MATH 146 Linear Algebra 1, Lecture Notes by Stephen New

# Chapter 5. The Dot and Cross Products in $\mathbf{R}^n$

**5.1 Definition:** Let $F$ be a field. For vectors $x,y\in F^n$ we define the **dot product** of $x$ and $y$ to be

$$x\cdot y=y^Tx=\sum_{i=1}^n x_iy_i\in F\,.$$

**5.2 Theorem:** *(Properties of the Dot Product) For all $x,y,z\in\mathbf{R}^n$ and all $t\in\mathbf{R}$ we have*
*(1) (Bilinearity)* $(x+y)\cdot z=x\cdot z+y\cdot z\ ,\ (tx)\cdot y=t(x\cdot y)$
$x\cdot(y+z)=x\cdot y+x\cdot z\ ,\ x\cdot(ty)=t(x\cdot y),$
*(2) (Symmetry) $x\cdot y=y\cdot x$, and*
*(3) (Positive Definiteness) $x\cdot x\geq 0$ with $x\cdot x=0$ if and only if $x=0$.*

Proof: The proof is left as an exercise.

**5.3 Definition:** For a vector $x\in\mathbf{R}^n$, we define the **length** (or **norm**) of $x$ to be

$$|x|=\sqrt{x\cdot x}=\sqrt{\sum_{i=1}^n {x_i}^2}.$$

We say that $x$ is a **unit vector** when $|x|=1$.

**5.4 Theorem:** *(Properties of Length) Let $x,y\in\mathbf{R}^n$ and let $t\in\mathbf{R}$. Then*
*(1) (Positive Definiteness) $|x|\geq 0$ with $|x|=0$ if and only if $x=0$,*
*(2) (Scaling) $|tx|=|t||x|$,*
*(3) $|x\pm y|^2=|x|^2\pm 2(x\cdot y)+|y|^2$.*
*(4) (The Polarization Identities) $x\cdot y=\frac{1}{2}\big(|x+y|^2-|x|^2-|y|^2\big)=\frac{1}{4}\big(|x+y|^2-|x-y|^2\big)$,*
*(5) (The Cachy-Schwarz Inequality) $|x\cdot y|\leq|x|\,|y|$ with $|x\cdot y|=|x|\,|y|$ if and only if the set $\{x,y\}$ is linearly dependent, and*
*(6) (The Triangle Inequality) $|x+y|\leq|x|+|y|$.*

Proof: We leave the proofs of Parts (1), (2) and (3) as an exercise, and we note that (4) follows immediately from (3). To prove part (5), suppose first that $\{x,y\}$ is linearly dependent. Then one of $x$ and $y$ is a multiple of the other, say $y=tx$ with $t\in\mathbf{R}$. Then

$$|x\cdot y|=|x\cdot(tx)|=|t(x\cdot x)|=|t|\,|x|^2=|x|\,|tx|=|x|\,|y|.$$

Suppose next that $\{x,y\}$ is linearly independent. Then for all $t\in\mathbf{R}$ we have $x+ty\neq 0$ and so

$$0\neq|x+ty|^2=(x+ty)\cdot(x+ty)=|x|^2+2t(x\cdot y)+t^2|y|^2.$$

Since the quadratic on the right is non-zero for all $t\in\mathbf{R}$, it follows that the discriminant of the quadratic must be negative, that is

$$4(x\cdot y)^2-4|x|^2|y|^2<0.$$

Thus $(x\cdot y)^2<|x|^2|y|^2$ and hence $|x\cdot y|<|x|\,|y|$. This proves part (5).

Using part (5) note that

$$|x+y|^2=|x|^2+2(x\cdot y)+|y|^2\leq|x+y|^2+2|x\cdot y|+|y|^2\leq|x|^2+2|x|\,|y|+|y|^2=\big(|x|+|y|\big)^2$$

and so $|x+y|\leq|x|+|y|$, which proves part (6).

**5.5 Definition:** For points $a, b \in \mathbf{R}^n$, we define the **distance** between $a$ and $b$ to be

$$\text{dist}(a,b) = |b - a|.$$

**5.6 Theorem:** *(Properties of Distance) Let $a, b, c \in \mathbf{R}^n$. Then*
*(1) (Positive Definiteness) $\text{dist}(a,b) \geq 0$ with $\text{dist}(a,b) = 0$ if and only if $a = b$,*
*(2) (Symmetry) $\text{dist}(a,b) = \text{dist}(b,a)$, and*
*(3) (The Triangle Inequality) $\text{dist}(a,c) \leq \text{dist}(a,b) + \text{dist}(b,c)$.*

Proof: The proof is left as an exercise.

