Chapter 3: Sequences

3.1 Definition: For $p \in \mathbf{Z}$, let $\mathbf{Z}_{\geq p} = \{k \in \mathbf{Z} | k \geq p\}$. A **sequence** in a set A is a function of the form $x : \mathbf{Z}_{\geq p} \to A$ for some $p \in \mathbf{Z}$. Given a sequence $x : \mathbf{Z}_{\geq p} \to A$, the k^{th} **term** of the sequence is the element $x_k = x(k) \in A$, and we denote the sequence x by

$$\langle x_k \rangle_{k \geq p} = \langle x_k | k \geq p \rangle = \langle x_p, x_{p+1}, x_{p+2}, \cdots \rangle.$$

Note that the range of the sequence $\langle x_k \rangle_{k \geq p}$ is the set $\{x_k\}_{k \geq p} = \{x_k | k \geq p\}$.

3.2 Definition: Let F be an ordered field and let $\langle x_k \rangle_{k \geq p}$ be a sequence in F. For $a \in F$ we say that the sequence $\langle x_k \rangle_{k \geq p}$ **converges** to a (or that the **limit** of $\langle x_k \rangle_{k \geq p}$ is equal to a), and we write $x_k \to a$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = a$, when

$$\forall\, 0 < \epsilon \in F\ \exists m \in \mathbf{Z}\ \forall k \in \mathbf{Z}_{\geq p}\ \big(k \geq m \Longrightarrow |x_k - a| \leq \epsilon\big).$$

We say that the sequence $\langle x_k \rangle_{k \geq p}$ **converges** (in F) when there exists $a \in F$ such that $\langle x_k \rangle_{k \geq p}$ converges to a. We say that the sequence $\langle x_k \rangle_{k \geq p}$ **diverges** (in F) when it does not converge (to any $a \in F$). We say that $\langle x_k \rangle_{k \geq p}$ **diverges to infinity**, or that the limit of $\langle x_k \rangle_{k \geq p}$ is equal to **infinity**, and we write $x_k \to \infty$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = \infty$, when

$$\forall r \in F\ \exists m \in \mathbf{Z}\ \forall k \in \mathbf{Z}_{\geq p}\ \big(k \geq m \Longrightarrow x_k \geq r\big).$$

Similarly we say that $\langle x_k \rangle_{k \geq p}$ **diverges to** $-\infty$, or that the limit of $\langle x_k \rangle_{k \geq p}$ is equal to **negative infinity**, and we write $x_k \to -\infty$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = -\infty$ when

$$\forall r \in \mathbf{R}\ \exists m \in \mathbf{Z}\ \forall k \in \mathbf{Z}_{\geq p}\ \big(k \geq m \Longrightarrow x_k \leq r\big).$$

3.3 Example: Let $\langle x_k \rangle_{k \geq 0}$ be the sequence in $\mathbf{R}$ given by $x_k = \frac{(-2)^k}{k!}$ for $k \geq 0$. Show that $\lim\limits_{k \to \infty} x_k = 0$.

Solution: Note that for $k \geq 2$ we have

$$|x_k| = \frac{2^k}{k!} = \left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{2}{k-1}\right)\left(\frac{2}{k}\right) \leq \frac{2}{1} \cdot \frac{2}{n} = \frac{4}{n}.$$

Given $\epsilon \in \mathbf{R}$ with $\epsilon > 0$, we can choose $m \in \mathbf{Z}_{\geq 2}$ with $m \geq \frac{4}{\epsilon}$ and then for all $k \geq m$ we have $|x_k - 0| = |x_k| \leq \frac{4}{k} \leq \frac{4}{m} \leq \epsilon$. Thus $\lim\limits_{k \to \infty} x_k = 0$, by the definition of the limit.

3.4 Example: Let $\langle a_k \rangle_{k \geq 0}$ be the **Fibonacci sequence** in $\mathbf{R}$, which is defined recursively by $a_0 = 0$, $a_1 = 1$ and by $a_k = a_{k-1} + a_{k-2}$ for $k \geq 2$. Show that $\lim\limits_{k \to \infty} a_k = \infty$.

Solution: We have $a_0 = 0$, $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. Note that $a_k \geq k - 1$ when $k \in \{0, 1, 2, 3\}$. Let $n \geq 4$ and suppose, inductively, that $a_k \geq k - 1$ for all $k \in \mathbf{Z}$ with $0 \leq k < n$. Then $a_n = a_{n-1} + a_{n-2} \geq (n-2) + (n-3) = n + n - 5 \geq n + 4 - 5 = n - 1$. By the Strong Principle of Induction, we have $a_n \geq n - 1$ for all $n \geq 0$. Given $r \in \mathbf{R}$ we can choose $m \in \mathbf{Z}_{\geq 0}$ with $m \geq r + 1$, and then for all $k \geq m$ we have $a_k \geq k - 1 \geq m - 1 \geq r$. Thus $\lim\limits_{k \to \infty} a_k = \infty$ by the definition of the limit.

3.5 Example: Let $x_k = (-1)^k$ for $k \geq 0$. Show that $\langle x_k \rangle_{k \geq 0}$ diverges.

Solution: Suppose, for a contradiction, that $\langle x_k \rangle_{k \geq 0}$ converges and let $a = \lim\limits_{k \to \infty} x_k$. By taking $\epsilon = \frac{1}{2}$ in the definition of the limit, we can choose $m \in \mathbf{Z}$ so the for all $k \in \mathbf{N}$, if $k \geq m$ then $|x_k - a| \leq \frac{1}{2}$. Choose $k \in \mathbf{N}$ with $2k \geq m$. Since $|x_{2k} - a| \leq \frac{1}{2}$ and $x_{2k} = (-1)^{2k} = 1$, we have $|1 - a| \leq \frac{1}{2}$ so that $\frac{1}{2} \leq a \leq \frac{3}{2}$. Since $|x_{2k+1} - a| \leq \frac{1}{2}$ and $x_{2k+1} = (-1)^{2k+1} = -1$, we also have $|-1 - a| \leq \frac{1}{2}$ which implies that $-\frac{3}{2} \leq a \leq -\frac{1}{2}$. But then we have $a \leq -\frac{1}{2}$ and $a \geq \frac{1}{2}$, which is not possible.

