MATH 146 Linear Algebra 1, Lecture Notes by Stephen New

Chapter 7. Module Homomorphisms and Linear Maps

7.1 Definition: Let R be a ring and let U and V be R-modules. An (R-module) **homomorphism** from U to V is a map $L:U\to V$ such that

$$L(x+y)=L(x)+L(y)\ \text{ and }\ L(tx)=t\,L(x)$$

for all $x,y\in U$ and all $t\in R$. A bijective homomorphism from U to V is called an **isomorphism** from U to V, a homomorphism from U to U is called an **endomorphism** of U, and an isomorphism from U to U is called an **automorphism** of U. We say that U is **isomorphic** to V, and we write $U\cong V$, when there exists an isomorphism $L:U\to V$. We use the following notation

$$\begin{aligned}\text{Hom}(U,V)=\text{Hom}_R(U,V)&=\big\{L:U\to V\big|L\text{ is a homomorphism}\big\},\\ \text{Iso}(U,V)=\text{Iso}_R(U,V)&=\big\{L:U\to V\big|L\text{ is an isomorphism}\big\},\\ \text{End}(U)=\text{End}_R(U)&=\big\{L:U\to U\big|L\text{ is an endomorphism}\big\},\\ \text{Aut}(U)=\text{Aut}_R(U)&=\big\{L:U\to U\big|L\text{ is an automorphism}\big\}.\end{aligned}$$

For $L,M\in\text{Hom}(U,V)$ and $t\in R$ we define $L+M$ and tM by

$$(L+M)(x)=L(x)+M(x)\ \text{ and }\ (tL)(x)=t\,L(x).$$

Using these operations, if R is commutative then the set $\text{Hom}(U,V)$ is an R-module. For $L\in\text{Hom}(U,V)$, the **image** (or **range** of L and the **kernel** (or **null set**) of L are the sets

$$\begin{aligned}\text{Image}(L)=\text{Range}(L)=L(U)&=\big\{L(x)\big|x\in U\big\}\ \text{ and}\\ \text{Ker}(L)=\text{Null}(L)=L^{-1}(0)&=\big\{x\in U\big|L(x)=0\big\}.\end{aligned}$$

When F is a field and U and V are vector spaces over F, an F-module homomorphism from U to V is also called a **linear map** from U to V.

7.2 Note: For an R-module homomorphism $L:U\to V$ and for $x\in U$ we have $L(0)=0$ and $L(-x)=-L(x)$, and for $t_i\in R$ and $x_i\in U$ we have $L\Big(\sum\limits_{i=1}^n t_ix_i\Big)=\sum\limits_{i=1}^n t_i\,L(x_i)$.

7.3 Definition: When G and H are groups, a map $L:G\to H$ is called a **group homomorphism** when $L(xy)=L(x)L(y)$ for al $x,y\in G$. A **group isomorphism** is a bijective group homomorphism. When R and S are rings, a map $L:R\to S$ is called a **ring homomorphism** when $L(x+y)=L(x)+L(y)$ and $L(xy)=L(x)L(y)$ for all $x,y\in R$. A **ring isomorphism** is a bijective ring homomorphism. When R is a ring and U and V are R-algebras, a map $L:U\to V$ is called an R**-algebra homomorphism** when $L(x+y)=L(x)+L(y)$, $L(xy)=L(x)L(y)$ and $L(tx)=t\,L(x)$ for all $x,y\in U$ and all $t\in R$. An R**-algebra isomorphism** is a bijective R-algebra homomorphism.

7.4 Theorem: *Let R be a ring and let U, V and W be R-modules.*

(1) If $L:U\to V$ and $M:V\to W$ are homomorphisms then so is the composite $ML:U\to W$.
(2) If $L:U\to V$ is an isomorphism, then so is the inverse $L^{-1}:V\to U$.

