Chapter 8. Complex Numbers

8.1 Definition: A **complex number** is a vector in $\mathbf{R}^2$. The **complex plane**, denoted by $\mathbf{C}$, is the set of complex numbers:

$$\mathbf{C} = \mathbf{R}^2 = \left\{(x,y)\,\middle|\, x \in \mathbf{R}, y \in \mathbf{R}\right\}.$$

In $\mathbf{C}$ we write $0 = (0,0)$, $1 = (1,0)$, $i = (0,1)$, and for $x, y \in \mathbf{R}$ we write $x = (x,0)$, $iy = yi = (o,y)$ and

$$x + iy = x + yi = (x,y).$$

If $z = x+iy$ with $x, y \in \mathbf{R}$ then x is called the **real** part of z and y is called the **imaginary** part of z, and we write

$$\operatorname{Re} z = x\ ,\text{and}\ \ \operatorname{Im} z = y\,.$$

8.2 Definition: We define the **sum** of two complex numbers to be the usual vector sum:

$$(a+ib) + (c+id) = (a+c) + i(b+d)\,,$$

where $a, b \in \mathbf{R}$. We define the **product** of two complex numbers by setting $i^2 = -1$ and by requiring the product to be commutative and associative and distributive over the sum:

$$(a+ib)(c+id) = ac + iad + ibc + i^2bd = (ac-bd) + i(ad+bc)\,.$$

8.3 Example: Let $z = 2+i$ and $w = 1+3i$. Find $z+w$ and zw.

Solution: $z+w = (2+i)+(1+3i) = (2+1)+i(1+3) = 3+4i$, and $zw = (2+i)(1+3i) = 2+6i+i-3 = -1+7i$.

8.4 Theorem: *The set of complex numbers is a field.*

Proof: We shall only verify that each non-zero complex number has an inverse. Let $z = a+ib$ where $a, b \in \mathbf{R}$. Suppose that $z \neq 0$ so $a^2+b^2 \neq 0$. For $x, y \in \mathbf{R}$ we have

$$\begin{aligned}(a+ib)(x+iy) = 1 &\iff (ax-by) + (ay+bx)\,i = 1 + 0\,i \\ &\iff \big(ax - by = 1 \text{ and } bx + ay = 0\big).\end{aligned}$$

We solve the pair of equations $ax - by = 1$ (1) and $bx + ay = 0$ (2). Multiply equation (1) by a and add b times Equation (2) to get $(a^2+b^2)x = a$, so we need $x = \frac{a}{a^2+b^2}$. Multiply Equation (2) by a and subtract b times Equation (1) to get $(a^2+b^2)y = -b$ so we need $y = \frac{-b}{a^2+b^2}$. Verify that when $x = \frac{a}{a^2+b^2}$ and $y = \frac{-b}{a^2+b^2}$ we do indeed have $(a+ib)(x+iy) = 1$. This shows that $(a+ib)^{-1}$ does exist and is given by

$$(a+ib)^{-1} = \tfrac{a}{a^2+b^2} - i\,\tfrac{b}{a^2+b^2}\,.$$

8.5 Example: Find $\dfrac{(4-i)-(1-2i)}{1+2i}$.

Solution: $\dfrac{(4-i)-(1-2i)}{1+2i} = \dfrac{3+i}{1+2i} = (3+i)(1+2i)^{-1} = (3+i)(\frac{1}{5} - \frac{2}{5}i) = 1-i.$

8.6 Definition: If $z = x + iy$ with $x, y \in \mathbf{R}$ then we define the **conjugate** of z to be

$$\overline{z} = x - iy\,.$$

and we define the **length** (or **magnitude**) of z to be

$$|z| = \sqrt{x^2 + y^2}\,.$$

8.7 Note: For z and w in $\mathbf{C}$ the following identities are all easy to verify.

$$\begin{aligned}
&\overline{\overline{z}} = z \\
&z + \overline{z} = 2\,\mathrm{Re}\, z\ , \quad z - \overline{z} = 2i\,\mathrm{Im}\, z \\
&z\overline{z} = |z|^2\ , \quad |\overline{z}| = |z| \\
&\overline{z+w} = \overline{z} + \overline{w}\ , \quad \overline{zw} = \overline{z}\,\overline{w}\ , \quad |zw| = |z||w|
\end{aligned}$$