3.6 Theorem: *(Independence of the Limit on the Initial Terms) Let $\langle x_k \rangle_{k \geq p}$ be a sequence in an ordered field F.*

(1) If $q \geq p$ and $y_k = x_k$ for all $k \geq q$, then $\langle x_k \rangle_{k \geq p}$ converges if and only if $\langle y_k \rangle_{k \geq q}$ converges, and in this case $\lim\limits_{k \to \infty} x_k = \lim\limits_{k \to \infty} y_k$.
(2) If $l \geq 0$ and $y_k = x_{k+l}$ for all $k \geq p$, then $\langle x_k \rangle_{k \geq p}$ converges if and only if $\langle y_k \rangle_{k \geq p}$ converges, and in this case $\lim\limits_{k \to \infty} x_k = \lim\limits_{k \to \infty} y_k$.

Proof: We prove Part (1) and leave the proof of Part (2) as an exercise. Let $q \geq p$ and let $y_k = x_k$ for $k \geq q$. Suppose $\langle x_k \rangle_{k \geq p}$ converges and let $a = \lim\limits_{k \to \infty} x_k$. Let $\epsilon > 0$. Choose $m \in \mathbf{Z}$ so that for all $k \in \mathbf{Z}_{\geq p}$, if $k \geq m$ then $|x_k - a| \leq \epsilon$. Let $k \in \mathbf{Z}_{\geq q}$ with $k \geq m$. Since $q \geq p$ we also have $k \in \mathbf{Z}_{\geq p}$ and so $|y_k - a| = |x_k - a| \leq \epsilon$. Thus $\langle y_k \rangle_{k \geq q}$ converges with $\lim\limits_{k \to \infty} y_k = a$. Conversely, suppose that $\langle y_k \rangle_{k \geq q}$ converges and let $a = \lim\limits_{k \to \infty} y_k$. Let $\epsilon > 0$. Choose $m_1 \in \mathbf{Z}$ so that for all $k \in \mathbf{Z}_{\geq q}$, if $k \geq m_1$ then $|y_k - a| \leq \epsilon$. Choose $m = \max\{m_1, q\}$. Let $k \in \mathbf{Z}_{\geq p}$ with $k \geq m$. Since $k \geq m$, we have $k \geq q$ and $k \geq m_1$ and so $|x_k - a| = |y_k - a| \leq \epsilon$. Thus $\langle x_k \rangle_{k \geq p}$ converges with $\lim\limits_{k \to \infty} x_k = a$.

3.7 Remark: Because of the above theorem, we often denote the sequence $\langle x_k \rangle_{k \geq p}$ simply as $\langle x_k \rangle$ (omitting the initial index p from our notation).

3.8 Theorem: *(Uniqueness of the Limit) Let $\langle x_k \rangle$ be a sequence in an ordered field F. If $\langle x_k \rangle$ has a limit (finite or infinite) then the limit is unique.*

Proof: Suppose, for a contradiction, that $x_k \to \infty$ and $x_k \to -\infty$. Since $x_k \to \infty$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow x_k \geq 1$. Since $x_k \to -\infty$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow x_k \leq -1$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then $x_k \geq 1$ and $x_k \leq -1$, which is not possible.

Suppose, for a contradiction, that $x_k \to \infty$ and $x_k \to a \in F$. Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| \leq 1$. Since $x_k \to \infty$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow x_k \geq a + 2$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then we have $|x_k - a| \leq 1$ so that $x \leq a + 1$ and we have $x_k \geq a + 2$, which is not possible. Similarly, it is not possible to have $x_k \to -\infty$ and $x_k \to a \in F$.

Finally suppose, for a contradiction, that $x_k \to a$ and $x_k \to b$ where $a, b \in F$ with $a \neq b$. Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| \leq \frac{|a-b|}{3}$. Since $x_k \to b$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |x_k - b| \leq \frac{|a-b|}{3}$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then we have $|x_k - a| \leq \frac{b-a}{3}$ and $|x_k - b| \leq \frac{b-a}{3}$ and so, using the Triangle Inequality, we have

$$|a - b| = |a - x_k + x_k - b| \leq |x_k - a| + |x_k - b| \leq \frac{|a-b|}{3} + \frac{|a-b|}{3} < |a - b|,$$

which is not possible.

3.9 Theorem: *(Basic Limits) In any ordered field F, for $a \in F$ we have*

$$\lim_{k\to\infty} a = a\ ,\ \lim_{k\to\infty} k = \infty \ \text{ and } \ \lim_{k\to\infty} \frac{1}{k} = 0.$$

Proof: The proof is left as an exercise.

3.10 Theorem: *(Operations on Limits) Let $\langle x_k\rangle$ and $\langle y_k\rangle$ be sequences in an ordered field F and let $c \in F$. Suppose that $\langle x_k\rangle$ and $\langle y_k\rangle$ both converge with $x_k \to a$ and $y_k \to b$. Then*
(1) $\langle c\,x_k\rangle$ converges with $c\,x_k \to ca$,
(2) $\langle x_k + y_k\rangle$ converges with $(x_k + y_k) \to a + b$,
(3) $\langle x_k - y_k\rangle$ converges with $(x_k - y_k) \to a - b$,
(4) $\langle x_k y_k\rangle$ converges with $x_k y_k \to ab$, and
(5) if $b \neq 0$ then $\langle x_k/y_k\rangle$ converges with $x_k/y_k \to a/b$.

