Chapter 4. Recursion and Induction

4.1 Theorem: *(Mathematical Induction) Let $F(n)$ be a statement about $n \in \mathbf{Z}$ and let $m \in \mathbf{Z}$. Suppose that $F(m)$ is true. Suppose that for all $n \in \mathbf{Z}$ with $n \geq m$, if $F(n)$ is true then $F(n+1)$ is true. Then $F(n)$ is true for all $n \in \mathbf{Z}$ with $n \geq m$.*

Proof: Let $S = \{k \in \mathbf{Z} | k \geq m \text{ and } F(k) \text{ is false}\}$. To prove that $F(n)$ is true for all $n \geq m$, we shall prove that $S = \emptyset$. Suppose, for a contradiction, that $S \neq \emptyset$. Since $S \neq \emptyset$ and S is bounded below by m, it follows from the Well-Ordering Property of $\mathbf{Z}$ that S has a minimum element. Let $a = \min(S)$. Since $a \in S$ it follows that $a \geq m$ and $F(a)$ is false. Since $F(m)$ is true and $F(a)$ is false, it follows that $a \neq m$. Since $a \geq m$ and $a \neq m$ it follows that $a > m$ and so $a - 1 \geq m$. We claim that $F(a-1)$ is true. Suppose, for a contradiction, that $F(a-1)$ is false. Since $a - 1 \geq m$ and $F(a-1)$ is false, it follows that $a-1 \in S$. Since $a = \min(S)$ and $a-1 \in S$, we have $a \leq a-1$. But we know that $a > a-1$ so we have obtained the desired contradiction (to the assumption that $F(a-1)$ is false). Thus $F(a-1)$ is true, as claimed. Since $a-1 \geq m$ and $F(a-1)$ is true, it follows by the hypothesis in the statement of the theorem that $F(a)$ is true. But, as mentioned earlier, since $a \in S$ we know that $F(a)$ is false, so we have obtained the desired contradiction (to the assumption that $S \neq \emptyset$). Thus $S = \emptyset$, as required.

4.2 Note: It follows, from the above theorem, that in order to prove that $F(n)$ is true for all $n \geq m$, we can do the following.

1. Prove that $F(m)$ is true (this is called proving the **base case**).
2. Let $n \geq m$ and suppose that $F(n)$ is true (this is called the **induction hypothesis**).
3. Prove that $F(n+1)$ is true.

Alternatively, we can prove that $F(n)$ is true for all $n \geq m$ as follows: prove that $F(m)$ is true, let $n > m$ and suppose that $F(n-1)$ is true, then prove that $F(n)$ is true.

4.3 Definition: For a sequence $(a_n)_{n \geq m}$, a formula for a_n in terms of n is called a **closed-form** formula for a_n, and a formula for a_n in terms of n along with previous terms a_k with $k < n$ is called a **recursion** formula for a_n.

4.4 Example: For the sequence $(a_n)_{n \geq 0}$ given in closed-form by $a_n = n^2$ for $n \geq 0$, we have

$$(a_n)_{n \geq 0} = (0, 1, 4, 9, 16, 25, 36, \cdots).$$

For the sequence $(a_n)_{n \geq 0}$ defined recursively by $a_0 = 2$ and $a_{n+1} = 2a_n - 1$ for $n \geq 0$, we have

$$(a_n)_{n \geq 0} = (2, 3, 5, 9, 17, 33, 62, \cdots).$$

The **Fibonacci sequence** is defined recursively by $a_0 = 0$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, so we have

$$(a_n)_{n \geq 0} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots).$$

4.5 Example: When we write

$$S_n = \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

what we really mean is that the sequence S_n is defined recursively by $S_m = a_m$ and $S_n = S_{n-1} + a_n$ for all $n > m$.

4.6 Example: When we write

$$P_n = \prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot \ldots \cdot a_n$$

what we really mean is that the sequence P_n is defined recursively by $P_m = a_m$ and $P_n = P_{n-1} \cdot a_n$ for $n > m$.

4.7 Example: When we say that $n!$ (read as n **factorial**) is defined for $n \in \mathbf{N}$ by $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, what we really mean is that $n!$ is defined recursively by $0! = 1$ and $n! = n \cdot (n-1)!$ for $n \geq 1$.

