Chapter 9. Cardinality

9.1 Definition: Let X and Y be sets and let $f : X \to Y$. Recall that the **domain** of f and the **range** of f are the sets

$$\text{Domain}(f) = X \ , \ \text{Range}(f) = f(X) = \big\{ f(x) \big| x \in X \big\} .$$

For $A \subseteq X$, the **image** of A under f is the set

$$f(A) = \big\{ f(x) \big| x \in A \big\}.$$

For $B \subseteq Y$, the **inverse image** of B under f is the set

$$f^{-1}(B) = \big\{ x \in X \big| f(x) \in B \big\}.$$

9.2 Definition: Let X, Y and Z be sets, let $f : X \to Y$ and let $g : Y \to Z$. We define the **composite** function $g \circ f : X \to Z$ by $(g \circ f)(x) = g\big(f(x)\big)$ for all $x \in X$.

9.3 Definition: We say that f is **injective** (or **one-to-one**, written as $1:1$) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$. We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is bijective, we define the **inverse** of f to be the function $f^{-1} : Y \to X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1}, and in this case we have $(f^{-1})^{-1} = f$.

9.4 Theorem: *Let $f : X \to Y$ and let $g : Y \to Z$. Then*
(1) if f and g are both injective then so is $g \circ f$,
(2) if f and g are both surjective then so is $g \circ f$, and
(3) if f and g are both invertible then so is $g \circ f$, and in this case $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: To prove Part (1), suppose that f and g are both injective. Let $x_1, x_2 \in X$. If $g(f(x_1)) = g(f(x_2))$ then since g is injective we have $f(x_1) = f(x_2)$, and then since f is injective we have $x_1 = x_2$. Thus $g \circ f$ is injective.

To prove Part (2), suppose that f and g are surjective. Given $z \in Z$, since g is surjective we can choose $y \in Y$ so that $g(y) = z$, then since f is surjective we can choose $x \in X$ so that $f(x) = y$, and then we have $g(f(x)) = g(y) = z$. Thus $g \circ f$ is surjective.

Finally, note that Part (3) follows from Parts (1) and (2).

9.5 Definition: For a set X, we define the **identity function** on X to be the function $I_X : X \to X$ given by $I_X(x) = x$ for all $x \in X$. Note that for $f : X \to Y$ we have $f \circ I_X = f$ and $I_Y \circ f = f$.

9.6 Definition: Let X and Y be sets and let $f : X \to Y$. A **left inverse** of f is a function $g : Y \to X$ such that $g \circ f = I_X$. Equivalently, a function $g : Y \to X$ is a left inverse of f when $g\big(f(x)\big) = x$ for all $x \in X$. A **right inverse** of f is a function $h : Y \to X$ such that $f \circ h = I_Y$. Equivalently, a function $h : Y \to X$ is a right inverse of f when $f\big(h(y)\big) = y$ for all $y \in Y$.

9.7 Theorem: *Let X and Y be nonempty sets and let $f : X \to Y$. Then*
(1) f is injective if and only if f has a left inverse,
(2) f is surjective if and only if f has a right inverse, and
(3) f is bijective if and only if f has a left inverse g and a right inverse h, and in this case we have $g = h = f^{-1}$.

Proof: To prove Part (1), suppose first that f is injective. Since $X \neq \emptyset$ we can choose $a \in X$ and then define $g : Y \to X$ as follows: if $y \in \text{Range}(f)$ then (using the fact that f is 1:1) we define $g(y)$ to be the unique element $x_y \in X$ with $f(x_y) = y$, and if $y \notin \text{Range}(f)$ then we define $g(y) = a$. Then for every $x \in X$ we have $y = f(x) \in \text{Range}(f)$, so $g(y) = x_y = x$, that is $g\big(f(x)\big) = x$. Conversely, if f has a left inverse, say g, then f is 1:1 since for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = g\big(f(x_1)\big) = g\big(f(x_2)\big) = x_2$.

To prove Part (2), suppose first that f is onto. For each $y \in Y$, choose $x_y \in X$ with $f(x_y) = y$, then define $g : X \to Y$ by $g(y) = x_y$ (we need the Axiom of Choice for this). Then g is a right inverse of f since for every $y \in Y$ we have $f\big(g(y)\big) = f(x_y) = y$. Conversely, if f has a right inverse, say g, then f is onto since given any $y \in Y$ we can choose $x = g(y)$ and then we have $f(x) = f\big(g(y)\big) = y$.