Proof: We prove Parts (4) and (5) leaving the proofs of the other parts as an exercise. First we prove Part (4). Note that for all k we have

$$|x_k y_k - ab| = |x_k y_k - x_k b + x_k b - ab| \leq |x_k y_k - x_k b| + |x_k b - ab| = |x_k|\,|y_k - b| + |b||x_k - a|.$$

Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| \leq 1$ and we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |x_k - a| \leq \frac{\epsilon}{2(1+|b|)}$. Since $y_k \to b$ we can choose $m_3 \in \mathbf{Z}$ so that $k \geq m_3 \Longrightarrow |y_k - b| \leq \frac{\epsilon}{2(1+|a|)}$. Let $m = \max\{m_1, m_2, m_3\}$ and let $k \geq m$. Then we have $|x_k - a| \leq 1$, $|x_k - a| \leq \frac{\epsilon}{2(1+|b|)}$ and $|x_k - b| \leq \frac{\epsilon}{2(1+|a|)}$. Since $|x_k - a| \leq 1$, we have $|x_k| = |x_k - a + a| \leq |x_k - a| + |a| \leq 1 + |a|$. By our above calculation (where we found a bound for $|x_k y_k - ab|$) we have

$$\begin{aligned}|x_k y_k - ab| &\leq |x_k||y_k - b| + |b||x_k - a| \leq (1 + |a|)|y_k - b| + (1 + |b|)|x_k - a| \\ &\leq (1 + |a|)\tfrac{\epsilon}{2(1+|a|)} + (1 + |b|)\tfrac{\epsilon}{2(1+|b|)} = \epsilon.\end{aligned}$$

Thus we have $x_k y_k \to ab$, by the definition of the limit.

To prove Part (5), suppose that $b \neq 0$. Since $y_k \to b \neq 0$, we can choose $m_1 \in \mathbf{Z}$ so that that $k \geq m_1 \Longrightarrow |y_k - b| \leq \frac{|b|}{2}$. Then for $k \geq m_1$ we have

$$|b| = |b - y_k + y_k| \leq |b - y_k| + |y_k| \leq \tfrac{|b|}{2} + |y_k|$$

so that

$$|y_k| \geq |b| - \tfrac{|b|}{2} = \tfrac{|b|}{2} > 0.$$

In particular, we remark that when $k \geq m_1$ we have $y_k \neq 0$ so that $\frac{1}{y_k}$ is defined. Note that for all $k \geq m_1$ we have

$$\left|\frac{1}{y_k} - \frac{1}{b}\right| = \frac{|b - y_k|}{|y_k|\,|b|} \leq \frac{|b - y_k|}{\frac{|b|}{2}\cdot|b|} = \frac{2}{|b|^2}\cdot|y_k - b|.$$

Let $\epsilon > 0$. Choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |y_k - b| \leq \frac{|b|^2\epsilon}{2}$. Let $m = \max\{m_1, m_2\}$. For $k \geq m$ we have $k \geq m_1$ and $k \geq m_2$ and so $|y_k| \geq \frac{|b|^2}{2}$ and $|y_k - b| \leq \frac{|b|^2\epsilon}{2}$ and so

$$\left|\frac{1}{y_k} - \frac{1}{b}\right| \leq \tfrac{2}{|b|^2}\cdot|y_k - b| \leq \tfrac{2}{|b|^2}\cdot\tfrac{|b|^2\epsilon}{2} = \epsilon.$$

This proves that $\lim\limits_{k\to\infty}\frac{1}{y_k} = \frac{1}{b}$. Using Part (4), we have $\lim\limits_{k\to\infty}\frac{x_k}{y_k} = \lim\limits_{k\to\infty}\left(x_k\cdot\frac{1}{y_k}\right) = a\cdot\frac{1}{b} = \frac{a}{b}$.

3.11 Example: Let $x_k = \frac{k^2+1}{2k^2+k+3}$ for $k \geq 0$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: We have $x_k = \frac{k^2+1}{2k^2+k+2} = \frac{1+\left(\frac{1}{k}\right)^2}{2+\frac{1}{k}+3\cdot\left(\frac{1}{k}\right)^2} \longrightarrow \frac{1+0^2}{2+0+3\cdot 0^2} = \frac{1}{2}$ where we used the Basic Limits $1 \to 1$, $2 \to 2$ and $\frac{1}{k} \to 0$ together with Operations on Limits.

3.12 Definition: The above theorem can be extended to include many situations involving infinite limits. To deal with these cases, given an ordered field F, we define the **extended ordered field** $\hat{F}$ to be the set

$$\hat{F} = F \cup \{-\infty, \infty\}.$$

We extend the order relation $<$ on F to an order relation on $\hat{F}$ by defining $-\infty < \infty$ and $-\infty < a$ and $a < \infty$ for all $a \in F$. We partially extend the operations $+$ and $\cdot$ to $\hat{F}$; for $a \in F$ we define

$$\infty + \infty = \infty \ , \ \infty + a = \infty \ , \ (-\infty) + (-\infty) = -\infty \ , \ (-\infty) + a \ ,$$
$$\infty \cdot \infty = \infty \ , (\infty)(-\infty) = -\infty \ , \ (-\infty)(-\infty) = \infty \ ,$$
$$\infty \cdot a = \begin{cases} \infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases} \quad \text{and } (-\infty)(a) = \begin{cases} -\infty \text{ if } a > 0, \\ \infty \text{ if } a < 0, \end{cases}$$