**5.7 Definition:** For nonzero vectors $0 \neq x, y \in \mathbf{R}^n$, we define the **angle** between $x$ and $y$ to be

$$\theta(x,y) = \cos^{-1}\left(\frac{x \cdot y}{|x|\,|y|}\right) \in [0, \pi].$$

Note that $\theta(x,y) = \frac{\pi}{2}$ if and only if $x \cdot y = 0$. For vectors $x, y \in \mathbf{R}^n$, we say that $x$ and $y$ are **orthogonal** when $x \cdot y = 0$.

**5.8 Theorem:** *(Properties of Angle) Let $0 \neq x, y \in \mathbf{R}^n$. Then*

*(1) $\theta(x,y) \in [0,\pi]$ with* $\begin{cases} \theta(x,y) = 0 \text{ if and only if } y = tx \text{ for some } t > 0, \text{ and} \\ \theta(x,y) = \pi \text{ if and only if } y = tx \text{ for some } t < 0, \end{cases}$
*(2) (Symmetry) $\theta(x,y) = \theta(y,x)$,*
*(3) (Scaling)* $\theta(tx,y) = \theta(x,ty) = \begin{cases} \theta(x,y) & \text{if } 0 < t \in \mathbf{R}, \\ \pi - \theta(x,y) & \text{if } 0 > t \in \mathbf{R}, \end{cases}$
*(4) (The Law of Cosines) $|y - x|^2 = |x|^2 + |y|^2 - 2|x|\,|y| \cos\theta(x,y)$,*
*(5) (Pythagoras' Theorem) $\theta(x,y) = \frac{\pi}{2}$ if and only if $|y - x|^2 = |x|^2 + |y|^2$, and*
*(6) (Trigonometric Ratios) if $(y - x) \cdot x = 0$ then $\cos\theta(x,y) = \frac{|x|}{|y|}$ and $\sin\theta(x,y) = \frac{|y-x|}{|y|}$.*

Proof: The Law of Cosines follows from the identity $|y - x|^2 = |y|^2 - 2(y \cdot x) + |x|^2$ and the definition of $\theta(x,y)$. Pythagoras' Theorem is a special case of the Law of Cosines. We Prove Part (6). Let $0 \neq x, y \in \mathbf{R}^n$ and write $\theta = \theta(x,y)$. Suppose that $(y - x) \cdot x = 0$. Then we have $y \cdot x - x \cdot x = 0$ so that $x \cdot y = |x|^2$, and so we have

$$\cos\theta = \frac{x \cdot y}{|x|\,|y|} = \frac{|x|^2}{|x|\,|y|} = \frac{|x|}{|y|}\,.$$

Also, by Pythagoras' Theorem we have $|x|^2 + |y - x|^2 = |y|^2$ so that $|y|^2 - |x|^2 = |y - x|^2$, and so

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{|x|^2}{|y|^2} = \frac{|y|^2 - |x|^2}{|y|^2} = \frac{|y - x|^2}{|y|^2}\,.$$

Since $\theta \in [0, \pi]$ we have $\sin\theta \geq 0$, and so taking the square root on both sides gives

$$\sin\theta = \frac{|y - x|}{|y|}\,.$$

**5.9 Definition:** For points $a, b, c \in \mathbf{R}^n$ with $a \neq b$ and $b \neq c$ we define

$$\angle abc = \theta(b - a, c - b).$$

## Orthogonal Complement and Orthogonal Projection in $\mathbf{R}^n$

**5.10 Definition:** Let $F$ be a field and let $U$, $V$ and $W$ be subspaces of $F^n$. Recall that

$$U+V=\left\{u+v\big|u\in U,v\in V\right\}$$

is a subspace of $F^n$. We say that $W$ is the **internal direct sum** of $U$ with $V$, and we write $W=U\oplus V$, when $W=U+V$ and $U\cap V=\{0\}$. As an exercise, show that $W=U\oplus V$ if and only if for every $x\in W$ there exist unique vectors $u\in U$ and $v\in V$ with $x=u+v$.