8.8 Note: We do *not* have inequalities between complex numbers. We can *only* write $a < b$ or $a \leq b$ in the case that a and b are both *real* numbers. But there are several inequalities between real numbers which concern complex numbers. For $z \in \mathbf{C}$ and $w \in C$,

$$\begin{aligned}
&|\mathrm{Re}\,(z)| \leq |z|\ , \quad |\mathrm{Im}\,(z)| \leq |z| \\
&|z + w| \leq |z| + |w|\ , \quad \text{this is called the } \textbf{triangle inequality} \\
&|z + w| \geq \big||z| - |w|\big|
\end{aligned}$$

The first two inequalities follow from the fact that $|z|^2 = |\mathrm{Re}\,(z)|^2 + |\mathrm{Im}\,(z)|^2$. We can then prove the triangle inequality as follows: $|z+w|^2 = (z+w)(\overline{z}+\overline{w}) = |z|^2+|w|^2+(w\overline{z}+z\overline{w}) = |z|^2 + |w|^2 + 2\mathrm{Re}\,(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z\overline{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$. The last inequality follows from the triangle inequality since $|z| = |z + w - w| \leq |z + w| + |w|$ and $|w| = |z + w - z| \leq |z + w| + |z|$. (Alternatively, the last two inequalities can be proven using the Law of Cosines).

8.9 Example: Given complex numbers a and b, describe the set $\big\{z \in \mathbf{C}\big||z-a| < |z-b|\big\}$.

Solution: Geometrically, this is the set of all z such that z is closer to a than to b, so it is the **half-plane** which contains a and lies on one side of the perpendicular bisector of the line segment ab.

8.10 Example: Given a complex number a, describe the set $\big\{z \in \mathbf{C}\big|1 < |z - a| < 2\big\}$.

Solution: $\big\{z\big||z - a| = 1\big\}$ is the circle centred at a of radius 1 and $\big\{z\big||z - a| = 2\big\}$ is the circle centred at a of radius 2, and $\big\{z \in \mathbf{C}\big|1 < |z - a| < 2\big\}$ is the region between these two circles. Such a region is called an **annulus**.

8.11 Example: Show that every non-zero complex number has exactly two complex square roots, and find a formula for the two square roots of $z = x + iy$.

Solution: Let $z = x + iy$ where $x, y \in \mathbf{R}$ with x and y not both zero. We need to solve $w^2 = z$ for $w \in \mathbf{C}$. Write $w = u + iv$ with $u, v \in \mathbf{R}$. We have

$$\begin{aligned} w^2 = z &\iff (u+iv)^2 = x + iy \iff (u^2 - v^2) + i(2uv) = x + iy \\ &\iff \big(u^2 - v^2 = x \text{ and } 2uv = y\big)\,. \end{aligned}$$

To solve this pair of equations for u, square both sides of the second equation to get $4u^2v^2 = y^2$, then multiply the first equation by $4u^2$ to get $4u^4 - 4u^2v^2 = 4x\,u^2$, that is $4u^4 - 4x\,u^2 - y^2 = 0$. By the quadratic formula,

$$u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2+y^2}}{2}\,.$$

In the case that $y \neq 0$, we must use the + sign so that the right side is non-negative, so we obtain

$$u = \pm\sqrt{\frac{x + \sqrt{x^2+y^2}}{2}}\,.$$

A similar calculation gives

$$v = \pm\sqrt{\frac{-x + \sqrt{x^2+y^2}}{2}}\,.$$

All four choices of sign will satisfy the equation $u^2 - v^2 = x$, but to satisfy $2uv = y$ notice that when $y > 0$, u and v have the same sign, and when $y < 0$, u and v have the opposite sign. It remains only to consider the case that $y = 0$, and we leave this case as an exercise. The final result is that