but other values, including $\infty + (-\infty)$, $\infty \cdot 0$ and $-\infty \cdot 0$ are left undefined in $\hat{F}$. In a similar way, we partially extend the inverse operations $-$ and $\div$ to $\hat{F}$. For example, for $a \in F$ we define

$$\infty - (-\infty) = \infty \, , -\infty - \infty = -\infty \, , \, \infty - a = \infty \, , -\infty - a = -\infty \, , \, a - \infty = -\infty \, , \, a - (-\infty) = \infty \, ,$$
$$\frac{a}{\infty} = 0 \ , \ \frac{\infty}{a} = \begin{cases} \infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases} \quad \text{and } \ \frac{-\infty}{a} = \begin{cases} -\infty \text{ if } a > 0 \\ \infty \text{ if } a < 0 \end{cases}$$

with other values, including $\infty - \infty$, $\frac{\infty}{\infty}$ and $\frac{\infty}{0}$, left undefined. The expressions which are left undefined in $\hat{F}$, including

$$\infty - \infty \ , \ \infty \cdot 0 \ , \ \frac{\infty}{\infty} \ , \ \frac{\infty}{0} \ , \ \frac{a}{0}$$

are known as **indeterminate forms**.

3.13 Theorem: *(Extended Operations on Limits) Let $\langle x_k \rangle$ and $\langle y_k \rangle$ be sequences in F. Suppose that $\lim\limits_{k\to\infty} x_k = u$ and $\lim\limits_{k\to\infty} y_k = v$ where $u, v \in \hat{F}$.*

(1) if $u + v$ is defined in $\hat{F}$ then $\lim\limits_{k\to\infty} (x_k + y_k) = u + v$,

(2) if $u - v$ is defined in $\hat{F}$ then $\lim\limits_{k\to\infty} (x_k - y_k) = u - v$,

(3) if $u \cdot v$ is defined in $\hat{F}$ then $\lim\limits_{k\to\infty} (x_k \cdot y_k) = u \cdot v$, and

(4) if u/v is defined in $\hat{F}$ then $\lim\limits_{k\to\infty} (x_k / y_k) = u/v$.

Proof: The proof is left as an exercise.

3.14 Theorem: *(Monotonic Surjective Functions) Let I and J be intervals in an ordered field F. Suppose $f : I \to J$ is increasing and surjective. Let $\langle x_k \rangle$ be a sequence in I. Then*
(1) If $x_k \to a \in I$ then $f(x_k) \to f(a) \in J$,
(2) if $x_k \to u$ where $u \in F \cup \{\infty\}$ is the right endpoint of I, then $f(x_k) \to v$ where $v \in F \cup \{\infty\}$ is the right endpoint of J, and
(3) if $x_k \to u$ where $u \in F \cup \{-\infty\}$ is the left endpoint of I then $f(x_k) \to v$ where $v \in F \cup \{-\infty\}$ is the left endpoint of J.

Analogous results hold when $f : I \to J$ is decreasing and surjective.

Proof: We prove Part (1). Let $a \in I$, suppose $x_k \to a$, and let $b = f(a) \in J$. Note that since f is surjective, it has a right inverse. Let $g : J \to I$ be a right inverse of f. Let $\epsilon > 0$. We consider several cases, depending on whether or not b is an endpoint of J. Suppose first that b is not an endpoint of J. Choose ϵ_0 with $0 < \epsilon_0 \leq \epsilon$ so that $[b - \epsilon_0, b + \epsilon_0] \subseteq J$. Note that since f is increasing we have $g(b - \epsilon_0) < a < g(b + \epsilon_0)$ (since $g(b - \epsilon_0) \geq a \implies b - \epsilon = f\big(g(b - \epsilon_0)\big) \leq f(a) = b$ which is impossible, and $a \geq g(b + \epsilon_0) \implies b = f(a) \geq f\big(g(b + \epsilon_0)\big) = b + \epsilon_0$ which is impossible). Since $x_k \to a$ we can choose $m \in \mathbf{Z}$ so that $k \geq m \implies g(b - \epsilon_0) \leq x_k \leq g(b + \epsilon_0)$. Then for $k \geq m$ we have $b - \epsilon_0 = f\big(g(b - \epsilon_0)\big) \leq f(x_k) \leq f\big(g(b + \epsilon_0)\big) = g + \epsilon_0$. Thus $f(x_k) \to b = f(a)$.

Next consider the case that b is equal to one (but not both) of the endpoints of J, say b is the right endpoint of J, and say the left endpoint of J is smaller than b. In this case, we choose ϵ_0 with $0 < \epsilon_0 \leq \epsilon$ so that $[b - \epsilon_0, b] \subseteq J$. Note that since f is increasing we have $g(b - \epsilon_0) < a$. Choose $m \in \mathbf{Z}$ so that $k \geq m \implies g(b - \epsilon_0) \leq x_k$. Then for $k \geq m$, since f is increasing we have $b - \epsilon_0 \leq f(x_k)$. Since b is the right endpoint of J, it follows that $b - \epsilon_0 \leq f(x_k) \leq b$ for all $k \geq m$, and so $f(x_k) \to b = f(a)$.

Finally, note that if b is equal to both the left and right endpoints of J, then we have $J = \{b\}$ and so $f(x_k) = b$ for all k, and hence $f(x_k) \to b$.