Proof: Suppose that $L:U\to V$ and $M:V\to W$ are R-module homomorphisms. Then for all $x,y\in U$ and all $t\in R$ we have

$$M(L(x+y)) = M(L(x)+L(y)) = M(L(x)) + M(L(y)) \text{ and}$$
$$M(L(tx)) = M(t\,L(x)) = t\,M(L(x)).$$

Suppose that $L:U\to V$ is an isomorphism. Then given $u,v\in V$ and $t\in R$, if we let $x=L^{-1}(u)$ and $y=L^{-1}(v)$ then we have

$$L^{-1}(u+v) = L^{-1}\big(L(x)+L(y)\big) = L^{-1}\big(L(x+y)\big) = x+y = L^{-1}(u)+L^{-1}(v) \text{ and}$$
$$L^{-1}(tu) = L^{-1}\big(t\,L(x)\big) = L^{-1}\big(L(tx)\big) = tx = t\,L^{-1}(u).$$

7.5 Corollary: *Let R be a ring. Then isomorphism is an equivalence relation on the class of all R-modules. This means that for all R-modules U, V and W we have*

(1) $U\cong U$,
(2) if $U\cong V$ then $V\cong U$, and
(3) if $U\cong V$ and $V\cong W$ then $U\cong W$.

7.6 Corollary: *When R is a commutative ring and U is an R-module,* $\text{End}(U)$ *is a ring under addition and composition, hence also an R-algebra, and* $\text{Aut}(U)$ *is a group under composition.*

7.7 Theorem: *Let $L:U\to V$ be an R-algebra homomorphism.*

(1) If U_0 is a submodule of U then $L(U)$ is a submodule of V. In particular, the image of L is a submodule of V.
(2) If V_0 is a submodule of V then $L^{-1}(V_0)$ is a submodule of U. In particular, the kernel of L is a submodule of U.

Proof: To prove Part (1), let U_0 be a submodule of U. Let $u,v\in L(U_0)$ and let $t\in R$. Choose $x,y\in U_0$ with $L(x)=u$ and $L(y)=v$. Since $x+y\in U_0$ and $L(x+y)=L(x)+L(y)=u+v$, it follows that $u+v\in L(U_0)$. Since $tx\in U_0$ and $L(tx)=t\,L(x)=tu$, it follows that $tu\in L(U_0)$. Thus $L(U_0)$ is closed under the module operations and so it is a submodule of V.

To prove Part (2), let V_0 be a submodule of V. Let $x,y\in L^{-1}(V_0)$ and let $t\in R$. Let $u=L(x)\in V_0$ and $v=L(y)\in V_0$. Since $L(x+y)=L(x)+L(y)=u+v\in V_0$ it follows that $x+y\in L^{-1}(V_0)$. Since $L(tx)=t\,L(x)=Lu\in V_0$ it follows that $tx\in L^{-1}(V_0)$. Thus $L^{-1}(V_0)$ is closed under the module operations and so it is a sub algebra of U.

7.8 Theorem: *Let $L:U\to V$ be an R-module homomorphism. Then*

(1) L is surjective if and only if $\text{Range}(L)=V$, *and*
(2) L is injective if and only if $\text{Ker}(L)=\{0\}$.

Proof: Part (1) is simply a restatement of the definition of subjectivity and does not require proof. To Prove Part (2), we begin by remarking that since $L(0)=0$ we have $\{0\}\subseteq\text{Ker}(L)$. Suppose L is injective. Then $x\in\text{Ker}(L)\implies L(x)=0\implies L(x)=L(0)\implies x=0$ and so $\text{Ker}(L)=\{0\}$. Suppose, conversely, that $\text{Ker}(L)=\{0\}$. Then $L(X)=L(y)\implies L(x)-L(y)=0\implies L(x-y)=0\implies x-y\in\text{Ker}(L)=\{0\}\implies x-y=0\implies x=y$ and so L is injective.

7.9 Example: The maps

$$L : P_n(R) \to R^{n+1} \text{ given by } L\Big(\sum_{i=0}^{n} a_i x^i\Big) = (a_0, a_1, \cdots, a_n)$$

$$L : R[x] \to R^{\infty} \text{ given by } L\Big(\sum_{i=0}^{n} a_i x^i\Big) = (a_0, a_1, \cdots, a_n, 0, 0, \cdots)$$

$$L : R[[x]] \to R^{\omega} \text{ given by } L\Big(\sum_{i=0}^{\infty} a_i x^i\Big) = (a_0, a_1, a_2, \cdots)$$

are all R-algebra isomorphisms, so we have $P_n(R) \cong R^{n+1}$, $R[x] \cong R^{\infty}$ and $R[[x]] \cong R^{\omega}$.