$$w = \begin{cases} \pm\left(\sqrt{\dfrac{x+\sqrt{x^2+y^2}}{2}} + i\sqrt{\dfrac{-x+\sqrt{x^2+y^2}}{2}}\right) &, \text{ if } y > 0, \\ \pm\left(\sqrt{\dfrac{x+\sqrt{x^2+y^2}}{2}} - i\sqrt{\dfrac{-x+\sqrt{x^2+y^2}}{2}}\right) &, \text{ if } y < 0, \\ \pm\sqrt{x} &, \text{ if } y = 0 \text{ and } x > 0, \\ \pm\, i\sqrt{|x|} &, \text{ if } y = 0 \text{ and } x < 0\,. \end{cases}$$

8.12 Note: When working with real numbers, for $0 < x \in \mathbf{R}$ it is customary to write $\sqrt{x}$ or $x^{1/2}$ to denote the unique positive square root of x. When working with complex numbers, for $0 \neq z \in \mathbf{C}$ we sometimes write $\sqrt{z}$ or $z^{1/2}$ to denote one of the two square roots of z, and we sometimes write $\sqrt{z}$ or $z^{1/2}$ to denote both square roots of z.

8.13 Example: Find $\sqrt{3-4i}$.

Solution: Using the formula derived in the previous example, we have

$$\sqrt{3-4i} = \pm\left(\sqrt{\tfrac{3+\sqrt{3^2+4^2}}{2}} - i\sqrt{\tfrac{-3+\sqrt{3^2+4^2}}{2}}\right) = \pm\left(\sqrt{\tfrac{3+5}{2}} - i\sqrt{\tfrac{-3+5}{2}}\right) = \pm(2-i)\,.$$

8.14 Note: The Quadratic Formula can be used for complex numbers. Indeed for $a, b, c, z \in \mathbf{C}$ with $a \neq 0$ we have

$$\begin{aligned} az^2 + bz + c = 0 &\Longleftrightarrow z^2 + \frac{b}{a}\,z + \frac{c}{a} = 0 \Longleftrightarrow z^2 + \frac{b}{2a}\,z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\ &\Longleftrightarrow \left(z + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \Longleftrightarrow z + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \\ &\Longleftrightarrow z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \end{aligned}$$

where $\sqrt{b^2 - 4ac}$ is being used to denote both square roots in the case that $b^2 - 4ac \neq 0$.

8.15 Example: Solve $i\,z^2 - (2+3i)z + 5(1+i) = 0$.

Solution: By the Quadratic Formula, we have

$$\begin{aligned} z &= \frac{(2+3i) + \sqrt{(2+3i)^2 - 20i(1+i)}}{2i} = \frac{(2+3i) + \sqrt{-5 + 12i + 20 - 20i}}{2i} \\ &= \frac{(2+3i) + \sqrt{15 - 8i}}{2i} \end{aligned}$$

and by the formula for square roots we have

$$\sqrt{15 - 8i} = \pm\left(\sqrt{\tfrac{15 + \sqrt{15^2 + 8^2}}{2}} - i\sqrt{\tfrac{-15 + \sqrt{15^2 + 8^2}}{2}}\right) = \pm\left(\sqrt{\tfrac{15+17}{2}} - i\sqrt{\tfrac{-15+17}{2}}\right) = \pm\,(4 - i)$$

and so

$$z = \frac{(2+3i) \pm (4-i)}{2i} = \frac{6+2i}{2i} \text{ or } \frac{-2+4i}{2i} = 1 - 3i \text{ or } 2 + i\,.$$