3.15 Corollary: *(Basic Elementary Functions Acting on Limits) Let $\langle x_k \rangle$ be a sequence in $\mathbf{R}$ and let $b \in \mathbf{R}$. Then*

(1) if $x_k \to a > 0$ then ${x_k}^b \to a^b$,

$$\text{if } x_k \to \infty \text{ then } \lim_{k\to\infty} {x_k}^b = \begin{cases} \infty \text{ if } b > 0 \\ 0 \text{ if } b < 0, \end{cases}$$

(2) if $x_k \to a$ and $b > 0$ then $b^{x_k} \to b^a$,

$$\text{if } x_k \to \infty \text{ and } b > 0 \text{ then } \lim_{k\to\infty} b^{x_k} = \begin{cases} \infty \text{ if } b > 1 \\ 0 \text{ if } 0 < b < 1, \end{cases}$$

(3) if $x_k \to a > 0$ and $b > 0$ then $\log_b x_k \to \log_b a$,

$$\text{if } x_k \to \infty \text{ and } b > 0 \text{ then } \lim_{k\to\infty} x_k = \begin{cases} \infty \text{ if } b > 1 \\ -\infty \text{ if } 0 < b < 1 \end{cases}$$

(4) if $x_k \to a$ then $\sin x_k \to \sin a$ and $\cos x_k \to \cos a$
if $x_k \to a$, where $a \neq \frac{\pi}{2} + 2\pi t$ with $t \in \mathbf{Z}$, then $\tan x_k \to \tan a$
(5) if $x_k \to a \in [-1, 1]$ then $\sin^{-1} x_k \to \sin^{-1} a$ and $\cos^{-1} x_k \to \cos^{-1} a$
if $x_k \to a$ then $\tan^{-1} x_k \to \tan^{-1} a$
if $x_k \to \infty$ then $\tan^{-1} x_k \to \frac{\pi}{2}$,
if $x_k \to -\infty$ then $\tan^{-1} x_k \to -\frac{\pi}{2}$.

Proof: All of these follow immediately from the previous theorem, except for the first statement in Part (4) (some care is needed when $\sin a = \pm 1$ or $\cos a = \pm 1$).

3.16 Example: Let $x_k = \frac{\sqrt{3k^2+1}}{k+2}$ for $k \geq 0$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: We have $x_k = \frac{\sqrt{3k^2+1}}{k+2} = \frac{\sqrt{3+\frac{1}{k^2}}}{1+2\cdot\frac{1}{k}} \longrightarrow \frac{\sqrt{3+0}}{1+2\cdot 0} = \sqrt{3}$ where we used Basic Limits, Operations on Limits, and Functions Acting on Limits (specifically, we used Part (1) of Corollary 3.15 with $b = \frac{1}{2}$).

3.17 Example: Let $x_k = \frac{1+3k}{\sqrt[3]{2+k-k^2}}$ for $k \geq 0$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: We have $x_k = \frac{1+3k}{\sqrt[3]{2+k-k^2}} = \frac{\frac{1}{k}+3}{\sqrt[3]{\frac{2}{k^2}+\frac{1}{k}-1}} \cdot k^{1/3} \longrightarrow \frac{0+3}{\sqrt[3]{0+0-1}} \cdot \infty = -1 \cdot \infty = -\infty$ where we used Basic Limits, Extended Operations, and Functions Acting on Limits.

3.18 Example: Let $x_k = \sin^{-1}\left(k - \sqrt{k^2+k}\right)$ for $k \geq 0$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: Note that $k - \sqrt{k^2+k} = \frac{k^2-(k^2+k)}{k+\sqrt{k^2+k}} = \frac{-k}{k+\sqrt{k^2+k}} = \frac{-1}{1+\sqrt{1+\frac{1}{k}}} \longrightarrow \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}$, and so $x_k = \sin^{-1}\left(k - \sqrt{k^2-k}\right) \longrightarrow \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

3.19 Theorem: *(Comparison) Let $\langle x_k\rangle$ and $\langle y_k\rangle$ be sequences in an ordered field F. Suppose that $x_k \leq y_k$ for all k. Then*
(1) if $x_k \to a$ and $y_k \to b$ then $a \leq b$,
(2) if $x_k \to \infty$ then $y_k \to \infty$, and
(3) if $y_k \to -\infty$ then $x_k \to -\infty$.

Proof: We prove Part (1). Suppose that $x_k \to a$ and $y_k \to b$. Suppose, for a contradiction, that $a > b$. Choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| \leq \frac{a-b}{3}$. Choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |y_k - b| \leq \frac{a-b}{3}$. Let $k = \max\{m_1, m_2\}$. Since $|x_k - a| \leq \frac{a-b}{3} < \frac{a-b}{2}$, we have $x_k > a - \frac{a-b}{2} = \frac{a+b}{2}$. Since $|y_k - b| \leq \frac{a-b}{3} < \frac{a-b}{2}$, we have $y_k < b + \frac{a-b}{2} = \frac{a+b}{2}$. This is not possible since $x_k \leq y_k$.

3.20 Example: Let $x_k = (\frac{3}{2} + \sin k)\ln k$ for $k \geq 1$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: For all $k \geq 1$ we have $\sin k \geq -1$ so $(\frac{3}{2} + \sin k) \geq \frac{1}{2}$ and hence $x_k \geq \frac{1}{2}\ln k$. Since $x_k \geq \frac{1}{2}\ln k$ for all $k \geq 1$ and $\frac{1}{2}\ln k \longrightarrow \frac{1}{2}\cdot\infty = \infty$, it follows that $x_k \to \infty$ by the Comparison Theorem.

3.21 Theorem: *(Squeeze) Let $\langle x_k\rangle$, $\langle y_k\rangle$ and $\langle z_k\rangle$ be sequences in an ordered field F.*
(1) If $x_k \leq y_k \leq z_k$ for all k and $x_k \to a$ and $z_k \to a$ then $y_k \to a$.
(2) If $|x_k| \leq y_k$ for all k and $y_k \to 0$ then $x_k \to 0$.