4.8 Example: Let $a_1 = 1$ and let $a_{n+1} = \frac{n}{n+1}\, a_n + 1$ for all $n \geq 1$. Find a closed-form formula for a_n.

Solution: Using the given recursion formula, the first few terms in the sequence $(a_n)_{n\geq 1}$ are as follows.

n	1	2	3	4	5	6
a_n	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$

It appears, from the table, that $a_n = \frac{n+1}{2}$ for all $n \geq 1$. Here is a proof by induction. When $n = 1$ we have $\frac{n+1}{2} = \frac{1+1}{2} = 1 = a_1 = a_n$. Let $n \geq 1$ be arbitrary and suppose, inductively, that $a_n = \frac{n+1}{2}$ (for this one particular value of n). Then we have

$$\begin{aligned} a_{n+1} &= \tfrac{n}{n+1}\, a_n + 1 \text{ by the recursion formula} \\ &= \tfrac{n}{n+1} \cdot \tfrac{n+1}{2} + 1 \text{ by the induction hypothesis} \\ &= \tfrac{n}{2} + 1 = \tfrac{(n+1)+1}{2}\ \text{, as required.} \end{aligned}$$

By induction, it follows that $a_n = \frac{n+1}{2}$ for all $n \geq 1$.

4.9 Example: Find $\prod\limits_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)$.

Solution: Let $P_n = \prod\limits_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)$ for $n \geq 2$. This means that the sequence $(P_n)_{n\geq 2}$ is defined recursively by $P_2 = 1 - \frac{1}{4} = \frac{3}{4}$ and $P_n = P_{n-1}\left(1 - \frac{1}{n^2}\right)$ for all $n \geq 3$. The first few values of P_n are as follows. $P_2 = 1 - \frac{1}{4} = \frac{3}{4}$, $P_3 = P_2\left(1 - \frac{1}{9}\right) = \frac{3}{4} \cdot \frac{8}{9} = \frac{2}{3}$, $P_4 = P_3\left(1 - \frac{1}{16}\right) = \frac{2}{3} \cdot \frac{15}{16} = \frac{5}{8}$, $P_5 = P_4\left(1 - \frac{1}{25}\right) = \frac{5}{8} \cdot \frac{24}{25} = \frac{3}{5}$, and $P_6 = P_5\left(1 - \frac{1}{36}\right) = \frac{3}{5} \cdot \frac{35}{36} = \frac{7}{12}$. It appears, from these first few values, that $P_n = \frac{n+1}{2n}$ for all $n \geq 2$. When $n = 2$ we have $\frac{n+1}{2n} = \frac{3}{4} = P_2$. Let $n \geq 3$ and suppose, inductively, that $P_{n-1} = \frac{n}{2(n-1)}$. Then

$$\begin{aligned} P_n &= P_{n-1}\left(1 - \tfrac{1}{n^2}\right)\ \text{, by the recursion formula for } P_n \\ &= \tfrac{n}{2(n-1)} \cdot \tfrac{n^2-1}{n^2}\ \text{, by the induction hypothesis} \\ &= \tfrac{n}{2(n-1)} \cdot \tfrac{(n-1)(n+1)}{n^2} = \tfrac{n+1}{2n}\ \text{, as required.} \end{aligned}$$

By induction, it follows that $P_n = \frac{n+1}{2n}$ for all $n \geq 2$.

4.10 Exercise: Find $\sum\limits_{k=1}^{n} k$ and find $\sum\limits_{k=1}^{n} k^3$.

4.11 Exercise: Let $(a_n)_{n\geq 0}$ be the Fibonacci sequence.

(a) Show that $a_0 + a_1 + a_2 + \cdots + a_n = a_{n+2} - 1$ for all $n \geq 0$.
(b) Show that ${a_0}^2 + {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2 = a_n a_{n+1}$ for all $n \geq 0$.
(c) Show that $a_{n-1}a_{n+1} = {a_n}^2 + (-1)^n$ for all $n \geq 1$.
(d) Show that ${a_{n-1}}^2 + {a_n}^2 = a_{2n-1}$ for all $n \geq 1$.

