Appendix 2: Exponential and Trigonometric Functions

2.1 Definition: Let X and Y be sets and let $f : X \to Y$. We say that f is **injective** (or **one-to-one**, written as $1\!:\!1$) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$. We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is bijective, we define the **inverse** of f to be the function $f^{-1} : Y \to X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1}, and in this case we have $(f^{-1})^{-1} = f$.

2.2 Example: Let $f(x) = \frac{1}{3}\sqrt{12x - x^2}$ for $0 \le x \le 6$. Show that f is injective and find a formula for its inverse function.

Solution: Note that when $0 \le x \le 6$ (indeed when $0 \le x \le 12$) we have $12x - x^2 = x(12 - x) \ge 0$, so that $\frac{1}{3}\sqrt{12x - x^2}$ exists, and we have $12x - x^2 = 36 - (x - 6)^2 \le 36$ so that $\frac{1}{3}\sqrt{12x - x^2} \le \frac{1}{3}\sqrt{36} = 2$. Thus if $0 \le x \le 6$ then $f(x) = \frac{1}{3}\sqrt{12x - x^2}$ exists and we have $0 \le f(x) \le 2$. Let $x, y \in \mathbf{R}$ with $0 \le x \le 6$ and $0 \le y \le 2$. Then we have

$$\begin{aligned} y = f(x) &\iff y = \tfrac{1}{3}\sqrt{12x - x^2} \\ &\iff 3y = \sqrt{12x - x^2} \\ &\iff 9y^2 = 12x - x^2 \quad \text{, since } y \ge 0 \\ &\iff x^2 - 12x + 9y^2 = 0 \\ &\iff x = \frac{12 \pm \sqrt{144 - 36y^2}}{2} = 6 \pm 3\sqrt{4 - y^2} \quad \text{, by the Quadratic Formula} \\ &\iff x = 6 - 3\sqrt{4 - y^2} \;\; \text{since } x \le 6. \end{aligned}$$

Verify that when $0 \le y \le 2$ we have $0 \le 4 - y^2 \le 4$ so that $\sqrt{4 - y^2}$ exists and we have $0 \le 6 - 3\sqrt{4 - y^2} \le 6$. Thus when we consider f as a function $f : [0, 6] \to [0, 2]$, it is bisectve and its inverse $f^{-1} : [0, 2] \to [0, 6]$ is given by $f^{-1}(y) = 6 - 3\sqrt{4 - y^2}$.

2.3 Definition: Let F be a field and let $f : A \subseteq F \to F$. We say that f is **even** when $f(-x) = f(x)$ for all $x \in F$ and we say that f is **odd** when $f(-x) = -f(x)$ for all $x \in F$.

2.4 Definition: Let F be an ordered field and let $f : A \subseteq\to F$. We say that f is **increasing** when it has the property that for all $x, y \in A$, if $x < y$ then $f(x) < f(y)$, and we say f is decreasing when for all $x, y \in A$ with $x < y$ we have $f(x) > f(y)$. We say that f is **monotonic** when f is either increasing or decreasing. Note that every monotonic function is injective.

2.5 Remark: We assume familiarity with exponential, logarithmic, trigonometric and inverse trigonometric functions. These functions can be defined rigorously. We shall give a brief description of how one can define the exponential and logarithmic function rigorously, and we shall provide an informal (non-rigorous) description of the trigonometric and inverse trigonometric functions.

2.6 Definition: Let us outline one possible way to define the value of x^y for suitable real numbers $x, y \in \mathbf{R}$. First we define $x^0 = 1$ for all $x \in \mathbf{R}$. Then for $n \in \mathbf{Z}$ with $n \geq 1$ we define x^n recursively by $x^n = x \cdot x^{n-1}$ for all $x \in \mathbf{R}$. Also, for $n \in \mathbf{Z}$ with $n \geq 1$ we define $x^{-n} = \frac{1}{x^n}$ for all $x \neq 0$. At this stage we have defined x^y for $y \in \mathbf{Z}$.

When $0 < n \in \mathbf{Z}$ is odd, for all $x \in \mathbf{R}$ we define $x^{1/n} = y$ where y is the unique real number such that $y^n = x$ (to be rigorous, one must prove that this number y exists and is unique). When $0 < n \in \mathbf{Z}$ is even, for $x \geq 0$ we define $x^{1/n} = y$ where y is the unique nonnegative real number such that $y^n = x$ (again, to be rigorous a proof is required). Also, for $0 < n \in \mathbf{Z}$ we define $x^{-1/n} = \frac{1}{x^{1/n}}$, which is defined for $x \neq 0$ if n is odd, and is defined for $x > 0$ when n is even. When $n, m \in \mathbf{Z}$ with $n > 0$ and $m > 0$ and $\gcd(n, m) = 1$, we define $x^{n/m} = (x^n)^{1/m}$, which is defined for all $x \in \mathbf{R}$ when m is odd and for $x \geq 0$ when m is even, and we define $x^{-n/m} = \frac{1}{x^{n/m}}$, defined for $x \neq 0$ when m is odd and for $x > 0$ when m is even. At this stage, we have defined x^y for $y \in \mathbf{Q}$.