**5.11 Definition:** Let $U\subseteq\mathbf{R}^n$ be a subspace. We define the **orthogonal complement** of $U$ in $\mathbf{R}^n$ to be

$$U^\perp=\left\{x\in\mathbf{R}^n\big|x\bullet u=0\text{ for all }u\in U\right\}.$$

**5.12 Theorem:** *(Properties of the Orthogonal Complement) Let $U\subseteq\mathbf{R}^n$ be a subspace, let $S\subseteq U$ and let $A\in M_{k\times n}(\mathbf{R})$. Then*
*(1) If $U=\mathrm{Span}\,(S)$ then $U^\perp=\left\{x\in\mathbf{R}^n\big|x\cdot u=0\text{ for all }u\in S\right\}$,*
*(2) $(\mathrm{Row}A)^T=\mathrm{Null}A$.*
*(3) $U^\perp$ is a vector space,*
*(4) $\dim(U)+\dim(U^\perp)=n$*
*(5) $U\oplus U^\perp=\mathbf{R}^n$,*
*(6) $(U^\perp)^\perp=U$,*
*(7) $(\mathrm{Null}A)^\perp=\mathrm{Row}A$.*

Proof: To prove part (1), let $T=\left\{x\in\mathbf{R}^n\big|x\bullet u=0\text{ for all }u\in S\right\}$. Note that $U^\perp\subseteq T$. Let $x\in T$. Let $u\in U=\mathrm{Span}\,(S)$, say $u=\sum\limits_{i=1}^n t_iu_i$ with each $t_i\in\mathbf{R}$ and each $u_i\in S$. Then $x\bullet u=x\bullet\sum\limits_{i=1}^n t_iu_i=\sum\limits_{i=1}^n t_i(x\bullet u_i)=0$. Thus $x\in U^\perp$ and so we have $T\subseteq U^\perp$.

To prove part (2), let $v_1,v_2,\cdots,v_n$ be the rows of $A$. Note that $Ax=\begin{pmatrix}x\bullet v_1\\ \vdots\\ x\bullet v_n\end{pmatrix}$ so we have $x\in\mathrm{Null}A\iff x\bullet v_i=0$ for all $i\iff x\in\mathrm{Span}\,\{v_1,v_2,\cdots,v_k\}^\perp=(\mathrm{Row}A)^\perp$ by part (1).

Part (3) follows from Part (2) since we can choose the matrix $A$ so that $U=\mathrm{Row}(A)$ and then we have $U^\perp=\mathrm{Null}(A)$ which is a vector space in $\mathbf{R}^n$.

Part (4) also follows from part (2) since if we choose $A$ so that $\mathrm{Row}A=U$ then we have $\dim(U)+\dim(U^\perp)=\dim\mathrm{Row}A+\dim(\mathrm{Row}A)^\perp=\dim\mathrm{Row}A+\dim\mathrm{Null}A=n$.

To prove part (5), in light of part (4), it suffices to show that $U\cap U^\perp=\{0\}$. Let $x\in U\cap U^\perp$. Since $x\in U^\perp$ we have $x\bullet u=0$ for all $u\in U$. In particular, since $x\in U$ we have $x\bullet x=0$, and hence $x=0$. Thus $U\cap U^\perp=\{0\}$ and so $U\oplus U^\perp=\mathbf{R}^n$.

To prove part (6), let $x\in U$. By the definition of $U^\perp$ we have $x\bullet v=0$ for all $v\in U^\perp$. By the definition of $(U^\perp)^\perp$ we see that $x\in(U^\perp)^\perp$. Thus $U\subseteq(U^\perp)^\perp$. By part (4) we know that $\dim U+\dim U^\perp=n$ and also that $\dim U^\perp+\dim(U^\perp)^\perp=n$. It follows that $\dim U=n-\dim U^\perp=\dim(U^\perp)^\perp$. Since $U\subseteq(U^\perp)^\perp$ and $\dim U=\dim(U^\perp)^\perp$ we have $U=(U^\perp)^\perp$, as required.

By parts (3) and (6) we have $(\mathrm{Null}A)^\perp=\left((\mathrm{Row}A)^\perp\right)^\perp=\mathrm{Row}A$, proving part (7).