8.16 Definition: If $z \neq 0$, we define the **angle** (or **argument**) of z to be the angle $\theta(z)$ from the positive x-axis counterclockwise to z. In other words, $\theta(z)$ is the angle such that

$$z = |z|\big(\cos\theta(z) + i\sin\theta(z)\big)\,.$$

8.17 Note: We can think of the angle $\theta(z)$ in several different ways. We can require, for example, that $0 \leq \theta(z) < 2\pi$ so that the angle is uniquely determined. Or we can allow $\theta(z)$ to be any real number, in which case the angle will be unique up to a multiple of 2π. Then again, we can think of $\theta(z)$ as the infinite set of real numbers $\theta(z) = \{\theta_0 + 2\pi k | k \in z\}$, that is we can regard $\theta(z)$ as an element of $\mathbf{R}/2\pi$, the set of real numbers modulo 2π.

8.18 Notation: For $\theta \in \mathbf{R}$ (or for $\theta \in \mathbf{R}/2\pi$) we shall write

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

8.19 Note: If $z \neq 0$ and we have $x = \mathrm{Re}\,(z)$, $y = \mathrm{Im}\,(z)$, $r = |z|$ and $\theta = \theta(z)$ then

$$\begin{aligned} &x = r\cos\theta\ , \quad y = r\sin\theta \\ &r = \sqrt{x^2 + y^2}\ , \quad \tan\theta = \frac{y}{x},\ \text{if } x \neq 0 \\ &z = re^{i\theta}\ , \quad \overline{z} = r\,e^{-i\,\theta}\ , \quad z^{-1} = \frac{1}{r}e^{-i\,\theta} \end{aligned}$$

We say that $x + i\,y$ is the **cartesian** form of z and $re^{i\,\theta}$ is the **polar** form.

8.20 Example: Let $z = -3 - 4i$. Express z in polar form.

Solution: We have $|z| = 5$ and $\tan\theta(z) = \frac{4}{3}$. Since $\theta(z)$ is in the third quadrant, we have $\theta(z) = \pi + \tan^{-1}\frac{4}{3}$. So $z = 5e^{i(\pi + \tan^{-1}(4/3))}$.

8.21 Example: Let $z = 10e^{i\,\tan^{-1}3}$. Express z in cartesian form.

Solution: $z = 10\left(\cos(\tan^{-1}3) + i\,\sin(\tan^{-1}3)\right) = 10\left(\frac{1}{\sqrt{10}} + i\frac{3}{\sqrt{10}}\right) = \sqrt{10} + 3\sqrt{10}\,i$.

8.22 Example: Find a formula for multiplication in polar coordinates.

Solution: For $z = re^{i\alpha}$ and $w = e^{i\beta}$ we have $zw = rs(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \big((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\big) = rs\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big)$ and so we obtain the formula

$$re^{i\alpha}se^{i\beta} = rs\,e^{i(\alpha+\beta)}\,.$$

8.23 Note: An immediate consequence of the above example is that

$$(r\,e^{i\,\theta})^n = r^n e^{i\,n\theta}$$

for $r, \theta \in \mathbf{R}$ and for $n \in \mathbf{Z}$. This result is known as **De Moivre's Law**.

8.24 Example: Find $(1+i)^{10}$.

Solution: This can be done in cartesian coordinates using the binomial theorem (which holds for complex numbers), but it is easier in polar coordinates. We have $1+i = \sqrt{2}e^{i\,\pi/4}$ so $(1+i)^{10} = (\sqrt{2}e^{i\,\pi/4})^{10} = (\sqrt{2})^{10}e^{i\,10\pi/4} = 32e^{i\,\pi/2} = 32i$.

8.25 Example: Find a formula for the n^{th} roots of a complex number. In other words, given $z = re^{i\theta}$, solve $w^n = z$.