7.10 Example: The map $L : M_{m\times n}(R) \to R^{m\cdot n}$ given by

$$L\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} = \big(a_{1,1}, a_{1,2}, \cdots, a_{1,n}, a_{2,1}, \cdots, a_{2,n}, \cdots, a_{m,1}, \cdots, a_{m,n}\big)$$

is an R-module isomorphism, so we have $M_{m\times n}(R) \cong R^{m\cdot n}$.

7.11 Example: Let A and B be sets with $|A| = |B|$, and let $g : A \to B$ be a bijection. Then the map $L : R^A \to R^B$ given by $L(f)(b) = f(g^{-1}(b))$, that is by $L(f) = fg^{-1}$, is an R-module isomorphism, and so we have $R^A \cong R^B$. In particular, if $|A| = n$ then we have $R^A \cong R^{\{1,2,\cdots,n\}} = R^n$, and if $|A| = \aleph_0$ then we have $R^A \cong R^{\{1,2,3,\cdots\}} = R^{\omega}$.

7.12 Example: Let $\mathcal{A} = (u_1, u_2, \cdots, u_n)$ be a finite ordered basis for a free R-module U. Then the map $\phi_{\mathcal{A}} : U \to R^n$ given by $\phi_{\mathcal{A}}(x) = [x]_{\mathcal{A}}$ is an R-module isomorphism, so we have $U \cong R^n$.

If $\mathcal{B} = (v_1, v_2, \cdots, v_n)$ is a finite ordered basis for another free R-module V, then the map $\phi_{\mathcal{B}}^{-1}\phi_{\mathcal{A}} : U \to V$ is an R-molule isomorphism, so we have $U \cong V$.

7.13 Example: Let R be a commutative ring. Let $\phi : \text{Hom}(R^n, R^m) \to M_{m\times n}(R)$ be the map given by $\phi(L) = [L] = \big(L(e_1), L(e_2), \cdots, L(e_n)\big) \in M_{m\times n}(R)$. Recall that the inverse of ϕ is the map $\psi : M_{m\times n}(R) \to \text{Hom}(R^n, R^m)$ given by $\psi(A) = L_A$ where $L_A(a) = Ax$. Note that ψ preserves the R-module operations because

$$\begin{aligned} \phi(A+B) &= L_{A+B} = L_A + L_B = \psi(A) + \psi(B) \text{ and} \\ \psi(tA) &= L_{tA} = t\,L_A = t\,\psi(A). \end{aligned}$$

Thus ϕ and ψ are R-module isomorphisms and we have $\text{Hom}(R^n, R^m) \cong M_{m\times n}(R)$. In the case that $m = n$, we also have

$$\psi(AB) = L_{AB} = L_A L_B = \psi(A)\psi(B)$$

and so the maps ϕ and ψ are in fact R-algebra isomorphisms so we have $\text{End}(R^n) \cong M_n(R)$ as R-algebras. By restricting ϕ and ψ to the invertible elements, we also obtain a group isomorphism $\text{Aut}(R^n) \cong GL_n(R)$.

7.14 Theorem: *Let R be a ring and let U and V be free R-modules. Then $U \cong V$ if and only if there exists a basis $\mathcal{A}$ for U and a basis $\mathcal{B}$ for V with $|\mathcal{A}| = |\mathcal{B}|$.*

Proof: Suppose that $U \cong V$. Let $\mathcal{A}$ be a basis for U and let $L : U \to V$ be an isomorphism. Let $\mathcal{B} = L(\mathcal{A}) = \big\{L(u)\big|u \in \mathcal{A}\big\}$. Since L is bijective we have $|\mathcal{A}| = |\mathcal{B}|$. Note that $\mathcal{B}$ spans V because given $y \in V$ we can choose $x \in U$ with $L(x) = y$, then write $x = \sum_{i=1}^{n} t_i u_i$ with $t_i \in R$ and $u_i \in \mathcal{A}$, and then we have

$$y = L(x) = L\Big(\sum_{i=1}^{n} t_i u_i\Big) = \sum_{i=1}^{n} t_i L(u_i) \in \text{Span}\,(\mathcal{B}).$$

It remains to show that $\mathcal{B}$ is linearly independent. Suppose that $\sum_{i=1}^{n} t_i v_i = 0$ where $t_i \in R$ and the v_i are distinct elements in $\mathcal{B}$. For each index i, choose $u_i \in \mathcal{A}$ with $L(u_i) = v_i$, and note that the elements u_i are distinct because L is bijective. We have

$$0 = \sum_{i=1}^{n} t_i v_i = \sum_{i=1}^{n} t_i L(u_i) = L\Big(\sum_{i=1}^{n} t_i u_i\Big).$$

Because L is injective, it follows that $\sum_{i=1}^{n} t_i u_i = 0$ and then, because $\mathcal{A}$ is linearly independent, it follows that each $t_i = 0$. Thus $\mathcal{B}$ is linearly independent, as required.