When $x > 1$ and $y \in \mathbf{R}$, we define $x^y = \sup\left\{x^t \middle| t \in \mathbf{Q}, t \leq y\right\}$ (to be rigorous, one needs to prove that the supremum exists and that when $y \in \mathbf{Q}$ this agrees with our previous definition). When $0 < x < 1$ and $y \in \mathbf{R}$ we define $x^y = \inf\left\{x^t \middle| t \in \mathbf{Q}, t \leq y\right\}$. Finally, we define $1^y = 1$ for all $y \in \mathbf{R}$ and we define $0^y = 0$ for all $y > 0$.

2.7 Theorem: *(Properties of Exponentials) Let $a, b, x, y \in \mathbf{R}$ with $a, b > 0$. Then*
(1) $a^0 = 1$,
(2) $a^{x+y} = a^b\, a^c$,
(3) $a^{x-y} = a^x / a^y$,
(4) $(a^x)^y = a^{xy}$,
(5) $(ab)^x = a^x b^x$.

Proof: We omit the proof.

2.8 Theorem: *(Properties of Power Functions)*
(1) When $a > 0$, the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^a$ is increasing and bijective and its inverse function is given by $f^{-1}(x) = x^{1/a}$.
(2) When $a < 0$, the function $f : (0, \infty) \to (0, \infty)$ given by $f(x) = x^a$ is decreasing and bijective and its inverse is given by $f^{-1}(x) = a^{1/x}$.

Proof: We omit the proof.

2.9 Definition: A function of the form $f(x) = x^a$ is called a **power function**.

2.10 Theorem: *(Properties of Exponential Functions)*
(1) When $a > 1$ *the function* $f : \mathbf{R} \to (0, \infty)$ *given by* $f(x) = a^x$ *is increasing and bijective.*
(2) When $0 < a < 1$ *the function* $f : \mathbf{R} \to (0, \infty)$ *given by* $f(x) = a^x$ *is decreasing and bijective.*

Proof: We omit the proof.

2.11 Definition: For $a > 0$ with $a \neq 1$, the function $f : \mathbf{R} \to (0, \infty)$ given by $f(x) = a^x$ is called the base a **exponential function**, its inverse function $f^{-1} : (0, \infty) \to \mathbf{R}$ is called the base a **logarithmic function**, and we write $f^{-1}(x) = \log_a x$. By the definition of the inverse function, we have $\log_a(a^x) = x$ for all $x \in \mathbf{R}$ and $e^{\log_a y} = y$ for all $y > 0$, and for all $x, y \in \mathbf{R}$ with $y > 0$ we have $y = a^x \iff x = \log_a y$.

2.12 Theorem: *(Properties of Logarithms) Let* $a, b, x, y \in (0, \infty)$. *Then*
(1) $\log_a 1 = 0$,
(2) $\log_a(xy) = \log_a x + \log_a y$,
(3) $\log_a(x/y) = \log_a x - \log_a y$,
(4) $\log_a(x^y) = y \log_a x$, *and*
(5) $\log_b x = \log_a x / \log_a b$,
(6) if $a > 1$, *the function* $g : (0, \infty) \to \mathbf{R}$ *given by* $g(x) = \log_a x$ *is increasing and bijective.*

Proof: The proof is left as an exercise.

2.13 Definition: There is a number $e \in \mathbf{R}$ called **natural base**, with $e \cong 2.71828$, which can be defined in many ways, for example we can define

$$e = \sup \left\{ \left(1 + \tfrac{1}{n}\right)^n \middle| 1 \leq n \in \mathbf{Z} \right\}$$