**5.13 Definition:** For a subspace $U \subseteq \mathbf{R}^n$ and a vector $x \in \mathbf{R}^n$, we define the **orthogonal projection** of $x$ onto $U$, denoted by $\mathrm{Proj}_U(x)$, as follows. Since $\mathbf{R}^n = U \oplus U^\perp$, we can choose unique vectors $u, v \in \mathbf{R}^n$ with $u \in U$, $v \in U^\perp$ and $x = u + v$. We then define

$$\mathrm{Proj}_U(x) = u.$$

Note that since $U = (U^\perp)^\perp$, for $u$ and $v$ as above we have $\mathrm{Proj}_{U^\perp}(x) = v$. When $y \in \mathbf{R}^n$ and $U = \mathrm{Span}\,\{y\}$, we also write $\mathrm{Proj}_y(x) = \mathrm{Proj}_U(x)$ and $\mathrm{Proj}_{y^\perp}(x) = \mathrm{Proj}_{U^\perp}(x)$.

**5.14 Theorem:** *Let $U \subseteq \mathbf{R}^n$ be a subspace and let $x \in \mathbf{R}^n$. Then $\mathrm{Proj}_U(x)$ is the unique point in $U$ which is nearest to $x$.*

Proof: Let $u, v \in \mathbf{R}^n$ with $u \in U$, $v \in V$ and $u + v = x$ so that $\mathrm{Proj}_U(x) = u$. Let $w \in U$ with $w \neq u$. Since $v \in U^\perp$ and $u, w \in U$ we have $v \cdot u = v \cdot w = 0$ and so $v \cdot (w - u) = v \cdot w - v \cdot u = 0$. Thus we have

$$\begin{aligned}|x - w|^2 &= |u + v - w|^2 = |v - (w - u)|^2 = \big(v - (w - u)\big) \cdot \big(v - (w - u)\big) \\ &= |v|^2 - 2\,v \cdot (w - u) + |w - u|^2 = |v|^2 + |w - u|^2 = |x - u|^2 + |w - u|^2 .\end{aligned}$$

Since $w \neq u$ we have $|w - u| > 0$ and so $|x - w|^2 > |x - u|^2$. Thus $|x - w| > |x - u|$, that is $\mathrm{dist}(x, w) > \mathrm{dist}(x, u)$, so $u$ is the vector in $U$ nearest to $x$, as required.

**5.15 Theorem:** *For any matrix $A \in M_{n \times l}(\mathbf{R})$ we have $\mathrm{Null}(A^T A) = \mathrm{Null}(A)$ and $\mathrm{Col}(A^T A) = \mathrm{Col}(A^T)$ so that $\mathrm{nullity}(A^T A) = \mathrm{nullity}(A)$ and $\mathrm{rank}(A^T A) = \mathrm{rank}(A)$.*

Proof: If $x \in \mathrm{Null}(A)$ then $Ax = 0$ so $A^T A x = 0$ hence $x \in \mathrm{Null}(A^T A)$. This shows that $\mathrm{Null}(A) \subseteq \mathrm{Null}(A^T A)$. If $x \in \mathrm{Null}(A^T A)$ then we have $A^T A x = 0$ which implies that $|Ax|^2 = (Ax)^T (Ax) = x^T A^T A x = 0$ and so $Ax = 0$. This shows that $\mathrm{Null}(A^T A) \subseteq \mathrm{Null}(A)$. Thus we have $\mathrm{Null}(A^T A) = \mathrm{Null}(A)$. It then follows that

$$\mathrm{Col}(A^T) = \mathrm{Row}(A) = \mathrm{Null}(A)^\perp = \mathrm{Null}(A^T A)^\perp = \mathrm{Row}(A^T A) = \mathrm{Col}\big((A^T A)^T\big) = \mathrm{Col}(A^T A).$$

**5.16 Theorem:** *Let $A \in M_{n \times l}(\mathbf{R})$, let $U = \mathrm{Col}(A)$ and let $x \in \mathbf{R}^n$. Then*

*(1) the matrix equation $A^T A\, t = A^T x$ has a solution $t \in \mathbf{R}^l$, and for any solution $t$ we have*

$$\mathrm{Proj}_U(x) = At,$$

*(2) if $\mathrm{rank}(A) = l$ then $A^T A$ is invertible and*

$$\mathrm{Proj}_U(x) = A(A^T A)^{-1} A^T x.$$

Proof: Note that $U^\perp = (\mathrm{Col} A)^\perp = \mathrm{Row}(A^T)^\perp = \mathrm{Null}(A^T)$. Let $u, v \in \mathbf{R}^n$ with $u \in U$, $v \in U^\perp$ and $u + v = x$ so that $\mathrm{Proj}_U(x) = u$. Since $u \in U = \mathrm{Col} A$ we can choose $t \in \mathbf{R}^l$ so that $u = At$. Then we have $x = u + v = At + v$. Multiply by $A^T$ to get $A^T = A^T A t + A^T v$. Since $v \in U^\perp = \mathrm{Null}(A^T)$ we have $A^T v = 0$ so $A^T A\, t = A^T x$. Thus the matrix equation $A^T A\, t = A^T x$ does have a solution $t \in \mathbf{R}^l$.

