MATH 138 Solutions to the Final Exam, Fall 2024

[10] **1:** (a) Find $\displaystyle\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution: Let $u=\sqrt{x}$ so that $u^2=x$ and $2u\,du=dx$. Then

$$\int_{x=0}^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx=\int_{i=0}^2 \frac{e^u}{u}\cdot 2u\,du=\int_{u=0}^2 2e^u\,du=\Big[2e^u\Big]_{u=0}^2=2e^2-2.$$

(b) Find $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$.

Solution: Let $2\sin\theta=x$ so that $2\cos\theta=\sqrt{4-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$\int_{x=0}^2 \frac{x^2\,dx}{\sqrt{4-x^2}}=\int_{\theta=0}^{\pi/2}\frac{(2\sin\theta)^2}{2\cos\theta}\cdot 2\cos\theta\;d\theta=\int_{\theta=0}^{\pi/2}4\sin^2\theta\;d\theta=\int_{\theta=0}^{\pi/2}2-2\cos 2\theta\;d\theta=\Big[2\theta-\sin 2\theta\Big]_{\theta=0}^{\pi/2}=\pi\,.$$

(c) Find $\displaystyle\int_1^\infty \frac{3x-2}{x^3+2x^2}\,dx$.

Solution: To get $\frac{3x-2}{x^3+2x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$, we need $Ax(x+2)+B(x+2)+Cx^2=3x-2$. Equate coefficients to get $A+C=0$, $2A+B=3$ and $2B=-2$, so we need $B=-1$, $A=2$ and $C=-2$. Thus

$$\int_{x=1}^\infty \frac{3x-2}{x^3+2x^2}\,dx=\int_{x=1}^\infty \tfrac{2}{x}-\tfrac{1}{x^2}-\tfrac{2}{x+2}\;dx=\Big[2\ln\tfrac{x}{x+2}+\tfrac{1}{x}\Big]_{x=1}^\infty=0-\big(2\ln\tfrac13+1\big)=2\ln 3-1\,.$$

[10] **2:** (a) Find the area of the region given by $0\le x\le\pi$, $0\le y\le \sin^5 x$.

Solution: Letting $u=\cos x$ so $du=-\sin x\,dx$, and using symmetry, the area is

$$\begin{aligned}A&=\int_{x=0}^\pi \sin^5 x\;dx=\int_{x=0}^\pi (1-\cos^2 x)^2\sin x\;dx=\int_{u=1}^{-1}-(1-u^2)^2\;du=2\int_{u=0}^1(1-u^2)^2\;du\\&=2\int_{u=0}^1 1-2u^2+u^4\;du=2\Big[u-\tfrac23u^3+\tfrac15u^5\Big]_{u=0}^1=2\big(1-\tfrac23+\tfrac15\big)=\tfrac{16}{15}.\end{aligned}$$

(b) Let R be the region given by $0\le x\le\frac{\pi}{2}$, $0\le y\le\cos x$. Find the volume of the solid obtained by revolving R about the y-axis.

Solution: Using cylindrical shells, and integrating by parts using $u=2\pi x$, $du=2\pi\,dx$, $v=\sin x$ and $dv=\cos x\,dx$, the volume is

$$V=\int_{x=0}^{\pi/2}2\pi x\cos x\,dx=\Big[2\pi x\sin x-\int 2\pi\sin x\;dx\Big]_{x=0}^{\pi/2}=\Big[2\pi x\sin x+2\pi\sin x\Big]_{x=0}^{\pi/2}=\pi^2-2\pi.$$

(c) Let C be the curve given by $y=x^2$ with $0\le x\le\sqrt6$. Find the area of the surface obtained by revolving C about the y-axis.

Solution: For $y=x^2$ we have $dL=\sqrt{1+(y')^2}\,dx=\sqrt{1+(2x)^2}\,dx=\sqrt{1+4x^2}\,dx$. Letting $u=1+4x$ so that $du=8x\,dx$, the surface area is

$$A=\int_{x=0}^{\sqrt6}2\pi x\,dL=\int_{x=0}^{\sqrt6}2\pi x\sqrt{1+4x^2}\;dx=\int_{u=1}^{25}\tfrac{\pi}{4}\,u^{1/2}\,du=\Big[\tfrac{\pi}{6}\,u^{3/2}\Big]_{u=1}^{25}=\tfrac{\pi}{6}(125-1)=\tfrac{62\pi}{3}.$$

[10] **3:** (a) Solve the initial value problem given by $y' = 3\sqrt{xy}$ with $y(1) = 4$.

