(b) \geq f(c)$. Note that since f is injective and $b \neq c$ we have $f(b) \neq f(c)$ and so $f(b) > f(c)$. Choose y with $f(c) < y < f(b)$. Since f is continuous on $[a, b]$ and on $[b, c]$, by the Intermediate Value Theorem, we can choose $x_1 \in [a, b]$ and $x_2 \in [b, c]$ with $f(x_1) = y = f(x_2)$. Since $y \neq f(b)$ we cannot have $x_1 = b$ or $x_2 \in b$ so we have $x_1 < b < x_2$ with $f(x_1) = f(x_2)$, which contradicts the fact that f is injective. Thus we cannot have $f(b) \geq f(c)$ and so we have $f(b) < f(c)$. A similar argument by contradiction shows that we cannot have $f(b) \leq f(a)$ and so we have $f(a) < f(b) < f(c)$, and so $f(b)$ lies between $f(a)$ and $f(c)$ as claimed. We have proven that for all $a, b, c \in I$ with $a < b < c$, $f(b)$ lies between $f(a)$ and $f(c)$. It follows from the above theorem that f is monotonic (hence strictly monotonic since it is injective). This proves Part (2).

To prove Part (3), suppose that $f : I \to J$ is strictly monotonic and let $g : J \to I$ be the inverse of f. Suppose that f is strictly increasing. Let $u, v \in J = f(I)$ with $u < v$. Let $a = g(u)$ and $b = g(v)$ so we have $u = f(a)$ and $v = f(b)$. Since f is strictly increasing, we must have $a < b$ (since $a = b \implies f(a) = f(b) \implies u = v$ and $a > b \implies f(a) > f(b) \implies u > v$). Thus $g(u) = a < b = g(v)$ and so g is strictly increasing. A similar argument shows that if f is strictly decreasing then so is g.

Part (4) follows from Part (3) by the Monotone Surjective Functions Theorem.

To prove Part (5), suppose that f is bijective and continuous and that f is differentiable at a with $f'(a) \neq 0$. By Part (4), we know that g is continuous at $b = f(a)$, and so as $y \to b$ in J we have $g(y) \to g(b)$ in I, and so for $x = g(y)$ we have

$$\frac{g(y) - g(b)}{y - b} = \frac{x - a}{f(x) - f(a)} = \frac{1}{\frac{f(x) - f(a)}{x - a}} \longrightarrow \frac{1}{f'(a)} \quad \text{as } y \to b.$$

5.12 Theorem: *(Derivatives of the Basic Elementary Functions) The basic elementary functions have the following derivatives.*

(1) $(x^a)' = a\,x^{a-1}$ *where* $a \in \mathbf{R}$ *and* $x \in \mathbf{R}$ *is such that* x^{a-1} *is defined,*

(2) $(a^x)' = \ln a \cdot a^x$ *where* $a > 0$ *and* $x \in \mathbf{R}$ *and*
$(\log_a x)' = \frac{1}{\ln a} \cdot \frac{1}{x}$ *where* $0 < a \neq 1$ *and* $x > 0$, *and in particular*
$(e^x)' = e^x$ *for all* $x \in \mathbf{R}$ *and* $(\ln x)' = \frac{1}{x}$ *for all* $x > 0$,

(3) $(\sin x)' = \cos x$ *and* $(\cos x)' = -\sin x$ *for all* $x \in \mathbf{R}$, *and*
$(\tan x)' = \sec^2 x$ *and* $(\sec x)' = \sec x \, \tan x$ *for all* $x \in \mathbf{R}$ *with* $x \neq \frac{\pi}{2} + k\pi, k \in \mathbf{Z}$,
$(\cot x)' = -\csc^2 x$ *and* $(\csc x)' = -\cot x \, \csc x$ *for all* $x \in \mathbf{R}$ *with* $x \neq \pi + k\pi, k \in \mathbf{Z}$,

(4) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ *and* $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$ *for* $|x| < 1$,
$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$ *and* $(\csc^{-1} x)' = \frac{-1}{x\sqrt{x^2-1}}$ *for* $|x| > 1$, *and*
$(\tan^{-1} x)' = \frac{1}{1+x^2}$ *and* $(\cot^{-1} x)' = \frac{-1}{1+x^2}$ *for all* $x \in \mathbf{R}$.