4.12 Theorem: *(Strong Mathematical Induction) Let $F(n)$ be a statement about $n \in \mathbf{Z}$ and let $m \in \mathbf{Z}$. Suppose that for all $n \in \mathbf{Z}$ with $n \geq m$, if $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < n$ then $F(n)$ is true. Then $F(n)$ is true for all $n \geq m$.*

Proof: Let $G(\ell)$ be the statement "$F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < \ell$". Note that $G(m)$ is true vacuously because there are no values of $k \in \mathbf{Z}$ with $m \leq k < m$. Let $\ell \geq m$ and suppose, inductively, that $G(\ell)$ is true or, in other words, suppose that $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < \ell$. Since $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < \ell$, it follows from the hypothesis in the statement of the theorem that $F(\ell)$ is true. Since $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < \ell$ and $F(\ell)$ is true, it follows that $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < \ell + 1$ or, in other words, it follows that $G(\ell + 1)$ is true, as required. By induction, it follows that $G(\ell)$ is true for all $\ell \geq m$.

Let $n \geq m$ be arbitrary. Since $G(\ell)$ is true for all $\ell \geq m$, in particular $G(n+1)$ is true, so $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < n+1$. Since $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < n+1$, in particular $F(n)$ is true. Since $n \geq m$ was arbitrary, it follows that $F(n)$ is true for all $n \geq m$.

4.13 Note: In order to prove that $F(n)$ is true for all $n \geq m$, we can do the following.

1. Let $n \geq m$ and suppose that $F(k)$ is true for all $k \in \mathbf{Z}$ with $m \leq k < n$.
2. Prove that $F(n)$ is true.

Although strong induction, used as above, does not require the proof that $F(m)$ is true (the base case), there are situations in which one ore more base cases must be verified to make this method of proof valid. For example, if a sequence $(x_n)_{n\geq 1}$ is defined by specifying the values of x_1 and x_2 and by giving a recursion formula for x_n in terms of x_{n-1} and x_{n-2} for all $n \geq 3$, then in order to prove that x_n satisfies the closed-form formula $x_n = f(n)$ for all $n \geq 1$ it suffices to prove that $x_1 = f(1)$ and $x_2 = f(2)$ (two base cases) and to prove that for all $n \geq 3$, if $x_{n-1} = f(n-1)$ and $x_{n-2} = f(x_{n-2})$ then $x_n = f(n)$.

4.14 Example: Let $a_0 = a_1 = 2$ and let $a_n = 2a_{n-1} + 3a_{n-2}$ for all $n \geq 2$. Find a closed-form formula for a_n.

Solution: The first few values of a_n are as follows.

n	0	1	2	3	4	5
a_n	2	2	10	26	82	242

It appears that $a_n = 3^n + (-1)^n$ for all $n \geq 0$. Here is a proof by induction. When $n = 0$ we have $3^n + (-1)^n = 3^0 + (-1)^0 = 1 + 1 = 2 = a_0 = a_n$ and when $n = 1$ we have $3^n + (-1)^n = 3^1 + (-1)^1 = 3 - 1 = 2 = a_1 = a_n$. Let $n \geq 2$ and suppose, inductively, that $a_k = 3^k + (-1)^k$ for all $k \in \mathbf{Z}$ with $0 \leq k < n$ (in particular for $k = n-1$ and $k = n-2$). Then

$$\begin{aligned} a_n &= 2a_{n-1} + 3a_{n-2} \text{ , by the recursion formula for } a_n \\ &= 2\big(3^{n-1} + (-1)^{n-1}\big) + 3\big(3^{n-2} + (-1)^{n-2}\big) \text{ , by the induction hypothesis} \\ &= 2 \cdot 3^{n-1} - 2(-1)^n + 3^{n-1} + 3(-1)^n = 3^n + (-1)^n \text{ , as required.} \end{aligned}$$

By induction, it follows that $a_n = 3^n + (-1)^n$ for all $n \geq 0$.

4.15 Note: One shortcoming with the method that we used in the above example is that we needed to guess a closed-form formula for the sequence and, for many sequences, such a closed-form formula can be extremely difficult to guess. For this reason it is useful to develop a method which allows us to calculate such a closed-form formula.