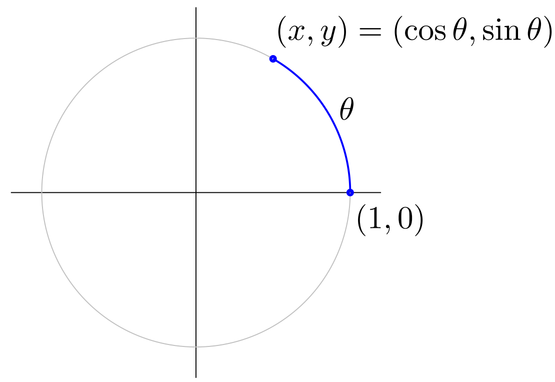
(to be rigorous, one must prove that the set $A = \left\{ (1 + \frac{1}{n})^n \middle| 1 \leq n \in \mathbf{Z} \right\}$ is bounded above). The logarithm to the base e is called the **natural logarithm**, and we write

$$\ln x = \log_e x \text{ for } x > 0.$$

The properties of exponentials and logarithms in Theorems 2.13 and 2.18 give

$$e^0 = 1 \ , \ a^{x+y} = e^x e^y \ , \ e^{x-y} = e^x / e^y \ , \ (e^x)^y = e^{xy},$$
$$\ln 1 = 0 \ , \ \ln(xy) = \ln x + \ln y \ , \ \ln(x/y) = \ln x - \ln y \ , \ \ln x^y = y \ln x$$
$$\log_a x = \frac{\ln x}{\ln a} \ \text{ and } \ a^x = e^{x \ln a}.$$

2.14 Definition: We define the trigonometric functions informally as follows. For $\theta \geq 0$, we define $\cos\theta$ and $\sin\theta$ to be the x- and y-coordinates of the point at which we arrive when we begin at the point $(1,0)$ and travel for a distance of θ units counterclockwise around the unit circle $x^2+y^2=1$. For $\theta \leq 0$, $\cos\theta$ and $\sin\theta$ are the x and y-coordinates of the point at which we arrive when we begin at $(1,0)$ and travel clockwise around the unit circle for a distance of $|\theta$ units. When $\cos\theta \neq 0$ we define $\sec\theta = 1/\cos\theta$ and $\tan\theta = \sin\theta/\cos\theta$, and when $\sin\theta \neq 0$ we define $\csc\theta = 1/\sin\theta$ and $\cot\theta = \cos\theta/\sin\theta$. (This definition is not rigorous because we did not define what it means to travel around the circle for a given distance).

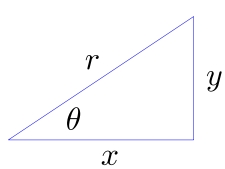


2.15 Definition: We define π, informally, to be the distance along the top half of the unit circle from $(1,0)$ to $(-1,0)$, and so we have $\cos\pi=-1$ and $\sin\pi=0$. By symmetry, the distance from $(1,0)$ to $(0,1)$ along the circle is equal to $\frac{\pi}{2}$ so we also have $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$.

2.16 Theorem: *(Basic Trigonometric Properties) For $\theta \in \mathbf{R}$ we have*
(1) $\cos^2\theta+\sin^2\theta=1$,
(2) $\cos(-\theta)=\cos\theta$ *and* $\sin(-\theta)=-\sin\theta$,
(3) $\cos(\theta+\pi)=-\cos\theta$ *and* $\sin(\theta+\pi)=-\sin\theta$,
(4) $\cos(\theta+2\pi)=\cos\theta$ *and* $\sin(\theta+2\pi)=\sin\theta$).

Proof: Informally, these properties can all be seen immediately from the above definitions. We omit a rigorous proof.

2.17 Theorem: *(Trigonometric Ratios) Let $\theta \in \left(0,\frac{\pi}{2}\right)$. For a right angle triangle with an angle of size θ and with sides of lengths x, y and r as shown, we have*



$$\cos\theta=\frac{x}{r}\ ,\ \sin\theta=\frac{y}{r}\ \text{ and }\ \tan\theta=\frac{y}{x}.$$

Proof: We can see this informally by scaling the picture in Definition 2.17 by a factor of r.

2.18 Theorem: *(Special Trigonometric Values) We have the following exact trigonometric values.*

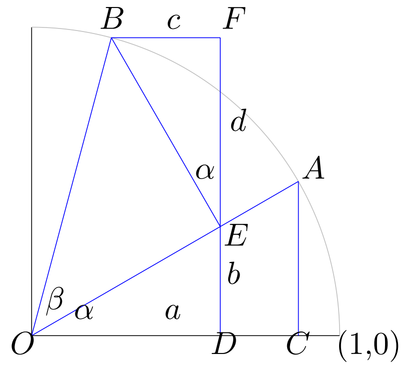
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Proof: This follows from the above theorem using certain particular right angled triangles.

2.19 Theorem: *(Trigonometric Sum Formulas) For $\alpha, \beta \in \mathbf{R}$ we have*

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ , and}$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Proof: Informally, we can prove this with the help of a picture. The picture below illustrates the situation when $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$.