Solution: Let $w = se^{i\alpha}$. We have $w^n = z \Longleftrightarrow (se^{i\alpha})^n = re^{i\theta} \Longleftrightarrow s^n e^{i\,n\alpha} = re^{i\theta} \Longleftrightarrow s^n = r$ and $n\alpha = \theta + 2\pi k$ for some $k \in \mathbf{Z} \Longleftrightarrow s = \sqrt[n]{r}$ and $\alpha = \dfrac{\theta + 2\pi k}{n}$ for some $k \in \mathbf{Z}$. Notice that when $z \neq 0$ there are exactly n solutions obtained by taking $0 \le k < n$. So we obtain the formula

$$(r\,e^{i\,\theta})^{1/n} = \sqrt[n]{r}\,e^{i\,(\theta+2\pi k)/n} \quad , \qquad k \in \{0, 1, \ldots, n-1\}\,.$$

In particular, $(r\,e^{i\,\theta})^{1/2} = \pm\sqrt{r}\,e^{i\,\theta/2}$. For $0 < a \in \mathbf{R}$ we have $z^2 = a \Longleftrightarrow z = \pm\sqrt{a}$, and for $0 > a \in \mathbf{R}$ we have $z^2 = a \Longleftrightarrow z = \pm\sqrt{|a|}\,i$.

8.26 Note: When working with complex numbers, for $0 \neq z \in \mathbf{C}$ and for $0 < n \in \mathbf{Z}$, we sometimes write $\sqrt[n]{z}$ or $w^{1/n}$ to denote one of the n solutions to $w^n = z$, and we sometimes write $\sqrt[n]{z}$ or $z^{1/n}$ to denote the set of all n^{th} roots.

8.27 Note: For $z, w \in \mathbf{C}$, the rule

$$(zw)^{1/n} = z^{1/n}w^{1/n}$$

does hold provided that $z^{1/n}$ is used to denote the set of all n^{th} roots, but it does not always hold when $z^{1/n}$ is used to denote one of the n^{th} roots. Consider the following amusing "proof" that $1 = -1$:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i^2 = -1\,.$$

8.28 Example: Find $\sqrt[3]{-2+2i}$.

Solution: Note that $-2+2i = 2\sqrt{2}\,e^{i\,3\pi/4}$, and so the formula for n^{th} roots gives

$$\begin{aligned}\sqrt[3]{-2+2i} &= \sqrt[3]{2\sqrt{2}\,e^{3\pi/4}} \\ &= \sqrt{2}\,e^{i\left(\pi/4 + \frac{2\pi}{3}\,k\right)}, k \in \{0,1,2\} \\ &= \sqrt{2}\,e^{i\,\pi/3}, \sqrt{2}\,e^{i\,11\pi/12}, \sqrt{2}\,e^{i\,19\pi/12}\,.\end{aligned}$$

8.29 Note: The remaining examples in this chapter illustrate situations in which we can use complex numbers as a tool to help solve certain problems which only involve real numbers.

8.30 Example: Let $x_0 = 1$ and $x_1 = 1$, and for $n \geq 2$ let $x_n = 2x_{n-1} - 5x_{n-2}$. Find a closed-form formula for x_n.

Solution: The characteristic polynomial for the recursion is $z^2 - 2z + 5 = 0$ which has (complex) roots $z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$. By the Linear Recursion Theorem (Theorem 2.47)

$$x_n = A(1+2i)^n + B(1-2i)^n$$

for some constants A and B. To get $x_0 = 1$ and $x_1 = 1$, we need $A + B = 1$ and $A(1+2i) + B(1-2i) = 1$. Solving these two equations gives $A = B = \frac{1}{2}$, so we have

$$\begin{aligned} x_n &= \tfrac{1}{2}\left((1+2i)^n + (1-2i)^n\right) = \tfrac{1}{2}\left(\left(\sqrt{5}\,e^{i\,\theta}\right)^n + \left(\sqrt{5}\,e^{-i\,\theta}\right)^n\right) = \tfrac{(\sqrt{5})^n}{2}\left(e^{i\,n\theta} + e^{-i\,n\theta}\right) \\ &= \tfrac{(\sqrt{5})^n}{2}\left(2\cos n\theta\right) = \left(\sqrt{5}\right)^n \cos n\theta \end{aligned}$$