4.16 Theorem: *(Quadratic Linear Recursion) Let $a, b, p, q \in \mathbf{R}$ with $q \neq 0$ and define $(x_n)_{n \geq m}$ recursively by $x_m = a$, $x_{m+1} = b$ and $x_n = px_{n-1} + qx_{n-2}$ for all $n \geq m+2$. Let $f(x) = x^2 - px - q$ and suppose that $f(x)$ factors as $f(x) = (x-u)(x-v)$ for some $u, v \in \mathbf{R}$ with $u \neq v$. Then there exist unique numbers $A, B \in \mathbf{R}$ such that $x_n = Au^n + Bv^n$ for all $n \geq m$.*

Proof: Since $x^2 - px - q = f(x) = (x-u)(x-v) = x^2 - (u+v)x + pq$ we have $u + v = p$ and $uv = -q$. Since $q \neq 0$ and $uv = -q$ it follows that $u \neq 0$ and $v \neq 0$.

In order to have $x_n = Au^n + Bv^n$ for all $n \geq m$ we must have $Au^m + Bv^m = x_m = a$ (1) and $Au^{m+1} + Bv^{m+1} = b$ (2). Multiplying Equation (1) by v and subtracting Equation (2) gives $A(vu^m - u^{m+1}) = av - b$, so we must choose $A = \frac{av-b}{u^m(v-u)}$. Multiplying Equation (1) by u and subtracting (2) gives $B(uv^m - v^{m+1}) = au - b$, so we must choose $B = \frac{au-b}{v^m(u-v)}$.

Let $A = \frac{av-b}{u^m(v-u)}$ and $B = \frac{au-b}{v^m(u-v)}$. We claim that $x_n = Au^n + Bv^n$ for all $n \geq m$. Here is a proof by induction. When $n = m$ we have

$$Au^n + Bv^n = Au^m + Bv^m = \tfrac{av-b}{v-u} + \tfrac{au-b}{u-v} = \tfrac{av-b-au+b}{v-u} = a = x_m = x_n$$

and when $n = m+1$ we have

$$Au^n + Bv^n = Au^{m+1} + Bv^{m+1} = \tfrac{(av-b)u}{v-u} + \tfrac{(au-b)v}{u-v} = \tfrac{auv-bu-auv+bv}{v-u} = b = x_{m+1} = x_n.$$

Let $n \geq m+2$ and suppose, inductively, that $x_k = Au^k + Bv^k$ for all $k \in \mathbf{Z}$ with $m \leq k < n$ (in particular for $k = n-1$ and $k = n-2$. Then

$$\begin{aligned} x_n &= px_{n-1} + qx_{n-2} \text{ , by the recursion formula for } x_n \\ &= (u+v)x_{n-1} - (uv)x_{n-2} \text{ , since } u+v = p \text{ and } uv = -q \\ &= (u+v)\big(Au^{n-1} + Bv^{n-1}\big) - (uv)\big(Au^{n-2} + Bv^{n-2}\big) \text{ , by the induction hypothesis} \\ &= A\big((u+v)u^{n-1} - (uv)u^{n-2}\big) + B\big((u+v)v^{n-1} - (uv)v^{n-2}\big) \\ &= Au^n + Bv^n \text{ , as required.} \end{aligned}$$

It follows, by induction, that $x_n = Au^n + Bv^n$ for all $n \geq m$.

4.17 Theorem: *(Linear Recursion) Let $a_0, a_1, a_2, \cdots, a_{d-1}$ and $c_0, c_1, c_2, \cdots, c_{d-1}$ be real (or complex) numbers with $c_0 \neq 0$. Let $(x_n)_{n \geq m}$ be the sequence defined recursively by $x_m = a_0$, $x_{m+1} = a_1$, $x_{m+2} = a_2$, $\cdots$, $x_{m+d-1} = a_{d-1}$ and*

$$x_n + c_{d-1}x_{n-1} + c_{d-2}x_{n-2} + \cdots + c_1 x_{n-d+1} + c_0 x_{n-d} = 0.$$

Let $f(x) = x^d + c_{d-1}x^{d-1} + c_{d-2}x^{d-2} + \cdots + c_1 x + c_0$ and suppose that $f(x)$ factors as $f(x) = \prod_{i=1}^{\ell} (x - u_i)^{k_i}$ where the u_i are distinct real (or complex) numbers. Then there exist unique polynomials $p_1(x), p_2(x), \cdots, p_\ell(x)$, with each $p_i(x)$ of degree less than k_i, such that

$$x_n = \sum_{i=1}^{\ell} p_i(n){u_i}^n \text{ for all } n \geq m.$$

Proof: This is a stronger version of the previous theorem, but we omit the proof.