In the picture, O is the origin, A is the point with coordinates $(\cos\alpha, \sin\alpha)$ and B is the point $(x,y) = \big(\cos(\alpha+\beta), \sin(\alpha+\beta)\big)$. In triangle ODE we see that $\cos\alpha = \frac{OD}{OE} = \frac{a}{\cos\beta}$ and $\sin\alpha = \frac{DE}{OE} = \frac{b}{\cos\beta}$, and so $a = \cos\alpha\cos\beta$, $b = \sin\alpha\cos\beta$. In triangle EFB, verify that the angle at E has size α, and so we have $\cos\alpha = \frac{EF}{EB} = \frac{d}{\sin\beta}$ and $\sin\alpha = \frac{BF}{BE} = \frac{c}{\sin\beta}$, and so $c = \sin\alpha\sin\beta$, $d = \cos\alpha\sin\beta$. The x and y-coordinates of the point B are $x = a - c$ and $y = b + d$, and so

$$\cos(\alpha+\beta) = x = a - c = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ , and}$$
$$\sin(\alpha+\beta) = y = b + d = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

This proves the theorem (informally) in the case that $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. One can then show that the theorem holds for all $\alpha, \beta \in \mathbf{R}$ by using the Basic Trigonometric Properties (2), (3) and (4).

2.20 Theorem: *(Double Angle Formulas) For all $x, y \in \mathbf{R}$ we have*

(1) $\sin 2x = 2\sin x\cos x$ *and* $\cos 2x = \cos^2 - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$*, and*

(2) $\cos^2 x = \dfrac{1+\cos 2x}{2}$ *and* $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

Proof: The proof is left as an exercise.

2.21 Theorem: *(Trigonometric Functions)*

(1) The function $f : [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos x$ is decreasing and bijective.
(2) The function $g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ given by $g(x) = \sin x$ is increasing and bijective.
(3) The function $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ given by $h(x) = \tan x$ is increasing and bijective.

Proof: We omit the proof.

2.22 Definition: The inverses of the functions f, g and h in the above theorem are called the **inverse cosine**, the **inverse sine**, and the **inverse tangent** functions. We write $f^{-1}(x) = \cos^{-1} x$, $g^{-1} = \sin^{-1} x$ and $h^{-1}(x) = \tan^{-1} x$. By the definition of the inverse function, we have

2.23 Definition: Let A and B be sets, let F be a field, let $c \in F$. Let $f : A \to F$ and $g : B \to F$. We define the functions cf , $f + g$, $f - g$, $f \cdot g$: $A \cap B \to F$ by

$$\begin{aligned}(cf)(x) &= c\,f(x)\\(f+g)(x) &= f(x)+g(x)\\(f-g)(x) &= f(x)-g(x)\\(f\cdot g)(x) &= f(x)g(x)\end{aligned}$$

for all $x \in A \cap B$, and for $C = \{x \in A \cap B \,|\, g(x) \neq 0\}$ we define $f/g : C \to F$ by

$$(f/g)(x) = f(x)/g(x)$$

for all $x \in C$.

2.24 Definition: A **polynomial function** over a field F is a function $f : F \to F$ which can be obtained from the functions 1 and x using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$ and $f \circ g$. In other words, a polynomial is a function of the form

$$f(x) = \sum_{i=0}^{n} c_i x^i = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

for some $n \in \mathbf{N}$ and some $c_i \in F$. The numbers c_i are called the **coefficients** of the polynomial and when $c_n \neq 0$ the number n is called the **degree** of the polynomial.

2.25 Definition: A **rational function** over a field F is a function $f : A \subseteq F \to F$ which can be obtained from the functions 1 and x using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$. In other words, a rational function is a function of the form

$$f(x) = p(x)/q(x)$$

for some polynomials p and q.

2.26 Definition: The functions 1, x, $x^{1/n}$ with $0 < n \in \mathbf{Z}$, e^x, $\ln x$, $\sin x$ and $\sin^{-1} x$, are called the **basic elementary functions**. An **elementary function** is any function $f : A \subseteq \mathbf{R} \to \mathbf{R}$ which can be obtained from the basic elementary functions using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$.

2.27 Example: The following functions are elementary

$$\begin{aligned}|x| &= \sqrt{x^2},\\\cos x &= \sin\left(x + \tfrac{\pi}{2}\right),\\\tan^{-1} x &= \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right),\\f(x) &= \frac{e^{\sqrt{x}+\sin x}}{\tan^{-1}(\ln x)}\end{aligned}$$

We shall see later that every elementary function is continuous in its domain, so any function which is discontinuos at a point in its domain cannot be elementary.