Now let $t \in \mathbf{R}^l$ be any solution to $A^T A\, t = A^t x$. Let $u = At$ and $v = x - u$. Note that $x = u + v$, $u = At \in \mathrm{Col}(A) = U$, and $A^T v = A^T (x - u) = A^T (x - At) = A^T x - A^T A\, t = 0$ so that $v \in \mathrm{Null}(A^T) = U^\perp$. Thus $\mathrm{Proj}_U(x) = u = At$, proving part (1).

Now suppose that $\mathrm{rank}(A) = l$. Since $A^T A \in M_{l \times l}(\mathbf{R})$ with $\mathrm{rank}(A^T A) = \mathrm{rank}(A) = l$, the matrix $A^T A$ is invertible. Since $A^T A$ is invertible, the unique solution $t \in \mathbf{R}^l$ to the matrix equation $A^T A\, t = A^T x$ is the vector $t = (A^T A)^{-1} A^T x$, and so from Part (1) we have $\mathrm{Proj}_U(x) = At = A(A^T A)^{-1} A^T x$, proving Part (2).

## The Volume of a Parallelotope

**5.17 Definition:** Given vectors $u_1, u_2, \cdots, u_k \in \mathbf{R}^n$, we define the **parallelotope** on $u_1, \cdots, u_k$ to be the set

$$P(u_1, \cdots, u_k) = \left\{ \sum_{j=1}^{k} t_i u_i \Big| 0 \leq t_i \leq 1 \text{ for all } i \right\}.$$

We define the **volume** of this parallelotope, denoted by $V(u_1, \cdots, u_k)$, recursively by $V(u_1) = |u_1|$ and

$$V(u_1, \cdots, u_k) = V(u_1, \cdots, u_{k-1}) \big| \text{Proj}_{U^\perp}(u_k) \big|$$

where $U = \text{Span}\{u_1, \cdots, u_{k-1}\}$.

**5.18 Theorem:** *Let $u_1, \cdots, u_k \in \mathbf{R}^n$ and let $A = (u_1, \cdots, u_k) \in M_{n \times k}(\mathbf{R})$. Then*

$$V(u_1, \cdots, u_n) = \sqrt{\det(A^T A)}.$$

Proof: We prove the theorem by induction on $k$. Note that when $k = 1$, $u_1 \in \mathbf{R}^n$ and $A = u_1 \in M_{n \times 1}(\mathbf{R})$, we have $V(u_1) = |u_1| = \sqrt{u_1 \cdot u_1} = \sqrt{{u_1}^T u_1} = \sqrt{A^T A}$, as required. Let $k \geq 2$ and suppose, inductively, that when $A = (u_1, \cdots, u_{k-1}) \in M_{n \times k-1}$ we have $\det(A^T A) > 0$ and $V(u_1, \cdots, u_{k-1}) = \sqrt{\det(A^T A)}$. Let $B = (u_1, \cdots, u_k) = (A, u_k)$. Let $U = \text{Span}\{u_1, \cdots, u_{k-1}\} = \text{Col}(A)$. Let $v = \text{Proj}_U(u_k)$ and $w = \text{Proj}_{U^\perp}(u_k)$. Note that $v \in U = \text{Col}(A)$ and $w \in U^\perp = \text{Null}(A^T)$. Then we have $u_k = v + w$ so that $B = (A, v + w)$. Since $v \in \text{Col}(A)$, the matrix $B$ can be obtained from the matrix $(A, w)$ by performing elementary column operations of the type $C_k \mapsto C_k + tC_i$. Let $E$ be the product of the elementary matrices corresponding to these column operations, and note that $B = (A, v + w) = (A, w)E$. Since the row operations $C_k \mapsto C_k + tC_i$ do not alter the determinant, $E$ is a product of elementary matrices of determinant 1, so we have $\det(E) = 1$. Since $\det(E) = 1$ and $w \in \text{Null}(A^T)$ we have

$$\begin{aligned}
\det(B^T B) &= \det\Big(E^T (A, w)^T (A, w) E\Big) = \det\left( \begin{pmatrix} A^T \\ w^T \end{pmatrix} (A \; w) \right) \\
&= \det \begin{pmatrix} A^T A & A^T w \\ w^T A & w^T w \end{pmatrix} = \begin{pmatrix} A^T A & 0 \\ 0 & |w|^2 \end{pmatrix} = \det(A^T A)\, |w|^2.
\end{aligned}$$