To prove Part (3), suppose first that f is bijective. The inverse function $f^{-1} : Y \to X$ is a left inverse for f because given $x \in X$ we can let $y = f(x)$ and then $f^{-1}(y) = x$ so that $f^{-1}\big(f(x)\big) = f^{-1}(y) = x$. Similarly, f^{-1} is a right inverse for f because given $y \in Y$ we can let x be the unique element in X with $y = f(x)$ and then we have $x = f^{-1}(y)$ so that $f\big(f^{-1}(y)\big) = f(x) = y$. Conversely, suppose that g is a left inverse for f and h is a right inverse for f. Since f has a left inverse, it is injective by Part (1). Since f has a right inverse, it is surjective by Part (2). Since f is injective and surjective, it is bijective. As shown above, the inverse function f^{-1} is both a left inverse and a right inverse. Finally, note that $g = f^{-1} = h$ because for all $y \in Y$ we have

$$g(y) = g\Big(f\big(f^{-1}(y)\big)\Big) = f^{-1}(y) = f^{-1}\Big(f\big(h(y)\big)\Big) = h(y)\,.$$

9.8 Corollary: *Let X and Y be sets. Then there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$.*

Proof: Suppose $f : X \to Y$ is an injective map. Then f has a left inverse. Let g be a left inverse of f. Since $g \circ f = I_X$, we see that f is a right inverse of g. Since g has a right inverse, g is surjective. Thus there is a surjective map $g : Y \to X$. Similarly, if $g : Y \to X$ is surjective, then it has a right inverse $f : X \to Y$ which is injective.

9.9 Definition: Let A and B be sets. We say that A and B have the **same cardinality**, and we write $|A| = |B|$, when there exists a bijective map $f : A \to B$ (or equivalently when there exists a bijective map $g : Y \to X$). We say that the cardinality of A is **less than or equal to** the cardinality of B, and we write $|A| \leq |B|$, when there exists an injective map $f : A \to B$ (or equivalently when there exists a surjective map $g : Y \to X$). We say that the cardinality of A is **less than** the cardinality of B, and we write $|A| < |B|$, when $|A| \leq |B|$ and $|A| \neq |B|$, (that is when there exists an injective map $f : A \to B$ but there does not exist a bijective map $g : A \to B$). We also write $|A| \geq |B|$ when $|B| \leq |A|$ and $|A| > |B|$ when $|B| < |A|$.

9.10 Example: The map $f : \mathbf{N} \to 2\mathbf{N}$ given by $f(k) = 2k$ is bijective, so $|2\mathbf{N}| = |N|$. The map $g : \mathbf{N} \to \mathbf{Z}$ given by $g(2k) = k$ and $g(2k+1) = -k-1$ for $k \in \mathbf{N}$ is bijective, so we have $|\mathbf{Z}| = |\mathbf{N}|$. The map $h : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ given by $h(k,l) = 2^k(2l+1) - 1$ is bijective, so we have $|\mathbf{N} \times \mathbf{N}| = |\mathbf{N}|$.

9.11 Theorem: *For all sets A, B and C,*

(1) $|A| = |A|$,
(2) if $|A| = |B|$ then $|B| = |A|$,
(3) if $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$,
(4) $|A| \leq |B|$ if and only if ($|A| = |B|$ or $|A| < |B|$), and
(5) if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$.

Proof: Part (1) holds because the identity function $I_A : A \to A$ is bijective. Part (2) holds because if $f : A \to B$ is bijective then so is $f^{-1} : B \to A$. Part (3) holds because if $f : A \to B$ and $g : B \to C$ are bijective then so is the composite $g \circ f : A \to C$. The rest of the proof is left as an exercise.

9.12 Definition: Let A be a set. For each $n \in \mathbf{N}$, let $S_n = \{0, 1, 2, \cdots, n-1\}$. For $n \in \mathbf{N}$, we say that the cardinality of A is equal to n, or that A **has** n **elements**, and we write $|A| = n$, when $|A| = |S_n|$. We say that A is **finite** when $|A| = n$ for some $n \in \mathbf{N}$. We say that A is **infinite** when A is not finite. We say that A is **countable** when $|A| = |\mathbf{N}|$.