Proof: First we prove Part (1) in the case that $a \in \mathbf{Q}$. When $n \in \mathbf{Z}^+$ and $f(x) = x^n$ we have

$$\begin{aligned}\frac{f(u)-f(x)}{u-x} &= \frac{u^n - x^n}{u-x} = \frac{(u-x)(u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1})}{u-x} \\ &= u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1} \longrightarrow n\,x^{n-1} \quad \text{as } u \to x.\end{aligned}$$

This shows that $(x^n)' = n\,x^{n-1}$ for all $x \in \mathbf{R}$ when $n \in \mathbf{Z}^+$. By the Reciprocal Rule, for $x \neq 0$ we have

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{(x^n)'}{(x^n)^2} = -\frac{n\,x^{n-1}}{x^{2n}} = -n\,x^{-n-1}.$$

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$ (when n is odd, $x^{1/n}$ is defined for all $x \in \mathbf{R}$, and when n is even, $x^{1/n}$ is defined only for $x \geq 0$). Since $f'(x) = (x^n)' = n\,x^{n-1}$ we have $f'(x) = 0$ when $x = 0$. By the Inverse Function Theorem, when $x \neq 0$ we have

$$(x^{1.n})' = g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n\,g(x)^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n\,x^{1-\frac{1}{n}}} = \tfrac{1}{n}\,x^{\frac{1}{n}-1}.$$

Finally, when $n \in \mathbf{Z}^+$ and $k \in \mathbf{Z}$ with $\gcd(k,n) = 1$, by the Chain Rule we have

$$(x^{k/n})' = \big((x^{1/n})^k\big)' = k(x^{1/n})^{k-1}(x^{1/n})' = k\,x^{\frac{k-1}{n}} \cdot \tfrac{1}{n}\,x^{\frac{1-n}{n}} = \tfrac{k}{n}\,x^{\frac{k}{n}-1}.$$

We have proven Part (1) in the case that $a \in \mathbf{Q}$.

Next we shall prove Part (2). For $f(x) = a^x$ where $a > 0$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h} - a^x}{h} = \frac{a^x a^h - a^x}{h} = a^x \cdot \frac{a^h - 1}{h}$$

and so we have $f'(x) = a^x \left(\lim\limits_{h \to 0} \frac{a^h - 1}{h}\right)$ provided that the limit exists and is finite. For $g(x) = \log_a x$, where $0 < a \neq 1$ and $x > 0$, we have

$$\frac{g(x+h)-g(x)}{h} = \frac{\log_a(x+h) - \log_a x}{h} = \frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \frac{\log_a\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}} = \tfrac{1}{x} \cdot \log_a \left(1 + \tfrac{h}{x}\right)^{x/h}$$

and so we have $g'(x) = \frac{1}{x} \cdot \log_a \left(\lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{x/h} \right)$ provided the limit exists and is finite. By letting $u = \frac{h}{x}$ we see that

$$\lim_{h \to 0^+} \left(1 + \tfrac{h}{x}\right)^{x/h} = \lim_{u \to \infty} \left(1 + \tfrac{1}{u}\right)^u = e$$

as you showed in Assignment 5. By letting $u = -\frac{h}{x}$, a similar argument shows that

$$\lim_{h \to 0^-} \left(1 + \tfrac{h}{x}\right)^{x/h} = \lim_{u \to \infty} \left(1 - \tfrac{1}{u}\right)^{-u} = e.$$

Thus the derivative $g'(x)$ does exist and we have

$$(\log_a x)' = g'(x) = \tfrac{1}{x} \log_a \left(\lim_{h \to 0} \left(1 + \tfrac{h}{x}\right)^{x/h} \right) = \tfrac{1}{x} \log_a e = \tfrac{1}{x} \cdot \tfrac{\ln e}{\ln a} = \tfrac{1}{x \ln a}.$$