Conversely, suppose that $\mathcal{A}$ is a basis for U and $\mathcal{B}$ is a basis for V with $|\mathcal{A}| = |\mathcal{B}|$. Let $g : A \to B$ be a bijection. Define a map $L : U \to V$ as follows. Given $x \in U$, write $x = \sum_{i=1}^{n} t_i u_i$ where $t_i \in R$ and the u_i are distinct elements in $\mathcal{A}$, and then define $L(x) = \sum_{i=1}^{n} t_i g(u_i)$. Note that L is an R-module homomorphism because for $r \in R$ and for $x = \sum_{i=1}^{n} s_i u_i$ and $y = \sum_{i=1}^{n} t_i u_i$ (where we are using the same elements u_i in both sums with some of the coefficients equal to zero), we have

$$L(rx) = L\Big(\sum_{i=1}^{n} (rs_i)u_i\Big) = \sum_{i=1}^{n} rs_i g(u_i) = r\sum_{i=1}^{n} s_i g(u_i) = rL(x)\ \text{, and}$$

$$L(x+y) = L\Big(\sum_{i=1}^{n} (s_i + t_i)u_i\Big) = \sum_{i=1}^{n} (s_i + t_i)g(u_i) = \sum_{i=1}^{n} s_i g(u_i) + \sum_{i=1}^{n} t_i g(u_i) = L(x) + L(y).$$

Also note that L is bijective with inverse $M : V \to U$ given by $M\Big(\sum_{i=1}^{n} t_i v_i\Big) = \sum_{i=1}^{n} t_i g^{-1}(v_i)$, where $t_i \in R$ and the v_i are distinct elements in $\mathcal{B}$.

7.15 Corollary: *Let F be a field and let U and V be vector spaces over F. Then*

$$U \cong V \iff \dim(U) = \dim(V).$$

7.16 Remark: When U and V are modules over a commutative ring R, we have

$$U \cong V \iff \text{rank}(U) = \text{rank}(V),$$

but we have not built up enough machinery to prove this result.

7.17 Theorem: *Let R be a ring, let U be a free R-module, and let V be any R-module. Let $\mathcal{A}$ be basis for U and, for each $u \in \mathcal{A}$, let $v_u \in V$. Then there exists a unique R-module homomorphism $L : U \to V$ with $L(u) = v_u$ for all $u \in \mathcal{A}$.*

Proof: Note that if $L : U \to V$ is an R-module homomorphism with $L(u) = v_u$ for all $u \in U$, then for $t_i \in R$ and $u_i \in \mathcal{A}$ we have

$$L\Big(\sum_{i=1}^n t_i u_i\Big) = \sum_{i=1}^n t_i L(u_i) = \sum_{i=1}^n t_i v_{u_i}.$$

This shows that the map L is unique and must be given by the above formula.

To prove existence, we define $L : U \to V$ by $L\Big(\sum_{i=1}^n t_i u_i\Big) = \sum_{i=1}^n t_i v_{u_i}$ where $t_i \in R$ and $u_i \in \mathcal{A}$, and we note that L is an R-module homomorphism because for $x = \sum_{i=1}^n s_i u_i$ and $y = \sum_{i=1}^n t_i u_i$ (using the same elements u_i in both sums) and $r \in R$ we have

$$L(rx) = L\Big(\sum_{i=1}^n r s_i u_i\Big) = \sum_{i=1}^n r s_i v_{u_i} = r \sum_{i=1}^n s_i v_{u_i} = r\, L(x) \text{ , and}$$
$$L(x+y) = L\Big(\sum_{i=1}^n (s_i+t_i) u_i\Big) = \sum_{i=1}^n (s_i+t_i) v_{u_i} = \sum_{i=1}^n s_i v_{u_i} + \sum_{i=1}^n t_i v_{u_i} = L(x) + L(y).$$