Proof: We prove Part (1). Suppose that $x_k \leq y_k \leq z_k$ for all k, and suppose that $x_k \to a$ and $z_k \to a$. Let $\epsilon > 0$. Choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| \leq \epsilon$, choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |z_k - a| \leq \epsilon$ and let $m = \max\{m_1, m_2\}$. Then for $k \geq m$ we have $a - \epsilon \leq x_k \leq y_k \leq z_k \leq a + \epsilon$ and so $|y_k - a| \leq \epsilon$. Thus $y_k \to a$, as required.

3.22 Example: Let $x_k = \frac{k+\tan^{-1}k}{2k+\sin k}$ for $k \geq 1$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: For all $k \geq 1$ we have $-\frac{\pi}{2} < \tan^{-1}k < \frac{\pi}{2}$ and $-1 \leq \sin k \leq 1$ and so

$$\frac{k - \frac{\pi}{2}}{2k+1} \leq \frac{k + \tan^{-1}k}{2k + \sin k} \leq \frac{k + \frac{\pi}{2}}{2k-1}.$$

As in previous examples, we have $\frac{k-\frac{\pi}{2}}{2k+1} \to \frac{1}{2}$ and $\frac{k+\frac{\pi}{2}}{2k-1} \to \frac{1}{2}$, and so $x_k = \frac{k+\tan^{-1}k}{2k+\sin k} \to \frac{1}{2}$ by the Squeeze Theorem.

3.23 Definition: Let $\langle x_k \rangle$ be a sequence in an ordered set X. We say that the sequence $\langle x_k \rangle$ is **bounded above** by $b \in X$ when $x_k \leq b$ for all k. We say that the sequence $\langle x_k \rangle$ is **bounded below** by $b \in X$ when $b \leq x_k$ for all k. We say $\langle x_k \rangle$ is **bounded above** when it is bounded above by some element $b \in X$, we say that $\langle x_k \rangle$ is **bounded below** when it is bounded below by some $b \in X$, and we say that $\langle x_k \rangle$ is **bounded** when it is bounded above and bounded below.

3.24 Definition: Let $\langle x_k \rangle$ be a sequence in an ordered field F. We say that $\langle x_k \rangle$ is **increasing** (for $k \geq p$) when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k \leq l$ then $x_k \leq x_l$. We say that $\langle x_k \rangle$ is **strictly increasing** (for $k \geq p$) when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k < l$ then $x_k < x_l$. Similarly, we say that $\langle x_k \rangle$ is **decreasing** when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k \leq l$ the $x_k \geq x_l$ and we say that $\langle x_k \rangle$ is **strictly decreasing** when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k < l$ the $x_k > x_l$. We say that $\langle x_k \rangle$ is **monotonic** when it is either increasing or decreasing.

3.25 Theorem: *(Monotonic Convergence) Let $\langle x_k \rangle$ be a sequence in* $\mathbf{R}$.
(1) Suppose $\langle x_k \rangle$ is increasing. If $\langle x_k \rangle$ is bounded above then $x_k \to \sup\{x_k\}$, and if $\langle x_k \rangle$ is not bounded above then $x_k \to \infty$.
(2) Suppose $\langle x_k \rangle$ is decreasing. If $\langle x_k \rangle$ is bounded below then $x_k \to \inf\{x_k\}$, and if $\langle x_k \rangle$ is not bounded below then $x_k \to -\infty$.

Proof: We prove Part (1) in the case that $\langle x_k \rangle_{k \geq p}$ is increasing and bounded above, say by $b \in \mathbf{R}$. Let $A = \{x_k | k \geq p\}$ (so A is the range of the sequence $\langle x_k \rangle$). Note that A is nonempty and bounded above (indeed b is an upper bound for A). By the Completeness Property of $\mathbf{R}$, A has a supremum in $\mathbf{R}$. Let $a = \sup\{x_k | k \geq p\}$. Note that $a \geq x_k$ for all $k \geq p$ and $a \leq b$, by the definition of the supremum. Let $\epsilon > 0$. By the Approximation Property of the supremum, we can choose an index $m \geq p$ so that the element $x_m \in A$ satisfies $a - \epsilon < x_m \leq a$. Since $\langle x_k \rangle$ is increasing, for all $k \geq m$ we have $x_k \geq x_m$, so we have $a - \epsilon \leq x_m \leq x_k \leq a$ and hence $|x_k - a| < \epsilon$. Thus $\lim\limits_{k\to\infty} x_k = a \leq b$.

3.26 Example: Let $x_1 = \frac{4}{3}$ and let $x_{k+1} = 5 - \frac{4}{x_n}$ for $k \geq 1$. Determine whether $\langle x_k \rangle$ converges, and if so then find the limit.

Solution: Suppose, for now, that $\langle x_k \rangle$ does converge, say $x_k \to a$. By Independence of Converge on Initial Terms, we also have $x_{k+1} \to a$. Using Operations on Limits, we have $a = \lim\limits_{k\to\infty} x_{k+1} = \lim\limits_{k\to\infty} \left(5 - \frac{4}{x_k}\right) = 5 - \frac{4}{a}$. Since $a = 5 - \frac{4}{a}$, we have $a^2 = 5a - 4$ or equivalently $(a-1)(a-4) = 0$. We have proven that if the sequence converges then its limit must be equal to 1 or 4.