Since $g(x) = \log_a x$ is differentiable with $g'(x) \neq 0$ it follows from the Inverse Function Theorem that $f(x) = a^x$ is differentiable with derivative

$$(a^x)' = f'(x) = \frac{1}{g'(f(x))} = \frac{1}{\frac{1}{f(x) \ln a}} = \ln a \cdot f(x) = \ln a \cdot a^x.$$

Now we return to the proof of Part (1), in the case that $a \notin \mathbf{Q}$. When $a > 0$ we have $a^x = e^{x \ln a}$ for all $x > 0$ and so by the Chain Rule

$$(x^a)' = \left(e^{a \ln x}\right)' = e^{a \ln x} (a \ln x)' = x^a \cdot \tfrac{a}{x} = a\, x^{a-1},$$

I may finish the proof later.

5.13 Definition: Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f : A \to F$ and let $a \in A$. We say that f has a **local maximum** value at a when

$$\exists \delta > 0\ \forall x \in A\ \Big(|x - a| \leq \delta \Longrightarrow f(x) \leq f(a) \Big).$$

Similarly, we say that f has a **local minimum** value at a when

$$\exists \delta > 0\ \forall x \in A\ \Big(|x - a| \leq \delta \Longrightarrow f(x) \geq f(a) \Big).$$

5.14 Theorem: *(Fermat's Theorem) Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f : A \to F$. Suppose that $a \in A$ is a limit point of A, both from above and from below. Suppose that f is differentiable at a and that f has a local maximum or minimum value at a. Then $f'(a) = 0$.*

Proof: We suppose that f has a local maximum value at a (the case that f has a local minimum value at a is similar). Choose $\delta > 0$ so that $|x - a| \leq \delta \Longrightarrow f(x) \leq f(a)$. For $x \in A$ with $a < x < a + \delta$, since $x > a$ and $f(x) \geq f(a)$ we have $\frac{f(x) - f(a)}{x - a} \geq 0$, and so

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \geq 0$$

by the Comparison Theorem. Similarly, for $x \in A$ with $a - \delta \leq x < a$, since $x < a$ and $f(x) \geq f(a)$ we have $\frac{f(x) - f(a)}{x - a} \leq 0$, and so

$$f'(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \leq 0.$$

5.15 Theorem: *(Mean Value Theorems) Let $a, b \in \mathbf{R}$ with $a < b$.*

(1) (Rolle's Theorem) If $f : [a, b] \to \mathbf{R}$ differentiable in (a, b) and continuous at a and b with $f(a) = 0 = f(b)$ then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

(2) (The Mean Value Theorem) If $f : [a, b] \to \mathbf{R}$ is differentiable in (a, b) and continuous at a and b then there exists a point $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(3) (Cauchy's Mean Value Theorem) If $f, g : [a, b] \to \mathbf{R}$ are differentiable in (a, b) and continuous at a and b, then there exists a point $c \in (a, b)$ such that

$$f'(c)\big(g(b) - g(a)\big) = g'(c)\big(f(b) - f(a)\big).$$

Proof: To Prove Rolle's Theorem, let $f : [a, b] \to \mathbf{R}$ be differentiable in (a, b) and continuous at a and b with $f(a) = 0 = f(b)$. If f is constant, then $f'(x) = 0$ for all $x \in [a, b]$, so we can choose any $c \in (a, b)$ and we have $f'(c) = 0$. Suppose that f is not constant. Either $f(x) > 0$ for some $x \in (a, b)$ or $f(x) < 0$ for some $x \in (a, b)$. Suppose that $f(x) > 0$ for some $x \in (a, b)$ (the case that $f(x) < 0$ for some $x \in (a, b)$ is similar). By the Extreme Value Theorem, f attains its maximum value at some point, say $c \in [a, b]$. Since $f(x) > 0$ for some $x \in (a, b)$, we must have $f(c) > 0$. Since $f(a) = f(b) = 0$ and $f(c) > 0$, we have $c \in (a, b)$. By Fermat's Theorem, we have $f'(c) = 0$. This completes the proof of Rolle's Theorem.