Solution: The DE is separable as we can write it as $y^{-1/2}\,dy = 3x^{1/2}\,dx$. Integrate both sides to get $2y^{1/2} = 2x^{3/2}+c$. To get $y(1) = 4$ we need $4 = 2+c$ so that $c = 2$, so the solution is given by $2y^{1/2} = 2x^{3/2}+2$, that is $y = (x^{3/2}+1)^2$.

(b) Solve the initial value problem given by $y' = x + y + 1$ with $y(0) = 1$.

Solution: The DE is linear as we can write it as $y' - y = x + 1$. An integrating factor is $\lambda = e^{\int -1\,dx} = e^{-x}$, and the solution is given by $y = e^x \int (x+1)e^{-x}\,dx$. Integrate by parts using $u = x+1$, $du = dx$, $v = -e^{-x}$ and $dv = e^{-x}\,dx$ to get

$$y = e^x \int (x+1)e^{-x}\,dx = e^x\Big(-(x+1)e^{-x} + \int e^{-x}\,dx\Big) = e^x\big(-(x+1)e^{-x} - e^{-x} + c\big) = ce^x - (x+2).$$

To get $y(0) = 1$ we need $1 = c - 2$ so that $c = 3$, so the solution is $y = 3\,e^x - (x+2)$.

(c) A tank initially contains 2 L of pure water. Brine (salty water), with a salt concentration of 3 gm/L, enters the tank at a rate of $r(t) = \frac{1}{t+1}$ L/min, where t is the time in minutes. The brine in the tank is kept well mixed, and drains from the tank at the same rate $r(t)$. Determine when the concentration of brine in the tank is 2 gm/L.

Solution: Let $S(t)$ be the amount of salt, in litres, at time t, in minutes. Taking $r_i = r_o = \frac{1}{t+1}$ and $c_i = 3$ and $c_o = \frac{S(t)}{2}$, the amount of salt satisfies the DE $S'(t) = r_{in}c_{in} - r_{out}c_{out} = \frac{3}{t+1} - \frac{S(t)}{2(t+1)}$. This DE is linear as we can write it as $S' + \frac{1}{2(t+1)}S = \frac{3}{t+1}$. An integrating factor is $\lambda = e^{\int \frac{1}{2(t+1)}\,dt} = e^{\frac{1}{2}\ln(t+1)} = (t+1)^{1/2}$ and the solution is $S(t) = (t+1)^{-1/2}\int 3(t+1)^{-1/2}\,dt = (t+1)^{-1/2}\big(6(t+1)^{1/2} + c\big)$. To get $S(0) = 0$ we need $6+c = 0$ so that $c = -6$, so the solution is $S(t) = (t+1)^{-1/2}\big(6(t+1)^{1/2} - 6\big) = 6 - \frac{6}{\sqrt{t+1}}$. The concentration 2 gm/L when the amount of salt is 4 gm, and we have

$$S(t) = 4 \iff 6 - \tfrac{6}{\sqrt{t+1}} = 4 \iff \tfrac{6}{\sqrt{t+1}} = 2 \iff \sqrt{t+1} = 3 \iff t = 8.$$

[10] **4:** (a) Determine, with proof, whether $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^2}$ converges.

Solution: We claim that $\sqrt{x} > \ln x$ for all $x > 0$. Let $f(x) = \sqrt{x} - \ln x$. Then $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x} = \frac{\sqrt{x}-2}{2x}$. Since $f'(4) = 0$ and $f'(x) < 0$ for $x \in (0,4)$ and $f'(x) > 0$ for $x \in (4,\infty)$, it follows that the minimum value of f is $f(4) = 2 - 2\ln 2$. Since $0 < \ln 2 < 1$ we have $0 < 2\ln 2 < 2$, and hence $f(x) \geq f(4) = 2 - 2\ln 2 > 0$ for all $x > 0$. This proves that $\sqrt{x} > \ln x$ for all $x > 0$. Thus for all $n > 1$ we have $0 < \ln n < \sqrt{n}$, hence $0 < (\ln n)^2 < n$, hence $\frac{1}{(\ln n)^2} > \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, it follows that $\sum \frac{1}{(\ln n)^2}$ diverges too, by the Comparison Test.