9.13 Note: When a set A is finite with $|A| = n$, and when $f : A \to S_n$ is a bijection, if we let $a_k = f^{-1}(k)$ for each $k \in S_n$ then we have $A = \{a_0, a_1, \cdots, a_{k-1}\}$ with the elements a_k distinct. Conversely, if $A = \{a_0, a_1, \cdots, a_{k-1}\}$ with the elements a_k all distinct, then we define a bijection $f : A \to S_n$ by $f(a_k) = k$. Thus we see that A is finite with $|A| = n$ if and only if A is of the form $A = \{a_0, a_1, \cdots, a_{n-1}\}$ with the elements a_k all distinct. Similarly, a set A is countable if and only if A is of the form $A = \{a_0, a_1, a_2, \cdots\}$ with the elements a_k all distinct.

9.14 Note: For $n \in \mathbf{N}$, if A is a finite set with $|A| = n + 1$ and $a \in A$ then $|A \setminus \{a\}| = n$. Indeed, if $A = \{a_0, a_1, \cdots, a_n\}$ with the elements a_i distinct, and if $a = a_k$ so that we have $A \setminus \{a\} = \{a_0, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n\}$, then we can define a bijection $f : S_n \to A \setminus \{a\}$ by $f(i) = a_i$ for $0 \leq i < k$ and $f(i) = a_{i+1}$ for $k \leq i < n$.

9.15 Theorem: *Let A be a set. Then the following are equivalent.*

(1) A is infinite.
(2) A contains a countable subset.
(3) $|\mathbf{N}| \leq |A|$
(4) There exists a map $f : A \to A$ which is injective but not surjective.

Proof: To prove that (1) implies (2), suppose that A is infinite. Since $A \neq \emptyset$ we can choose an element $a_0 \in A$. Since $A \neq \{a_0\}$ we can choose an element $a_1 \in A \setminus \{a_0\}$. Since $A \neq \{a_0, a_1\}$ we can choose $a_3 \in A \setminus \{a_0, a_1\}$. Continue this procedure: having chosen distinct elements $a_0, a_1, \cdots, a_{n-1} \in A$, since $A \neq \{a_0, a_1, \cdots, a_{n-1}\}$ we can choose $a_n \in A \setminus \{a_0, a_1, \cdots, a_{n-1}\}$. In this way, we obtain a countable set $\{a_0, a_1, a_2, \cdots\} \subseteq A$.

Next we show that (2) is equivalent to (3). Suppose that A contains a countable subset, say $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with the element a_i distinct. Since the a_i are distinct, the map $f : \mathbf{N} \to A$ given by $f(k) = a_k$ is injective, and so we have $|\mathbf{N}| \leq |A|$. Conversely, suppose that $|\mathbf{N}| \leq |A|$, and chose an injective map $f : \mathbf{N} \to A$. Considered as a map from $\mathbf{N}$ to $f(\mathbf{N})$, f is bijective, so we have $|\mathbf{N}| = |f(\mathbf{N})|$ hence $f(\mathbf{N})$ is a countable subset of A.

Next, let us show that (2) implies (4). Suppose that A has a countable subset, say $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with the element a_i distinct. Define $f : A \to A$ by $f(a_k) = a_{k+1}$ for all $k \in \mathbf{N}$ and by $f(b) = b$ for all $b \in A \setminus \{a_0, a_1, a_2, \cdots\}$. Then f is injective but not surjective (the element a_0 is not in the range of f).