Now we use Rolle's Theorem to prove the Mean Value Theorem. Suppose that $f : [a, b] \to \mathbf{R}$ is differentiable in (a, b) and continuous at a and b. Let $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$. Then g is differentiable in (a, b) with $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ and g is continuous at a and b with $g(a) = 0 = g(b)$. By Rolle's Theorem, we can choose $c \in (a, b)$ so that $f'(c) = 0$, and then $g'(c) = \frac{f(b)-f(a)}{b-a}$, as required.

Finally, we use the Mean Value Theorem to Prove Cauchy's Mean Value Theorem. Suppose that $f, g : [a, b] \to \mathbf{R}$ are both differentiable in (a, b) and continuous at a and b. Let $h(x) = f(x)\big(g(b) - g(a)\big) - g(x)\big(f(b) - f(a)\big)$. Then h is differentiable in (a, b) and continuous at a and b. with $h(a) = f(a)g(b) - g(a)f(b) = h(b)$. By the Mean Value Theorem, we can choose $c \in (a, b)$ so that $h'(c) = \frac{h(b)-h(a)}{b-a} = 0$, and then we have $f(c)\big(g(b) - g(a)\big) - g(c)\big(f(b) - f(a)\big) = 0$, as required.

5.16 Corollary: *Let $a, b \in \mathbf{R}$ with $a < b$. Let $f : [a, b] \to \mathbf{R}$. Suppose that f is differentiable in (a, b) and continuous at a and b.*

(1) If $f'(x) \geq 0$ for all $x \in (a, b)$ then f is increasing on $[a, b]$.
(2) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$.
(3) If $f'(x) \leq 0$ for all $x \in (a, b)$ then f is decreasing on $[a, b]$.
(4) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.
(5) if $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.
(6) If $g : [a, b] \to \mathbf{R}$ is continuous at a and b and differentiable in (a, b) with $g'(x) = f'(x)$ for all $x \in (a, b)$, then for some $c \in \mathbf{R}$ we have $g(x) = f(x) + c$ for all $x \in (a, b)$.

Proof: We prove Part (1) (the proofs of the other parts are similar. Suppose that $f'(x) \geq 0$ for all $x \in (a, b)$. Let $a \leq x < y \leq b$. Choose $c \in (x, y)$ so that $f'(c) = \frac{f(y)-f(x)}{y-x}$. Then $f(y) - f(x) = f'(c)(y - x) \geq 0$ and so $f(y) \geq f(x)$. Thus f is increasing on $[a, b]$.

5.17 Corollary: *(The Second Derivative Test) Let I be an interval in $\mathbf{R}$, let $f : I \to \mathbf{R}$ and let $a \in I$. Suppose that f is differentiable in I with $f'(a) = 0$.*
(1) If $f''(a) > 0$ then f has a local minimum at a.
(2) If $f''(a) < 0$ then f has a local maximum at a.

Proof: The proof is left as an exercise.

5.18 Theorem: *(l'Hôpital's Rule) Let I be a non degenerate interval in $\mathbf{R}$. Let $a \in I$, or let a be an endpoint of I. Let $f, g : I \setminus \{a\} \to \mathbf{R}$. Suppose that f and g are differentiable in $I \setminus \{a\}$ with $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. Suppose either that $\lim\limits_{x \to a} f(x) = 0 = \lim\limits_{x \to a} g(x)$ or that $\lim\limits_{x \to a} g(x) = \pm\infty$. Suppose that $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = u \in \hat{\mathbf{R}}$. Then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = u$.*

Similar results hold for limits $x \to a^+$, $x \to a^-$, $x \to \infty$ and $x \to -\infty$.