4.18 Example: Let $x_0 = 4$, $x_1 = -1$ and $x_n = 3x_{n-1} + 10x_{n-2}$ for $n \geq 2$. Find a closed-form formula for x_n.

Solution: The first few terms x_n are as follows.

n	0	1	2	3	4
4	-1	37	101	673	3029

It seems difficult to guess a closed-form formula for x_n from the information in the above table. Instead, we make use of the above theorem with $p = 3$ and $q = 10$. Let

$$f(x) = x^2 - px - q = x^2 - 3x - 10 = (x-5)(x+2).$$

From the above theorem, we know that for some $A, B \in \mathbf{R}$ we have $x_n = A(5)^n + B(-2)^n$ for all $n \geq 0$. In particular, we must have $A(5)^0 + B(-2)^0 = x_0$, that is $A + B = 4$ (1) and we must have $A(5)^1 + B(-2)^1 = x_1$, that is $5A - 2B = -1$ (2). Multiply Equation (1) by 2 and add Equation (2) to get $7A = 7$ so that $A = 1$, and multiply Equation (1) by 5 and subtract Equation (2) to get $7B = 21$ so that $B = 3$. Thus $x_n = 5^n + 3(-2)^n$ for all $n \geq 0$.

4.19 Exercise: Find a closed-form formula for the terms of the Fibonacci sequence.

4.20 Note: Suppose that we choose k of n objects, When the objects are chosen with replacement (so that repetition is allowed) and the order of the chosen objects matters (so the chosen objects form an ordered k-tuple), the number of ways to choose k of n objects is equal to n^k (since we have n choices for each of the k objects). For example, the number of ways to roll 3 six-sided dice is equal to $6^3 = 216$.

When the objects are chosen without replacement (so that the k chosen objects are distinct) and the order matters, the number of ways to choose k of n objects is equal to $n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ (since we have n choices for the first object and $n-1$ choices for the second object and so on). In particular, the number of ways to arrange n objects in order (to form an ordered n-tuple) is equal to $n!$.

When the objects are chosen without replacement and the order does not matter (so the chosen objects form a k-element set), the number of ways to choose k of n objects is equal to $\frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$ (since each k-element set can be ordered in $k!$ ways to form $k!$ ordered k-tuples, and there are $\frac{n!}{(n-k)!}$ such ordered k-tuples). For example, the number of 4-element subsets of the set $\{1,2,3,4,5,6,7\}$ is equal to $\frac{7!}{4!3!} = \frac{7\cdot6\cdot5\cdot4}{4\cdot3\cdot2\cdot1} = 7\cdot5 = 35$.

4.21 Definition: For $n, k \in \mathbf{N}$ with $0 \leq k \leq n$, we define the **binomial coefficient** $\binom{n}{k}$, read as "n choose k", by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

4.22 Theorem: *(Pascal's Triangle) For $k, n \in \mathbf{N}$ with $0 \leq k \leq n$ we have*

$$\binom{n}{0} = \binom{n}{n} = 1\,,\ \binom{n}{k} = \binom{n}{n-k} \text{ and } \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Proof: The formulas $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{k} = \binom{n}{n-k}$ are immediate from the definition of $\binom{n}{k}$ (since $0! = 1$) and we have

$$\begin{aligned}\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{(k+1)n!}{(k+1)!(n-k)!} + \frac{(n-k)n!}{(k+1)!(n-k)!} \\ &= \frac{(k+1+n-k)n!}{(k+1)!(n-k)!} = \frac{(n+1)!}{(k+1)!((n+1)-(k+1))!} = \binom{n+1}{k+1}.\end{aligned}$$

4.23 Exercise: Make a table displaying the values $\binom{n}{k}$ for $0 \leq k \leq n \leq 10$. The table forms a triangle of positive integers in which each entry is obtained by adding two of the entries above.