7.18 Corollary: *Let R be a commutative ring, let U be a free R-module with basis $\mathcal{A}$ and let V be an R-module. Then the map $\phi : \mathrm{Hom}(U,V) \to V^{\mathcal{A}}$, given by $\phi(L)(u) = L(u)$ for all $u \in \mathcal{A}$, is an R-module isomorphism, and so we have $\mathrm{Hom}(U,V) \cong V^{\mathcal{A}}$.*

Proof: The above theorem states that the map ϕ is bijective, and we note that ϕ is an R-module homomorphism because for $L, M \in \mathrm{Hom}(U,V)$ and $t \in R$ we have

$$\phi(L+M)(u) = (L+M)(u) = L(u) + M(u) = \phi(L)(u) + \phi(M)(u) \text{ , and}$$
$$\phi(tL)(u) = (tL)(u) = t\, L(u) = t\phi(L)(u)$$

for all $u \in \mathcal{A}$ hence $\phi(L+M) = \phi(L) + \phi(M)$ and $\phi(tL) = t\phi(L)$.

7.19 Example: For a module U over a commutative ring R, the **dual module** of U is the R-module

$$U^* = \mathrm{Hom}(U,R).$$

By the above corollary, when U is a free module with basis $\mathcal{A}$. we have $U^* \cong R^{\mathcal{A}}$, and if $|\mathcal{A}| = n$ then we have $U^* \cong R^n \cong U$. When U is a vector space over a field F, U^* is called the **dual vector space** of U.

7.20 Definition: Let R be a commutative ring and let U and V be free R-modules with finite bases. Let $\mathcal{A}$ and $\mathcal{B}$ be finite ordered bases for U and V respectively, with $|\mathcal{A}| = n$ and $|\mathcal{B}| = m$. For $L \in \mathrm{Hom}(U,V)$, we define the **matrix** of L with respect to $\mathcal{A}$ and $\mathcal{B}$ to be the matrix

$$[L]_{\mathcal{B}}^{\mathcal{A}} = \big[\phi_{\mathcal{B}} L\, \phi_{\mathcal{A}}^{-1}\big] \in M_{m\times n}(R).$$

When $L \in \mathrm{End}(U)$ we write $[L]_{\mathcal{A}}$ for the matrix $[L]_{\mathcal{A}}^{\mathcal{A}} \in M_n(R)$.

7.21 Theorem: *Let R be a commutative ring. Let $\mathcal{A}$ and $\mathcal{B}$ be finite ordered bases for free R-modules U and V, respectively, with $|\mathcal{A}| = n$ and $|\mathcal{B}| = m$. Let $L \in \mathrm{Hom}(U,V)$. Then*

(1) $[L]_{\mathcal{B}}^{\mathcal{A}}$ is the matrix such hat $[L]_{\mathcal{B}}^{\mathcal{A}}[u]_{\mathcal{A}} = \big[L(u)\big]_{\mathcal{B}}$ for all $u \in U$, and
(2) if $\mathcal{A} = (u_1, u_2, \cdots, u_n)$ and $\mathcal{B} = (v_1, v_2, \cdots, v_m)$ then

$$[L]_{\mathcal{B}}^{\mathcal{A}} = \Big(\big[L(u_1)\big]_{\mathcal{B}}, \big[L(u_2)\big]_{\mathcal{B}}, \cdots, \big[L(u_n)_{\mathcal{B}}\big]\Big) \in M_{m\times n}(R).$$

Proof: Part (1) holds because for $u \in U$ and $x = \phi_{\mathcal{A}}(u) = [u]_{\mathcal{A}}$ we have

$$[L]_{\mathcal{B}}^{\mathcal{A}}[u]_{\mathcal{A}} = \big[\phi_{\mathcal{B}} L\, \phi_{\mathcal{A}}^{-1}\big]\phi_{\mathcal{A}}(u) = \big(\phi_{\mathcal{B}} L\, \phi_{\mathcal{A}}^{-1}\big)\big(\phi_{\mathcal{A}}(u)\big)\phi_{\mathcal{B}}\big(L(u)\big) = \big[L(u)\big]_{\mathcal{B}}.$$