Finally, to prove that (4) implies (1) we shall prove that if A is finite then every injective map $f : A \to A$ is surjective. We prove this by induction on the cardinality of A. The only set A with $|A| = 0$ is the set $A = \emptyset$, and then the only function $f : A \to A$ is the empty function, which is surjective. Since that base case may appear too trivial, let us consider the next case. Let $n = 1$ and let A be a set with $|A| = 1$, say $A = \{a\}$. The only function $f : A \to A$ is the function given by $f(a) = a$, which is surjective. Let $n \geq 1$ and suppose, inductively, that for every set A with $|A| = n$, every injective map $f : A \to A$ is surjective. Let B be a set with $|B| = n+1$ and let $g : B \to B$ be injective. Suppose, for a contradiction, that g is not surjective. Choose an element $b \in B$ which is not in the range of g so that we have $g : B \to B \setminus \{b\}$. Let $A = B \setminus \{b\}$ and let $f : A \to A$ be given by $f(x) = g(x)$ for all $x \in A$. Since $g : B \to A$ is injective and $f(x) = g(x)$ for all $x \in A$, f is also injective. Again since g is injective, there is no element $x \in B \setminus \{b\}$ with $g(x) = g(b)$, so there is no element $x \in A$ with $f(x) = g(b)$, and so f is not surjective. Since $|A| = n$ (by the above note), this contradicts the induction hypothesis. Thus f must be surjective. By the Principle of Induction, for every $n \in \mathbf{N}$ and for every set A with $|A| = n$, every injective function $f : A \to A$ is surjective.

9.16 Corollary: *Let A and B be sets.*

(1) If A is countable then A is infinite.
(2) When $|A| \leq |B|$, if B is finite then so is A (equivalently if A is infinite then so is B).
(3) If $|A| = n$ and $|B| = m$ then $|A| = |B|$ if and only if $n = m$.
(4) If $|A| = n$ and $|B| = m$ then $|A| \leq |B|$ if and only if $n \leq m$.
(5) When one of the two sets A and B is finite, if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Proof: Part (1) is immediate: if A is countable then A contains a countable subset (itself), so A is infinite, by Theorem 4.15.

To prove Part (2), suppose that $|A| \leq |B$ and that $|A|$ is infinite. Since A is infinite, we have $|\mathbf{N}| \leq |A|$ (by Theorem 4.15). Since $|\mathbf{N}| \leq |A|$ and $|A| \leq |B|$ we have $|\mathbf{N}| \leq |B|$ (by Theorem 4.11). Since $|\mathbf{N}| \leq |B|$, B is infinite (by Theorem 4.15 again).

To Prove Part (3), suppose that $|A| = n$ and $|B| = m$. If $n = m$ then we have $S_n = S_m$ and so $|A| = |S_n| = |S_m| = |B|$. Conversely, suppose that $|A| = |B|$. Suppose, for a contradiction, that $n \neq m$, say $n > m$, and note that $S_m \subsetneqq S_n$. Since $|A| = |B|$ we have $|S_n| = |A| = |B| = |S_m|$ so we can choose a bijection $f : S_n \to S_m$. Since $S_m \subsetneqq S_n$, we can consider f as a function $f : S_n \to S_n$ which is injective but not surjective. This contradicts Theorem 4.16, and so we must have $n = m$. This proves Part (3).

To prove Part (4), we again suppose that $|A| = n$ and $|B| = m$. If $n \leq m$ then $S_n \subseteq S_m$ so the inclusion map $I : S_n \to S_m$ is injective and we have $|A| = |S_n| \leq |S_m| = |B|$. Conversely, suppose that $|A| \leq |B|$ and suppose, for a contradiction, that $n > m$. Since $|A| \leq |B|$ we have $|S_n| = |A| \leq |B| = |S_m|$ so we can choose an injective map $f : S_n \to S_m$. Since $n > m$ we have $S_m \subsetneqq S_n$ so we can consider f as a map $f : S_n \to S_n$, and this map is injective but not surjective. This contradicts Theorem 2.16, and so $n \leq m$.

Finally, to prove Part (5) we suppose that one of the two sets A and B is finite, and that $|A| \leq |B|$ and $|B| \leq |A|$. If A is finite then, since $|B| \leq |A|$, Part (2) implies that B is finite. If B is finite then, since $|A| \leq |B|$, Part (2) implies that A is finite. Thus, in either case, we see that A and B are both finite. Since A and B are both finite with $|A| \leq |B|$ and $|B| \leq |A|$, we must have $|A| = |B|$ by Parts (3) and (4).