By the induction hypothesis, we can take the square root on both sides to get

$$\sqrt{\det(B^T B)} = \sqrt{\det(A^T A)}\, |w| = V(u_1, \cdots, u_{k-1})\ |w| = V(u_1, \cdots, u_k).$$

## The Cross Product in $\mathbf{R}^n$

**5.19 Definition:** Let $F$ be a field. For $n \geq 2$ we define the **cross product**

$$X : \prod_{k=1}^{n-1} F^n \to F^n$$

as follows. Given vectors $u_1, u_2, \cdots, u_{n-1} \in F^n$, we define $X(u_1, u_2, \cdots, u_{n-1}) \in F^n$ to be the vector with entries

$$X(u_1, u_2, \cdots, u_{n-1})_j = (-1)^{n+j} |A^{(j)}|$$

where $A^{(j)} \in M_{n-1}(F)$ is the matrix obtained from $A = (u_1, u_2, \cdots, u_{n-1}) \in M_{n \times n-1}(F)$ by removing the $j^{\text{th}}$ row. Given a vector $u \in F^2$ we write $X(u)$ as $u^\times$, and given two vectors $u, v \in F^3$ we write $X(u, v)$ as $u \times v$.

**5.20 Example:** Given $u \in F^2$ we have

$$u^\times = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^\times = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}.$$

Given $u, v \in F^3$ we have

$$u \times v = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \\ -\begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \\ \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}.$$

**5.21 Note:** Because the determinant is $n$-linear, alternating and skew-symmetric, it follows that the cross product is $(n-1)$-linear, alternating and skew-symmetric. Thus for $u_i, v, w \in F^n$ and $t \in F$ we have

(1) $X(u_1, \cdots, v + w, \cdots, u_{n-1}) = X(u_1, \cdots, v, \cdots, u_{n-1}) + X(u_1, \cdots, w, \cdots, u_{n-1})$,
(2) $X(u_1, \cdots, t\, u_k, \cdots, u_{n-1}) = t\, X(u_1, \cdots, u_k, \cdots, u_{n-1})$,
(3) $X(u_1, \cdots, u_k, \cdots, u_l, \cdots, u_{n-1}) = -X(u_1, \cdots, u_l, \cdots, u_k, \cdots, u_{n-1})$.

**5.22 Definition:** Recall that for $u_1, \cdots, u_n \in \mathbf{R}^n$, the set $\{u_1, \cdots, u_n\}$ is a basis for $\mathbf{R}^n$ if and only if $\det(u_1, \cdots, u_n) \neq 0$. For an ordered basis $\mathcal{A} = (u_1, \cdots, u_n)$, we say that $\mathcal{A}$ is **positively oriented** when $\det(u_1, \cdots, u_n) > 0$ and we say that $\mathcal{A}$ is **negatively oriented** when $\det(u_1, \cdots, u_n) < 0$.

**5.23 Theorem:** *Let $u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1}, w \in \mathbf{R}^n$. Then*

*(1) $X(u_1, \cdots, u_{n-1}) \cdot w = \det(u_1, \cdots, u_{n-1}, w)$,*
*(2) $X(u_1, \cdots, u_{n-1}) = 0$ if and only if $\{u_1, \cdots, u_{n-1}\}$ is linearly dependent.*
*(3) When $w = X(u_1, \cdots, u_{n-1}) \neq 0$ we have $\det(u_1, \cdots, u_{n-1}, w) > 0$ so that the $n$-tuple $(u_1, \cdots, u_{n-1}, w)$ is a positively oriented basis for $\mathbf{R}^n$,*
*(4) $X(u_1, \cdots, u_{n-1}) \cdot X(v_1, \cdots, v_{n-1}) = \det(A^T B)$ where $A = (u_1, \cdots, u_{n-1}) \in M_{n \times n-1}(\mathbf{R})$ and $B = (v_1, \cdots, v_{n-1}) \in M_{n \times n-1}(\mathbf{R})$, and*
*(5) $\big|X(u_1, \cdots, u_{n-1})\big|$ is equal to the volume of the parallelipiped on $u_1, \cdots, u_{n-1}$.*

Proof: I may include a proof later.