where $\theta = \theta(1+2i) = \tan^{-1} 2$. Thus we obtain

$$x_n = (\sqrt{5})^n \cos\left(n \tan^{-1} 2\right) .$$

8.31 Example: Find $\displaystyle\sum_{i=0}^{n} \binom{3n}{3i}$.

Solution: Let $\alpha = e^{i\,2\pi/3}$. Note that $1 + \alpha + \alpha^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$. By the Binomial Theorem we have

$$\begin{aligned} (1+1)^{3n} &= \binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \binom{3n}{4} + \cdots + \binom{3n}{3n} \\ (1+\alpha)^{3n} &= \binom{3n}{0} + \binom{3n}{1}\alpha + \binom{3n}{2}\alpha^2 + \binom{3n}{3} + \binom{3n}{4}\alpha + \cdots + \binom{3n}{3n} \\ (1+\alpha^2)^{3n} &= \binom{3n}{0} + \binom{3n}{1}\alpha^2 + \binom{3n}{2}\alpha + \binom{3n}{3} + \binom{3n}{4}\alpha^2 + \cdots + \binom{3n}{3n} \end{aligned}$$

Adding these three equations gives $(1+1)^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n} = 3\displaystyle\sum_{i=0}^{n}\binom{3n}{3i}$. Note that $1 + \alpha = 1 - \frac{1}{2} + \frac{\sqrt{3}}{2}i = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i\,\pi/3}$ and similarly $1 + \alpha^2 = e^{-i\,\pi/3}$, and so

$$\begin{aligned} \sum_{i=0}^{n}\binom{3n}{3i} &= \tfrac{1}{3}\left((1+1)^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n}\right) = \tfrac{1}{3}\left(2^{3n} + \left(e^{i\,\pi/3}\right)^{3n} + \left(e^{-i\,\pi/3}\right)^{3n}\right) \\ &= \tfrac{1}{3}\left(2^{3n} + e^{i\,n\pi} + e^{-i\,n\pi}\right) = \frac{2^{3n} + 2(-1)^n}{3} . \end{aligned}$$

8.32 Note: The Fundamental Theorem of Algebra states that every non-constant polynomial over **C** has a root in **C**. It follows that every such polynomial factors into linear factors over **C**. If a polynomial $f(x)$ has real coefficients, and α is a complex root of f so that $f(\alpha) = 0$, then we have $f(\overline{\alpha}) = \overline{f(\alpha)} = 0$ so that $\overline{\alpha}$ is also a root of f. Notice that in this case

$$(x - \alpha)(x - \overline{\alpha}) = x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha} = x^2 - 2\mathrm{Re}\,(\alpha) + |\alpha|^2 ,$$

which has real coefficients. It follows that every non-constant polynomial over **R** factors into linear and quadratic factors over **R**.

8.33 Example: Let $f(x)=x^4+2x^2+4$. Solve $f(z)=0$ for $z\in\mathbf{C}$, factor $f(z)$ over the complex number, and then factor $f(x)$ over the real numbers.

Solution: By the quadratic formula, $f(z)=0$ when $z^2=-1\pm\sqrt{3}\,i$ or in polar coordinates $z=2e^{\pm i\,2\pi/3}$. Thus the roots of f are $z=\pm\sqrt{2}e^{\pm i\,\pi/3}$, and so f factors over $\mathbf{C}$ as

$$z^4+2z^2+4=\big(z-\sqrt{2}e^{i\,\pi/3}\big)\big(z-\sqrt{2}e^{-i\,\pi/3}\big)\big(z+\sqrt{2}e^{i\,\pi/3}\big)\big(z+\sqrt{2}e^{-i\,\pi/3}\big)\,.$$