The first few terms of the sequence are $x_1 = \frac{4}{3}$, $x_2 = 2$ and $x_3 = 3$. Since the terms appear to be increasing, we shall try to prove that $1 \leq x_n \leq x_{n+1} \leq 4$ for all $n \geq 1$. This is true when $n = 1$. Suppose it is true when $n = k$. Then we have

$$\begin{aligned} 1 \leq x_k \leq & x_{k+1} \leq 4 \Longrightarrow 1 \geq \tfrac{1}{x_k} \geq \tfrac{1}{x_{k+1}} \geq \tfrac{1}{4} \Longrightarrow -4 \leq -\tfrac{4}{x_k} \leq -\tfrac{4}{x_{k+1}} \leq -1 \\ & \Longrightarrow 1 \leq 5 - \tfrac{4}{x_k} \leq 5 - \tfrac{4}{x_{k+1}} \leq 4 \Longrightarrow 1 \leq x_{k+1} \leq x_{k+2} \leq 4. \end{aligned}$$

Thus, by the Principle of Induction, we have $1 \leq x_n \leq x_{n+1} \leq 4$ for all $n \geq 1$.

Since $x_n \leq x_{n+1}$ for all $n \geq 1$, the sequence is increasing, and since $x_n \leq 4$ for all $n \geq 1$, the sequence is bounded above by 4. By the Monotone Convergence Theorem, the sequence does converge. By the first paragraph, we know the limit must be either 1 or 4, and since the sequence starts at $x_1 = 2$ and increases, the limit must be 4.

3.27 Theorem: *(The Nested Interval Theorem) Let $I_0, I_1, I_2, \cdots$ be nonempty, closed bounded intervals in* $\mathbf{R}$*. Suppose that $I_0 \supseteq I_1 \supset I_2 \supset \cdots$. Then $\bigcap\limits_{k=0}^{\infty} I_k \neq \emptyset$.*

Proof: For each $k \geq 1$, let $I_k = [a_k, b_k]$ with $a_k < b_k$. For each k, since $I_k \subseteq I_{k+1}$ we have $a_{k+1} \leq a_k < b_k \leq b_{k+1}$. Since $a_k \geq a_{k+1}$ for all k, the sequence $\langle a_k \rangle$ is increasing. Since $a_k < b_k \leq b_{k-1} \leq \cdots \leq b_1$ for all k, the sequence $\langle a_k \rangle$ is bounded above by b_1. Since $\langle a_k \rangle$ is increasing and bounded above, it converges. Let $a = \sup\{a_k\} = \lim\limits_{k\to\infty} a_k$. Similarly, $\langle b_k \rangle$ is decreasing and bounded below by a_1, and so it converges. Let $b = \inf\{b_k\} = \lim\limits_{k\to\infty} b_k$. Fix $m \geq 1$. For all $k \geq m$ we have $a_m < b_m \leq b_{m+1} \leq \cdots \leq b_k$. Since $a_k \leq b_k$ for all k, by the Comparison Theorem we have $a \leq b$, and so the interval $[a, b]$ is not empty. Since $\langle a_k \rangle$ is increasing with $a_k \to a$, it follows (we leave the proof as an exercise) that $a_k \leq a$ for all $k \geq 1$. Similarly, we have $b_k \geq b$ for all $k \geq 1$ and so $[a, b] \subseteq [a_k, b_k] = I_k$. Thus $[a, b] \subseteq \bigcap\limits_{k=1}^{\infty} I_k$, and so $\bigcap\limits_{k=1}^{\infty} I_k \neq \emptyset$.

3.28 Definition: Let $\langle x_k \rangle_{k \geq p}$ be a sequence in a set X Given a strictly increasing function $f : \mathbf{Z}_{\geq q} \to \mathbf{Z}_{\geq p}$, write $k_l = f(l)$ and let $y_l = x_{k_l}$ for all $l \geq q$. Then the sequence $\langle y_l \rangle_{l \geq q}$ is called a **subsequence** of the sequence $\langle x_k \rangle_{k \geq p}$. In other words, a subsequence of $\langle x_k \rangle_{k \geq p}$ is a sequence of the form

$$\left\langle x_{k_q}, x_{k_{q+1}}, x_{k_{q+2}}, \cdots \right\rangle \text{ with } p \leq k_q < k_{q+1} < k_{q+2} < \cdots .$$

Given a bijective function $f : \mathbf{Z}_{\geq q} \to \mathbf{Z}_{\geq p}$, write $k_l = f(l)$ and let $y_l = x_{k_l}$ for $l \geq 1$. Then the sequence $\langle y_l \rangle_{l \geq q}$ is called a **rearrangement** of the sequence $\langle x_k \rangle$.

3.29 Theorem: *Let $\langle x_k \rangle$ be a sequence in an ordered field F. Suppose that $x_k \to a$. Then*

(1) every subsequence of $\langle x_k \rangle$ converges to a, and
(2) every rearrangement of $\langle x_k \rangle$ converges to a.

Proof: We shall prove Parts (1) and (2) simultaneously. Let $f : \mathbf{Z}_{\geq q} \to \mathbf{Z}_{\geq p}$ be an injective map. Write $k_l = f(l)$ and let $y_l = x_{k_l}$ for $k \geq l$. Let $\epsilon > 0$. Choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| \leq \epsilon$. Since f is injective, there are only finitely many indices l with $p \leq f(l) < m_1$. Choose $m \in \mathbf{Z}$ with m larger than every such index l. Then for $l \geq m$ we have $k_l = f(l) \geq m_1$ and so $|y_l - a| = |x_{k_l} - a| \leq \epsilon$.