9.17 Theorem: *Let A be a set. Then $|A| \leq |\mathbf{N}|$ if and only if A is finite or countable.*

Proof: First we claim that every subset of $\mathbf{N}$ is either finite or countable. Let $A \subseteq \mathbf{N}$ and suppose that A is not finite. Since $A \neq \emptyset$, we can set $a_0 = \min A$ (using the Well-Ordering Property of $\mathbf{N}$). Note that $\{0, 1, \cdots, a_0\} \cap A = \{a_0\}$. Since $A \neq \{a_0\}$ (so the set $A \setminus \{a_0\}$ is nonempty) we can set $a_1 = \min A \setminus \{a_0\}$. Then we have $a_0 < a_1$ and $\{0, 1, 2, \cdots, a_1\} \cap A = \{a_0, a_1\}$. Since $A \neq \{a_0, a_1\}$ we can set $a_2 = \min A \setminus \{a_0, a_1\}$. Then we have $a_0 < a_1 < a_2$ and $\{0, 1, 2, \cdots, a_3\} \cap A = \{a_0, a_1, a_2\}$. We continue the procedure: having chosen $a_0, a_1, \cdots, a_{n-1} \in A$ with $a_0 < a_1 < \cdots < a_{n-1}$ such that $A \cap \{0, 1, \cdots, a_{n-1}\} = \{a_0, a_1, \cdots, a_{n-1}\}$, since $A \neq \{a_0, a_1, \cdots, a_{n-1}\}$ we can set $a_n = \min A \setminus \{a_0, a_1, \cdots, a_{n-1}\}$, and then we have $a_0 < a_1 < \cdots < a_{n-1} < a_n$ and $A\{0, 1, 2, \cdots, a_n\} \cap A = \{a_0, a_1, \cdots, a_n\}$. In this way, we obtain a countable set $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with $a_0 < a_1 < a_2 < \cdots$ with the property that for all $m \in \mathbf{N}$, $\{0, 1, 2, \cdots, a_m\} \cap A = \{a_0, a_1, \cdots, a_m\}$. Since $0 \leq a_0 < a_1 < a_2 < \cdots$, it follows (by induction) that $a_k \geq k$ for all $k \in \mathbf{N}$. It follows in turn that $A \subseteq \{a_0, a_1, a_2 \cdots\}$ because given $m \in A$, since $m \leq a_m$ we have

$$m \in \{0, 1, 2, \cdots, m\} \cap A \subseteq \{0, 1, 2, \cdots, a_m\} \cap A = \{a_0, a_1, \cdots, a_m\}.$$

Thus $A = \{a_0, a_1, a_2, \cdots\}$ and the elements a_i are distinct, so A is countable. This proves our claim that every subset of $\mathbf{N}$ is either finite or countable.

Now suppose that $|A| \leq |N|$ and choose an injective map $f : A \to \mathbf{N}$. Since f is injective, when we consider it as a map $f : A \to f(A)$, it is bijective, and so $|A| = |f(A)|$. Since $f(A) \subseteq \mathbf{N}$, the previous paragraph shows that $f(A)$ is either finite or countable. If $f(A)$ is finite with $|f(A)| = n$ then $|A| = |f(A)| = |S_n|$, and if $f(A)$ is countable then we have $|A| = |f(A)| = |\mathbf{N}|$. Thus A is finite or countable.

9.18 Theorem: *Let A be a set. Then*

(1) $|A| < |\mathbf{N}|$ if and only if A is finite,
(2) $|\mathbf{N}| < |A|$ if and only if A is neither finite nor countable, and
(3) if $|A| \leq |\mathbf{N}|$ and $|\mathbf{N}| \leq |A|$ then $|A| = |\mathbf{N}|$.

Proof: Part (1) follows from Theorem 4.15 because

$$\begin{aligned} |A| < |\mathbf{N}| &\iff (|A| \leq |\mathbf{N}| \text{ and } |A| \neq |\mathbf{N}|) \\ &\iff (A \text{ is finite or countable and } A \text{ is not countable}) \\ &\iff A \text{ is finite} \end{aligned}$$

and Part (2) follows from Theorem 4.17 because

$$\begin{aligned} |\mathbf{N}| < |A| &\iff (|\mathbf{N}| \leq |A| \text{ and } |\mathbf{N}| \neq |A|) \\ &\iff (A \text{ is not finite and } A \text{ is not countable.}) \end{aligned}$$

To prove Part (3), suppose that $|A| \leq |\mathbf{N}|$ and $|\mathbf{N}| \leq |A|$. Since $|A| \leq |\mathbf{N}|$, we know that A is finite or countable by Theorem 4.17. Since $|\mathbf{N}| \leq |A|$, we know that that A is infinite by Theorem 4.15. Since A is finite or countable and A is not finite, it follows that A is countable. Thus $|A| = |\mathbf{N}|$.