Part (2) follows from Part (1) because for each index k, the k^{th} column of $[L]_{\mathcal{B}}^{\mathcal{A}}$ is

$$[L]_{\mathcal{B}}^{\mathcal{A}}(e_k) = [L]_{\mathcal{B}}^{\mathcal{A}}[u_k]_{\mathcal{A}} = \big[L(u_k)\big]_{\mathcal{B}}.$$

7.22 Theorem: *Let R be a commutative ring. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be finite ordered bases for free R-modules U, V and W, respectively.*

(1) For $L, M \in \mathrm{Hom}(U,V)$ and $t \in R$ we have $\big[L+M\big]_{\mathcal{B}}^{\mathcal{A}} = [L]_{\mathcal{B}}^{\mathcal{A}} + [M]_{\mathcal{B}}^{\mathcal{A}}$ and $\big[t\,L\big]_{\mathcal{B}}^{\mathcal{A}} = t\,[L]_{\mathcal{B}}^{\mathcal{A}}$.
(2) For $L \in \mathrm{Hom}(U,V)$ and $M \in \mathrm{Hom}(V,W)$ we have $\big[ML\big]_{\mathcal{C}}^{\mathcal{A}} = [M]_{\mathcal{C}}^{\mathcal{B}}[L]_{\mathcal{B}}^{\mathcal{A}}$.

Proof: We prove Part (2), leaving the (similar) proof of Part (1) as an exercise. Let $L \in \mathrm{Hom}(U,V)$ andn let $M \in \mathrm{Hom}(V,W)$. Say $|\mathcal{A}| = n$, $|\mathcal{B}| = m$ and $|\mathcal{C}| = l$. Let $x \in R^n$. Choose $u \in U$ with $[u]_{\mathcal{A}} = \phi_{\mathcal{A}}(u) = x$. Then

$$\big[ML\big]_{\mathcal{C}}^{\mathcal{A}}x = \big[ML\big]_{\mathcal{C}}^{\mathcal{A}}[u]_{\mathcal{A}} = \big[M(L(u))\big]_{\mathcal{C}} = \big[M\big]_{\mathcal{C}}^{\mathcal{B}}\big[L(u)\big]_{\mathcal{B}} = \big[M\big]_{\mathcal{C}}^{\mathcal{B}}\big[L\big]_{\mathcal{B}}^{\mathcal{A}}[u]_{\mathcal{A}} = \big[M\big]_{\mathcal{C}}^{\mathcal{B}}\big[L\big]_{\mathcal{B}}^{\mathcal{A}}x.$$

Since $\big[ML\big]_{\mathcal{C}}^{\mathcal{A}}x = \big[M\big]_{\mathcal{C}}^{\mathcal{B}}\big[L\big]_{\mathcal{B}}^{\mathcal{A}}x$ for all $x \in R^n$ it follows that $\big[ML\big]_{\mathcal{C}}^{\mathcal{A}} = \big[M\big]_{\mathcal{C}}^{\mathcal{B}}\big[L\big]_{\mathcal{B}}^{\mathcal{A}}$.

7.23 Corollary: *Let R be a commutative ring. Let $\mathcal{A}$ and $\mathcal{B}$ be finite ordered bases for free R-modules U and V. Then the map $\phi_{\mathcal{B}}^{\mathcal{A}} : \mathrm{Hom}(U,V) \to M_{m\times n}(R)$ given by $\phi_{\mathcal{B}}^{\mathcal{A}}(L) = [L]_{\mathcal{B}}^{\mathcal{A}}$ is an R-module isomorphism, and the map $\phi_{\mathcal{A}} : \mathrm{End}(U) \to M_n(R)$ given by $\phi_{\mathcal{A}}(L) = [L]_{\mathcal{A}}$ is an R-algebra isomorphism which restricts to a group isomorphism $\phi_{\mathcal{A}} : \mathrm{Aut}(U) \to GL_n(R)$.*

7.24 Corollary: *(Change of Basis) Let R be a commutative ring. Let U and V be free R-modules. Let $\mathcal{A}$ and $\mathcal{C}$ be two ordered bases for U with $|\mathcal{A}| = |\mathcal{C}| = n$ and let $\mathcal{B}$ and $\mathcal{D}$ be two ordered bases for V with $|\mathcal{B}| = |\mathcal{D}| = m$. For $L \in \mathrm{Hom}(U,V)$ and $u \in U$ we have*

$$[u]_{\mathcal{C}} = [I_U]_{\mathcal{C}}^{\mathcal{A}}[u]_{\mathcal{A}} \quad\text{and}\quad [L]_{\mathcal{D}}^{\mathcal{C}} = [I_V]_{\mathcal{D}}^{\mathcal{B}}[L]_{\mathcal{B}}^{\mathcal{A}}[I_U]_{\mathcal{A}}^{\mathcal{C}}$$

where I_U and I_V are the identity maps on U and V.