3.30 Theorem: *(Bolzano-Weirstrass) Every bounded sequence in* $\mathbf{R}$ *has a convergent subsequence.*

Proof: Let $\langle x_k \rangle$ be a bounded sequence in $\mathbf{R}$. Choose $a, b \in \mathbf{R}$ with $a \leq x_k$ for all k and $x_k \leq b$ for all k. Then we have $x_k \in [a, b]$ for all k. We define a sequence of nonempty closed intervals recursively as follows. Let $I_0 = [a_0, b_0] = [a, b]$. Note that $I_0 = \left[a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right]$. Let $I_1 = [a_1, b_1]$ be equal to one of the two intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, chosen in such a way that there are infinitely many indices k with $x_k \in I_1$. Suppose we have chosen intervals $I_j = [a_j, b_j]$ with $b_j - a_j = \frac{1}{2^j}(b - a)$ for $1 \leq j \leq n$, such that $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ and such that for each index j, there are infinitely many indices k with $x_k \in I_j$. Note that $I_n = [a_n, b_n] = \left[a_n, \frac{a_n+b_n}{2}\right] \cup \left[\frac{a_n+b_n}{2}, b_n\right]$. Let I_{n+1} be equal to one of the two intervals $\left[a_n, \frac{a_n+b_n}{2}\right]$ and $\left[\frac{a_n+b_n}{2}, b_n\right]$, chosen in such a way that there are infinitely many indices k with $x_k \in I_{n+1}$. In this way, we obtain a sequence $\langle I_j \rangle_{j \geq 0}$ of nonempty closed intervals.

By the Nested Interval Theorem, $\bigcap_{j=0}^{\infty} I_j$ is not empty. Choose a point c with $c \in I_n$ for every $n \geq 0$.

We shall now construct a subsequence of $\langle x_k \rangle$ which converges to c. Since for each $j \geq 0$ there exist infinitely many indices k with $x_k \in I_j$, we can construct a subsequence of $\langle x_k \rangle$ as follows. Choose k_0 so that $x_{k_0} \in I_0$, then choose $k_1 > k_0$ so that $x_{k_1} \in I_1$, then choose $k_2 > k_1$ with $x_{k_2} \in I_2$, and so on. In this way, we obtain a subsequence $\langle x_{k_j} \rangle_{j \geq 0}$ of $\langle x_k \rangle$ with $x_{k_j} \in I_j$ for all $j \geq 0$. We claim that $x_{k_j} \to c$ as $j \to \infty$. Let $\epsilon > 0$ Choose $m \in \mathbf{Z}$ so that $\frac{1}{2^m}(b-a) \leq \epsilon$. For $j \geq m$, since $c \in [a,b] \subseteq [a_j, b_j]$ and $x_{k_j} \in [a_j, b_j]$, it follows that

$$|x_{k_j} - c| = \max\{x_{k_j}, c\} - \min\{x_{k_j}, c\} \leq b_j - a_j = \frac{1}{2^j}(b-a) \leq \frac{1}{2^m}(b-a) \leq \epsilon.$$

Thus $x_{k_j} \to c$ as $j \to \infty$, as claimed.

3.31 Definition: Let $\langle x_k \rangle_{k \geq p}$ be a sequence in an ordered field F. We say that $\langle x_k \rangle$ is **Cauchy** when

$$\forall \epsilon > 0 \; \exists m \in \mathbf{Z} \; \forall k, l \in \mathbf{Z}_{\geq p} \; \big(k, l \geq m \Longrightarrow |x_k - x_l| \leq \epsilon\big).$$

3.32 Theorem: *(Cauchy Criterion for Convergence)*

(1) For a sequence $\langle x_k \rangle$ in an ordered field F, if $\langle x_k \rangle$ converges then it is Cauchy.
(2) For a sequence $\langle x_k \rangle$ in $\mathbf{R}$, if $\langle x_k \rangle$ is Cauchy then it converges.

Proof: To prove Part (1), let $\langle x_k \rangle$ be a sequence in an ordered field F and suppose that $x_k \to a$. Let $\epsilon > 0$ and choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |x_k - a| \leq \frac{\epsilon}{2}$. Then for $k, l \geq m$ we h ave

$$|x_k - x_l| = |x_k - a + a - x_l| \leq |x_k - a| + |a - x_l| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\langle x_k \rangle$ is Cauchy.

To prove Part (2), let $\langle x_k \rangle_{k \geq p}$ be a sequence in $\mathbf{R}$ and suppose that $\langle x_k \rangle$ is Cauchy. We claim that $\langle x_k \rangle$ is bounded. Since $\langle x_k \rangle$ is Caucy, we can choose $m \in \mathbf{Z}$ so that $k, l \geq m \Longrightarrow |x_k - x_l| \leq 1$. In particular, for all $k \geq m$ we have $|x_k - x_m| \leq 1$ and so $|x_k| = |x_k - x_m + x_m| \leq |x_k - x_m| + |x_m| \leq 1 + |x_m|$. It follows that $\langle x_k \rangle$ is bounded by $b = \max\big\{|x_p|, |x_{p+1}|, \cdots, |x_{m-1}|\,,\, 1 + |x_m|\big\}$.

Because $\langle x_k \rangle$ is bounded, it has a convergent subsequence, by the Bolzano Weierstrass Theorem. Let $\langle x_{k_j} \rangle$ be a convergent subsequence of $\langle x_k \rangle$ and let $a = \lim_{j \to \infty} x_{k_j}$. We claim that $x_k \to a$. Let $\epsilon > 0$. Since $\langle x_k \rangle$ is Cauchy, we can choose $m \in \mathbf{Z}$ so that $k, l \geq m \Longrightarrow |x_k - x_l| \leq \frac{\epsilon}{2}$. Since $x_{k_j} \to a$ we can choose $m_0 \in \mathbf{Z}$ so that $j \geq m_0 \Longrightarrow |x_{k_j} - a| \leq \frac{\epsilon}{2}$. Choose an index $j \geq m_0$ so that $k_j \geq m$. Then for all $k \geq m$ we have

$$|x_k - a| = |x_k - x_{k_j} + x_{k_j} - a| \leq |x_k - x_{k_j}| + |x_{k_j} - a| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $x_k \to a$, as claimed.