Since $\big(z-\sqrt{2}e^{i\,\pi/3}\big)\big(z-\sqrt{2}e^{-i\,\pi/3}\big)=z^2-\sqrt{2}\,z+2$ and $\big(z+\sqrt{2}e^{i\,\pi/3}\big)\big(z+\sqrt{2}e^{-i\,\pi/3}\big)=z^2+\sqrt{2}\,z+2$, we see that over $\mathbf{R}$, f factors as

$$f(x)=(x^2-\sqrt{2}\,x+2)(x^2+\sqrt{2}\,x+2)\,.$$

8.34 Note: Historically, complex numbers first arose in the study of cubic equations. An equation of the form $ax^3+bx^2+cx+d=0$, where $a,b,c,d\in\mathbf{C}$ with $a\neq 0$ can be solved as follows. First, divide by a to obtain an equation of the form $x^3+Bx^2+Cx+D=0$. Next, make the substitution $x=y-\frac{B}{3}$ and rewrite the equation in the form $y^3+py+q=0$. Then make the substitution $y=z-\frac{p}{3z}$ to convert the equation to the form $z^3+q-\frac{p^3}{27}z^{-3}=0$. Finally, multiply by z^3 to obtain $z^6+qz^3-\frac{p^3}{27}$ and solve for z^3 using the Quadratic Formula.

8.35 Example: Let $f(x)=x^3+3x^2+4x+1$. Note that $f'(x)=3x^2+6x+4=3(x+1)^2+1>0$, so f is increasing and hence has exactly one real root. Find the real root of f.

Solution: Let $x=y-1$. Then $x^3+3x^2+4x+1=(y-1)^3+3(y-1)^2+4(y-1)+1=y^3+y-1$. Let $y=z-\frac{1}{3z}$. Then $y^3+y-1=(z-\frac{1}{3}z^{-1})^3+(z-\frac{1}{3}z^{-1})-1=z^3-1-\frac{1}{27}z^{-3}$. We solve $z^6-z^3-\frac{1}{27}=0$ using the quadratic formula, and obtain $z^3=\frac{1\pm\sqrt{\frac{31}{27}}}{2}$. If $z=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}$ then $r\,z^{-1}=-\frac{1}{3}\sqrt[3]{\frac{2}{1+\sqrt{\frac{31}{27}}}}=-\frac{1}{3}\sqrt[3]{\frac{2(1-\sqrt{\frac{31}{27}})}{1-\frac{31}{27}}}=\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$. Similarly, if $z=\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$ then $r\,z^{-1}=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}$. In either case we have $y=z+rz^{-1}=\sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}}+\sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}}$, and $x=y-1=\sqrt[3]{\frac{\sqrt{\frac{31}{27}}+1}{2}}-\sqrt[3]{\frac{\sqrt{\frac{31}{27}}-1}{2}}-1$. (We did not use complex numbers in this example).

8.36 Example: Find the three real roots of $f(x)=x^3-3x+1$.

Solution: Let $x=z+z^{-1}$ so that $f(x)=(z+z^{-1})^3-3(z+z^{-1})+1=z^3+1+z^{-3}$. Multiply by z^3 and solve $z^6+z^3+1=0$ to get $z^3=\frac{-1\pm\sqrt{3}\,i}{2}=e^{\pm i\,2\pi/3}$. If $z^3=e^{i\,2\pi/3}$ then $z=e^{i\,2\pi/9}$, $e^{i\,8\pi/9}$ or $e^{i\,14\pi/9}$ and so $x=z+z^{-1}=z+\overline{z}=2\mathrm{Re}\,(z)=2\cos(\frac{2\pi}{9})$, $2\cos(\frac{8\pi}{9})$ or $2\cos(\frac{14\pi}{9})$. If $z^3=e^{-i\,2\pi/3}$ then we obtain the same values for x. Thus the three real roots are $2\cos(40°)$, $-2\cos(20°)$ and $2\cos(80°)$.