9.19 Definition: Let A be a set. When A is countable we write $|A| = \aleph_0$. When A is finite we write $|A| < \aleph_0$. When A is infinite we write $|A| \geq \aleph_0$. When A is either finite or countable we write $|A| \leq \aleph_0$ and we say that A is **at most countable**. when A is neither finite nor countable we write $|A| > \aleph_0$ and we say that A is **uncountable**.

9.20 Theorem:

(1) If A and B are countable sets, then so is $A \times B$.
(2) If A and B are countable sets, then so is $A \cup B$.
(3) If $A_0, A_1, A_2, \cdots$ are countable sets, then so is $\bigcup_{k=0}^{\infty} A_k$.
(4) $\mathbf{Q}$ is countable.

Proof: To prove Parts (1) and (2), let $A = \{a_0, a_1, a_2, \cdots\}$ with the a_i distinct and let $B = \{b_0, b_1, b_2, \cdots\}$ with the b_i distinct. Since every positive integer can be written uniquely in the form $2^k(2l+1)$ with $k, l \in \mathbf{N}$, the map $f : A \times B \to \mathbf{N}$ given by $f(a_k, b_l) = 2^k(2l+1)-1$ is bijective, and so $|A \times B| = |\mathbf{N}|$. This proves Part (1). Since the map $g : \mathbf{N} \to A \cup B$ given by $g(k) = a_k$ is injective, we have $|\mathbf{N}| \leq |A \cup B|$. Since the map $h : \mathbf{N} \to A \cup B$ given by $h(2k) = a_k$ and $h(2k+1) = b_k$ is surjective, we have $|A \cup B| \leq |N|$. Since $|\mathbf{N}| \leq |A \cup B|$ and $|A \cup B| \leq |\mathbf{N}|$, we have $|A \cup B| = |\mathbf{N}|$ by Part (3) of Theorem 4.18. This proves (2).

To prove Part (3), for each $k \in \mathbf{N}$, let $A_k = \{a_{k0}, a_{k1}, a_{k2}, \cdots\}$ with the a_{ki} distinct. Since the map $f : \mathbf{N} \to \bigcup_{k=0}^{\infty} A_k$ given by $f(k) = a_{0,k}$ is injective, $|\mathbf{N}| \leq \left|\bigcup_{k=0}^{\infty} A_k\right|$. Since $\mathbf{N} \times \mathbf{N}$ is countable by Part (1), and since the map $g : \mathbf{N} \times \mathbf{N} \to \bigcup_{k=0}^{\infty} A_k$ given by $g(k, l) = a_{k,l}$ is surjective, we have $\left|\bigcup_{k=0}^{\infty} A_k\right| \leq |\mathbf{N} \times \mathbf{N}| = |\mathbf{N}|$. By Part (3) of Theorem 4.18, we have $\left|\bigcup_{k=0}^{\infty} A_k\right| = |\mathbf{N}|$, as required.

Finally, we prove Part (4). Since the map $f : \mathbf{N} \to \mathbf{Q}$ given by $f(k) = k$ is injective, we have $|\mathbf{N}| \leq |\mathbf{Q}|$. Since the map $g : \mathbf{Q} \to \mathbf{Z} \times \mathbf{Z}$, given by $g\left(\frac{a}{b}\right) = (a, b)$ for all $a, b \in \mathbf{Z}$ with $b > 0$ and $\gcd(a, b) = 1$, is injective, and since $\mathbf{Z} \times \mathbf{Z}$ is countable, we have $|\mathbf{Q}| \leq |\mathbf{Z} \times \mathbf{Z}| = |\mathbf{N}|$. Since $|\mathbf{N}| \leq |\mathbf{Q}|$ and $|\mathbf{Q}| \leq |\mathbf{N}|$, we have $|\mathbf{Q}| = |\mathbf{N}|$, as required.