Proof: By Part (1) of Theorem 7.21 we have $[u]_{\mathcal{C}} = \big[I_U(u)\big]_{\mathcal{C}} = [I_U]_{\mathcal{C}}^{\mathcal{A}}[u]_{\mathcal{A}}$ and by Part (2) of Theorem 7.22

$$\big[L\big]_{\mathcal{D}}^{\mathcal{C}} = \big[I_V\,L\,I_U\big]_{\mathcal{D}}^{\mathcal{C}} = [I_V]_{\mathcal{D}}^{\mathcal{B}}[L]_{\mathcal{B}}^{\mathcal{A}}[I_U]_{\mathcal{A}}^{\mathcal{C}}.$$

7.25 Definition: Let $\mathcal{A} = (u_1, u_2, \cdots, u_n)$ and $\mathcal{B} = (v_1, v_2, \cdots, v_n)$ be two finite ordered bases for a module U over a commutative ring R. The matrix

$$[I]_{\mathcal{B}}^{\mathcal{A}} = \big([u_1]_{\mathcal{B}}, [u_2]_{\mathcal{B}}, \cdots, [u_n]_{\mathcal{B}}\big) \in M_n(R)$$

is called the **change of basis matrix** from $\mathcal{A}$ to $\mathcal{B}$. Note that $[I]_{\mathcal{B}}^{\mathcal{A}}$ is invertible with

$$\Big([I]_{\mathcal{B}}^{\mathcal{A}}\Big)^{-1} = [I]_{\mathcal{A}}^{\mathcal{B}}.$$

7.26 Note: Let $\mathcal{A}$ and $\mathcal{B}$ be two finite ordered bases, with $|\mathcal{A}| = |\mathcal{B}|$, for a free module U over a commutative ring R. For $L \in \text{End}(U)$, the Change of Basis Theorem gives

$$[L]_{\mathcal{B}} = [L]_{\mathcal{B}}^{\mathcal{B}} = [I]_{\mathcal{B}}^{\mathcal{A}}[L]_{\mathcal{A}}^{\mathcal{A}}[I]_{\mathcal{A}}^{\mathcal{B}}.$$

If we let $A = [L]_{\mathcal{A}}$ and $B = [L]_{\mathcal{B}}$ and $P = [I]_{\mathcal{B}}^{\mathcal{A}}$ then the formula becomes

$$B = PAP^{-1}.$$

7.27 Note: Given a finite ordered basis $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ for a free R-module U over a commutative ring R, and given an invertible matrix $P \in GL_n(R)$, if we choose $u_k \in U$ with $[u_k]_{\mathcal{B}} = Pe_k$, then $\mathcal{A} = \{u_1, u_2, \cdots, u_n\}$ is an ordered basis for U such that $[I]_{\mathcal{B}}^{\mathcal{A}} = P$. Thus every invertible matrix $P \in GL_n(R)$ is equal to a change of basis matrix.

7.28 Definition: Let R be a commutative ring. For $A, B \in M_n(R)$, we say that A and B are **similar**, and we write $A \sim B$, when $B = PAP^{-1}$ for some $P \in GL_n(R)$.

7.29 Note: Let R be a commutative ring, and let $A, B \in M_n(R)$ with $A \sim B$. Choose $P \in GL_n(R)$ so that $B = PAP^{-1}$. Then we have

$$\det(B) = \det\big(PAP^{-1}\big) = \det(P)\det(A)\det(P)^{-1} = \det(A).$$

Thus similar matrices have the same determinant.

7.30 Definition: Let F be a field and let U be a finite dimensional vector space over F. For $L \in \text{End}(U)$, we define the **determinant** of L to be

$$\det(L) = \det\big([L]_{\mathcal{A}}\big)$$

where $\mathcal{A}$ is any ordered basis for U.