Proof: We give the proof for $x \to a^+$ (assuming that a is a limit point of I from the right) and that $u \in \mathbf{R}$. Suppose first that $\lim\limits_{x \to a^+} f(x) = 0 = \lim\limits_{x \to a^+} g(x)$. Choose $b \in I$ with $a < b$. Extend the maps f and g to obtain maps $f, g : [a, b] \to \mathbf{R}$ by defining $f(a) = 0 = g(b)$. Note that f and g are continuous at a since $\lim\limits_{x \to a^+} f(x) = 0$ and $\lim\limits_{x \to a^+} g(x) = 0$. Let $\langle x_k \rangle$ be a sequence in $(a, b]$ with $x_k \to a$. For each index k, by Cauchy's Mean Value Theorem we can choose $c_k \in (a, x_k)$ so that $f'(c_k)\big(g(x_k) - g(a)\big) = g'(c_k)\big(f(x_k) - f(a)\big)$. Since $f(a) = 0 = g(a)$, this simplifies to $f'(c_k)g(x_k) = g'(c_k)f(x_k)$ and so we have $\frac{f(x_k)}{g(x_k)} = \frac{f'(c_k)}{g'(c_k)}$. Since $a < c_k < x_k$ and $x_k \to a$, we have $c_k \to a$ by the Squeeze Theorem. Since $\lim\limits_{x \to a^+} \dfrac{f'(x)}{g'(x)} = u$ and $c_k \to a$, we have $\frac{f(x_k)}{g(x_k)} = \frac{f'(c_k)}{g'(c_k)} \to u$ by the Sequential Characterization of Limits. We have shown that for every sequence $\langle x_k \rangle$ in $(a, b]$ with $x_k \to a$ we have $\frac{f(x_k)}{g(x_k)} \to u$, and it follows that $\lim\limits_{x \to a^+} \dfrac{f(x)}{g(x)} = u$ by the Sequential Characterization of Limits.

Now suppose that $\lim\limits_{x \to a^+} g(x) = \infty$. Since $\lim\limits_{x \to a^+} g(x) = \infty$ we can choose $b \in I$ with $b > a$ so that $g(x) > 0$ for all $x \in (a, b]$. Let $\langle x_k \rangle$ be a sequence in $(a, b]$ with $x_k \to a$. For each pair of indices k, l, by Cauchy's Mean Value Theorem we can choose $c_{kl} \in (a, x_k)$ so that $f'(c_{kl})\big(g(x_k) - g(x_l)\big) = g'(c_{kl})\big(f(x_k) - f(x_l)\big)$. Divide both sides by $g'(c_{kl})g(x_l)$ to get

$$\frac{f'(c_{kl})}{g'(c_{kl})} \frac{g(x_k)}{g(x_l)} - \frac{f'(c_{kl})}{g'(c_{kl})} = \frac{f(x_k)}{g(x_l)} - \frac{f(x_l)}{g(x_l)}.$$

so we have

$$\frac{f(x_l)}{g(x_l)} = \frac{f'(c_{kl})}{g'(c_{kl})} + \frac{f(x_k)}{g(x_l)} - \frac{f'(c_{kl})}{g'(c_{kl})} \frac{g(x_k)}{g(x_l)}.$$

Let $\epsilon > 0$. Since $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = u$ we can choose $\delta > 0$ so that $|x - a| \leq \delta \implies \left| \frac{f'(x)}{g'(x)} - u \right| \leq \frac{\epsilon}{3}$. Since $x_k \to a$ we can choose $m \in \mathbf{Z}^+$ so $k \geq m \implies |x_k - a| \leq \delta$. Note that when $k, l \geq m$, since c_{kl} lies between x_k and x_l we also have $|c_{kl} - a| \leq \delta$ so $\left| \frac{f'(c_{kl})}{g'(c_{kl})} - u \right| \leq \min\left\{1, \frac{\epsilon}{3}\right\}$. Fix $k \geq m$. Choose l large enough so that $\left| \frac{f(x_k)}{g(x_l)} \right| \leq \frac{\epsilon}{3}$ and $\left| \frac{f'(c_{kl})}{g'(c_{kl})} \frac{g(x_k)}{g(x_l)} \right| \leq \frac{\epsilon}{3}$. Then we have

$$\left| \frac{f(x_l)}{g(x_l)} - u \right| \leq \left| \frac{f'(c_{kl})}{g'(c_{kl})} - u \right| + \left| \frac{f(x_k)}{g(x_l)} \right| + \left| \frac{f'(c_{kl})}{g'(c_{kl})} \frac{g(x_k)}{g(x_l)} \right| \leq \epsilon.$$