9.21 Definition: For a set A, let $\mathcal{P}(A)$ denote the **power set** of A, that is the set of all subsets of A, and let 2^A denote the set of all functions from A to $S_2 = \{0, 1\}$.

9.22 Theorem:

(1) For every set A, $\left|\mathcal{P}(A)\right| = \left|2^A\right|$.
(2) For every set A, $|A| < \left|\mathcal{P}(A)\right|$.
(3) $\mathbf{R}$ is uncountable.

Proof: Let A be any set. Define a map $g : \mathcal{P}(A) \to 2^A$ as follows. Given $S \in \mathcal{P}(A)$, that is given $S \subseteq A$, we define $g(S) \in 2^A$ to be the map $g(S) : A \to \{0, 1\}$ given by

$$g(S)(a) = \begin{cases} 1 \text{ if } a \in S, \\ 0 \text{ if } a \notin S. \end{cases}$$

Define a map $h : 2^A \to \mathcal{P}(A)$ as follows. Given $f \in 2^A$, that is given a map $f : A \to \{0, 1\}$, we define $h(f) \in \mathcal{P}(A)$ to be the subset

$$h(f) = \left\{a \in A \middle| f(a) = 1\right\} \subseteq A.$$

The maps g and h are the inverses of each other because for every $S \subseteq A$ and every $f : A \to \{0, 1\}$ we have

$$f = g(S) \iff \forall a \in A \;\; f(a) = g(S)(a) \iff \forall a \in A \;\; f(a) = \begin{cases} 1 \text{ if } a \in S, \\ 0 \text{ if } a \notin S, \end{cases}$$
$$\iff \forall a \in A \;\big(f(a) = 1 \iff a \in S\big) \iff \left\{a \in A \middle| f(a) = 1\right\} = S \iff h(f) = S.$$

This completes the proof of Part (1).

Let us prove Part (2). Again we let A be any set. Since the the map $f : A \to \mathcal{P}(A)$ given by $f(a) = \{a\}$ is injective, we have $|A| \leq \left|\mathcal{P}(A)\right|$. We need to show that $|A| \neq \left|\mathcal{P}(A)\right|$.

Let $g : A \to \mathcal{P}(A)$ be any map. Let $S = \big\{a \in A \big| a \notin g(a)\big\}$. Note that S cannot be in the range of g because if we could choose $a \in A$ so that $g(a) = S$ then, by the definition of S, we would have $a \in S \Longleftrightarrow a \notin g(a) \Longleftrightarrow a \notin S$ which is not possible. Since S is not in the range of g, the map g is not surjective. Since g was an arbitrary map from A to $\mathcal{P}(A)$, it follows that there is no surjective map from A to $\mathcal{P}(A)$. Thus there is no bijective map from A to $\mathcal{P}(A)$ and so we have $|A| \neq \big|\mathcal{P}(A)\big|$, as desired.

Finally, we shall prove that $\mathbf{R}$ is uncountable using the fact (which we did not prove) that every real number has a unique decimal expansion which does not end with an infinite string of 9's. We define a map $g : 2^{\mathbf{N}} \to \mathbf{R}$ as follows. Given $f \in 2^{\mathbf{N}}$, that is given a map $f : \mathbf{N} \to \{0,1\}$, we define $g(f)$ to be the real number $g(f) \in [0,1)$ with the decimal expansion $g(f) = 0.f(0)f(1)f(2)f(3)\cdots$ (for those who have seen infinite series, this is the number $g(f) = \sum_{k=0}^{\infty} f(k)10^{-k-1}$). By the uniqueness of decimal expansions, the map g is injective, so we have $\big|2^{\mathbf{N}}\big| \leq |\mathbf{R}|$. Thus $|\mathbf{N}| < \big|\mathcal{P}(\mathbf{N})\big| = \big|2^{\mathbf{N}}\big| \leq |\mathbf{R}|$, and so $\mathbf{R}$ is uncountable, by Part (2) of Theorem 4.18.