4.24 Notation: Let R be a ring and let $a \in R$. For $k \in \mathbf{Z}^+$ we write $ka = a + a + \cdots + a$ with k terms in the sum, and we write $(-k)a = k(-a)$, and we write $a^k = a \cdot a \cdot \ldots \cdot a$ with k terms in the product. For $0 \in \mathbf{Z}$ we write $0a = 0$ and $a^0 = 1$. When $a \in R$ is a unit, for $k \in \mathbf{Z}^+$ we write $a^{-k} = (a^{-1})^k$.

4.25 Exercise: Let R be a ring and let $a, b \in R$. Show that for all $k, l \in \mathbf{Z}$ we have $(-k)a = -(ka)$, $(k+l)a = ka + la$ and $(ka)(lb) = (kl)(ab)$. Show that for all $k, l \in \mathbf{Z}^+$ we have $a^{k+l} = a^k a^l$. Show that if $ab = ba$ then for all $k, l \in \mathbf{Z}^+$ we have $(ab)^k = a^k b^k$. Show that if a is a unit, then for all $k, l \in \mathbf{Z}$ we have $a^{-k} = (a^k)^{-1}$ and $a^{k+l} = a^k a^l$.

4.26 Theorem: *(Binomial Theorem) Let R be a ring, let $a, b \in R$ with $ab = ba$, and let $n \in \mathbf{N}$. Then*

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a\, b^{n-1} + \binom{n}{n} b^n .\end{aligned}$$

Proof: We shall prove, by induction, that $(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ for all $n \geq 0$.

When $n = 0$ we have $\sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{0}{0} a^0 b^0 = 1 = (a+b)^0 = (a+b)^n$.

When $n = 1$ we have $\sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a + b = (a+b)^1 = (a+b)^n$.

Let $n \geq 1$ and suppose, inductively that $(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Then

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= (a+b)\Big(\binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a\, b^{n-1} + \binom{n}{n} b^n \Big) \\ &= \binom{n}{0} a^{n+1} + \binom{n}{1} a^n b + \binom{n}{2} a^{n-1} b^2 + \cdots + \binom{n}{n-1} a^2\, b^{n-1} + \binom{n}{n} a\, b^n \\ &\qquad + \binom{n}{0} a^n b + \binom{n}{1} a^{n-1} b^2 + \cdots + \binom{n}{n-2} a^2 b^{n-1} + \binom{n}{n-1} a\, b^n + \binom{n}{n} b^{n+1} \\ &= a^{n+1} + \left(\binom{n}{0} + \binom{n}{1}\right) a^n b + \left(\binom{n}{1} + \binom{n}{2}\right) a^{n-1} b + \cdots \\ &\qquad + \left(\binom{n}{n-2} + \binom{n}{n-1}\right) a^2 b^{n-1} + \left(\binom{n}{n-1} + \binom{n}{n}\right) a\, b^n + b^{n+1} \\ &= \binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^n b + \binom{n+1}{2} a^{n-1} b^2 + \cdots + \binom{n+1}{n-1} a^2\, b^n + \binom{n+1}{n+1} a\, b^n \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k+1} a^{n+1-k} b^k\end{aligned}$$

as required, since $\binom{n}{0} = 1 = \binom{n+1}{0}$ and $\binom{n}{n} = 1 = \binom{n+1}{n+1}$ and $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ for all k with $0 \leq k \leq n$. By induction, we have $(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ for all $n \geq 0$.

Finally note that, by interchanging a and b, we also have $(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ for all $n \geq 0$.

4.27 Example: Expand $(x+2)^6$.

Solution: By the Binomial Theorem, we have

$$\begin{aligned}(x+2)^6 &= \binom{6}{0}x^6 + \binom{6}{1}x^5 2^1 + \binom{6}{2}x^4 2^2 + \binom{6}{3}x^3 2^3 + \binom{6}{4}x^2 2^4 + \binom{6}{5}x^1 2^5 + \binom{6}{6}2^6 \\ &= x^6 + 6\cdot 2\,x^5 + 15\cdot 4\,x^4 + 20\cdot 8\,x^3 + 15\cdot 16\,x^2 + 6\cdot 32\,x + 64 \\ &= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64.\end{aligned}$$