9.23 Theorem: *(Cantor - Schroeder - Bernstein) Let A and B be sets. Suppose that $|A| \leq |B|$ and $|B| \leq |A|$. Then $|A| = |B|$*

Proof: We sketch a proof. Choose injective functions $f : A \to B$ and $g : B \to A$. Since the functions $f : A \to f(A)$, $g : B \to g(B)$ and $f : g(B) \to f\big(g(B)\big)$ are bijective we have $|A| = |f(A)|$ and $|B| = |g(B)| = \big|f(g(B))\big|$. Also note that $f\big(g(B)\big) \subseteq f(A) \subseteq B$. Let $X = f\big(g(B)\big)$, $Y = f(A)$ and $Z = B$. Then we have $X \subseteq Y \subseteq Z$ and we have $|X| = |Z|$ and we need to show that $|Y| = |Z|$. The composite $h = f \circ g : Z \to X$ is a bijection. Define sets Z_n and Y_n for $n \in \mathbf{N}$ recursively by

$$Z_0 = Z,\ Z_n = h(Z_{n-1}) \ \text{ and } \ Y_0 = Y\ ,\ Y_n = h(Y_{n-1}).$$

Since $Y_0 = Y$, $Z_0 = Z$, $Z_1 = h(Z_0) = h(Z) = X$ and $X \subseteq Y \subseteq Z$, we have

$$Z_1 \subseteq Y_0 \subseteq Z_0.$$

Also note that for $1 \leq n \in \mathbf{N}$,

$$Z_n \subseteq Y_{n-1} \subseteq Z_{n-1} \Longrightarrow h(Z_n) \subseteq h(Y_{n-1}) \subseteq h(Z_{n-1}) \Longrightarrow Z_{n+1} \subseteq Y_n \subseteq Z_n.$$

By the Induction Principle, it follows that $Z_n \subseteq Y_{n-1} \subseteq Z_{n-1}$ for all $n \geq 1$, so we have

$$Z_0 \supseteq Y_0 \supseteq Z_1 \supseteq Y_1 \supseteq Z_2 \supseteq Y_2 \supseteq \cdots$$

Let $U_n = Z_n \setminus Y_n$, $U = \bigcup_{n=1}^{\infty} U_n$ and $V = Z \setminus U$. Define $H : Z \to Y$ by

$$H(x) = \begin{cases} h(x) \text{ if } x \in U, \\ \ \ x \quad \text{if } x \in V. \end{cases}$$

Verify that H is bijective.

9.24 Example: Show that $|\mathbf{R}| = \left|2^{\mathbf{N}}\right|$.

Solution: $g : 2^{\mathbf{N}} \to \mathbf{R}$ as follows: for $f \in 2^{\mathbf{N}}$ we let $g(f)$ be the real number $g(f) \in [0,1)$ with decimal expansion $g(f) = 0.f(0)f(1)f(2)\cdots$.. Then g is injective so $\left|2^{\mathbf{N}}\right| \leq |\mathbf{R}|$. Define $h : 2^{\mathbf{N}} \to [0,1)$ as follows: for $f \in 2^{\mathbf{N}}$ let $h(f)$ be the real number $h(f) \in [0,1]$ with binary expansion $h(f) = 0.f(0)f(1)f(2)\cdots$. Then h is surjective so we have $\left|[0,1]\right| \leq \left|2^{\mathbf{N}}\right|$. The map $k : \mathbf{R} \to [0,1]$ given by $k(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$ is injective so we have $|\mathbf{R}| \leq \left|[0,1]\right|$. Since $|\mathbf{R}| \leq \left|[0,1]\right| \leq \left|2^{\mathbf{N}}\right|$ and $\left|2^{\mathbf{N}}\right| \leq |\mathbf{R}|$, we have $|\mathbf{R}| = \left|2^{\mathbf{N}}\right|$ by the Cantor-Schroeder-Bernstein Theorem.

9.25 Theorem: *Let A and B be finite sets, let A^B be the set of all functions $f : A \to B$, and let $\mathcal{P}(A)$ be the power set of A (that is the set of all subsets of A). Then*

(1) if A and B are disjoint then $|A \cup B| = |A| \cup |B|$,
(2) $|A \times B| = |A| \cdot |B|$,
(3) $\left|A^B\right| = |A|^{|B|}$, and
(3) $|\mathcal{P}(A)| = 2^{|A|}$.

Proof: The proof is left as an exercise.