4.28 Example: Find the coefficient of x^8 in the expansion of $\left(5x^3 - \frac{2}{x^2}\right)^{11}$.

Solution: By the Binomial Theorem, we have

$$\left(5x^3 - \tfrac{2}{x^2}\right)^{11} = \sum_{k=0}^{11}\binom{11}{k}(5x^3)^{11-k}(2x^{-2})^k = \sum_{k=0}^{11}(-1)^k\binom{11}{k}5^{11-k}2^k\,x^{3(11-k)-2k}.$$

In order to get $3(11-k)-2k = 8$, we need $33-5k=8$, so we choose the term with $k=5$. Thus the coefficient of x^8 is equal to

$$(-1)^5\binom{11}{5}5^6\,2^5 = -\tfrac{11\cdot 10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2\cdot 1}5^6\,2^5 = -11\cdot 3\cdot 7\cdot 10^6 = -231{,}000{,}000.$$

4.29 Example: Find $\sum_{k=0}^{n}\binom{2n}{2k}\frac{1}{2^k}$.

Solution: Notice that

$$\begin{aligned}\left(1+\tfrac{1}{\sqrt2}\right)^{2n} &= \binom{2n}{0}+\binom{2n}{1}\tfrac{1}{\sqrt2}+\binom{2n}{2}\tfrac12+\binom{2n}{3}\tfrac{1}{2\sqrt2}+\binom{2n}{4}\tfrac{1}{2^2}+\cdots+\binom{2n}{2n-1}\tfrac{1}{2^{n-1}\sqrt2}+\binom{2n}{2n}\tfrac{1}{2^n}\\ \left(1-\tfrac{1}{\sqrt2}\right)^{2n} &= \binom{2n}{0}-\binom{2n}{1}\tfrac{1}{\sqrt2}+\binom{2n}{2}\tfrac12-\binom{2n}{3}\tfrac{1}{2\sqrt2}+\binom{2n}{4}\tfrac{1}{2^2}-\cdots-\binom{2n}{2n-1}\tfrac{1}{2^{n-1}\sqrt2}+\binom{2n}{2n}\tfrac{1}{2^n}\end{aligned}$$

Add these and divide by 2 to get

$$\tfrac12\left(\left(1+\tfrac{1}{\sqrt2}\right)^{2n}+\left(1-\tfrac{1}{\sqrt2}\right)^{2n}\right) = \binom{2n}{0}+\binom{2n}{2}\tfrac12+\binom{2n}{4}\tfrac{1}{2^2}+\cdots+\binom{2n}{2n}\tfrac{1}{2^n} = \sum_{k=0}^{n}\binom{2n}{2k}\tfrac{1}{2^k}.$$

4.30 Exercise: Let n be a positive integer. By calculating $\sum_{k=0}^{n}\left((k+1)^{m+1}-k^{m+1}\right)$ in two different ways, find a recursion formula for the sum $S_m = \sum_{k=0}^{n}k^m$.

4.31 Exercise: There are n points on a circle around a disc. Each of the $\binom{n}{2}$ pairs of points is connected by a line segment. No three of these line segments have a common point of intersection. Determine the number of regions into which the disc is divided by the line segments.

4.32 Exercise: Let $n \in \mathbf{Z}^+$. Show that every positive real number has a unique positive n^{th} root. When n is odd, show that every real number has a unique real n^{th} root.

4.33 Notation: When $n \in \mathbf{Z}^+$ and $x \in \mathbf{R}$ (with $x \geq 0$ in the case that n is even) we denote the unique n^{th} root of x by $x^{1/n}$ or by $\sqrt[n]{x}$. In the case $n=2$ and $x \geq 0$, we also write $x^{1/2}$ as $\sqrt{x}$.

4.34 Exercise: (The Quadratic Formula) Show that for all $a,b,c,x \in \mathbf{R}$ with $a \neq 0$ we have

$$ax^2+bx+c=0 \Longleftrightarrow b^2-4ac \geq 0 \text{ and } x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Be careful to notice (and prove) any familiar algebraic properties of $\mathbf{R}$, which you need to use in your proof, but which have not yet been mentioned in these course notes.