(b) Prove that if $\sum\limits_{n\geq 1} |a_n|$ converges then $\sum\limits_{n\geq 1} a_n$ converges.

Solution: Suppose that $\sum |a_n|$ converges. Note that for all n we have $-|a_n| \leq a_n \leq |a_n|$ and hence $0 \leq a_n + |a_n| \leq 2|a_n|$. If $\sum |a_n|$ converges then $\sum 2|a_n|$ converges by linearity, and hence $\sum (a_n + |a_n|)$ converges too, by comparison. Since $\sum |a_n|$ and $\sum (a_n + |a_n|)$ both converge, it follows that $\sum a_n$ converges too, by linearity (because $a_n = (a_n + |a_n|) - |a_n|$).

(c) Let $a_n > 0$ for all $n \in \mathbb{Z}^+$ and suppose that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$ with $0 \leq r < 1$. Prove that $\sum\limits_{n\geq 1} a_n$ converges.

Solution: Choose $s \in \mathbb{R}$ with $r < s < 1$. Since $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$, by taking $\epsilon = s - r$ we can choose $N \in \mathbb{Z}^+$ so that $n \geq N \implies \left|\frac{a_{n+1}}{a_n} - r\right| \leq s - r$. Then when $n \geq N$ we have $\frac{a_{n+1}}{a_n} \leq r + (s - r) = s$ so that $a_{n+1} \leq s\,a_n$. In particular, we have $a_{N+1} \leq s\,a_N$, and $a_{N_2} \leq s\,a_{N+1} = s^2 a_N$ and $a_{N+2} \leq s\,a_{N+2} \leq s^3 a_N$ and so on, so that in general $a_{N+k} \leq s^k\,a_N$ for all $k \geq 0$. Since $\sum s^k a_N$ converges (it is geometric with ratio $s < 1$) and $a_{N+k} \leq s^k a_N$ for all $k \geq 0$, it follows that $\sum a_{N+k}$ converges by the Comparison Test. Thus $\sum a_n$ also converges (since the first finitely many terms do not affect convergence).

[10] **5:** (a) Find the Taylor polynomial of degree 3 centred at 0 for $f(x) = e^x\sqrt{1+2x}$.

Solution: For all x with $|2x|<1$ we have

$$\begin{aligned} e^x(1+2x)^{1/2} &= \left(1+x+\tfrac{1}{2!}x^2+\tfrac{1}{3!}x^3+\cdots\right)\left(1+\tfrac12(2x)+\tfrac{(\frac12)(-\frac12)}{2!}(2x)^2+\tfrac{(\frac12)(-\frac12)(-\frac32)}{3!}(2x)^3+\cdots\right) \\ &= \left(1+x+\tfrac12x^2+\tfrac16x^3+\cdots\right)\left(1+x-\tfrac12x^2+\tfrac12x^3+\cdots\right) \\ &= 1+2x-x^2+\tfrac23x^3+\cdots \end{aligned}$$

and so the 3$^{\text{rd}}$ Taylor polynomial is $T_3(x) = 1+2x+x^2+\frac23x^3$.

(b) Approximate the value of $\ln\frac34$ so that the absolute error is $E \le \frac{1}{100}$.

Solution: We give two solutions. For the first solution, note that for all $|x|<1$ we have

$$\begin{aligned} \frac{1}{1+x} &= 1-x+x^2-x^3+\cdots \\ \ln(1+x) &= x-\tfrac12x^2+\tfrac13x^3-\tfrac14x^4+\cdots \end{aligned}$$

and hence, by taking $x=\frac13$, we have

$$\ln\tfrac34 = -\ln\tfrac43 = -\ln\left(1+\tfrac13\right) = -\tfrac13+\tfrac{1}{2\cdot3^2}-\tfrac{1}{3\cdot3^3}-\tfrac{1}{4\cdot3^4}+\cdots \cong -\tfrac13+\tfrac{1}{2\cdot3^2} = -\tfrac{5}{18}$$

with absolute error $E \le \frac{1}{3\cdot3^3} = \frac{1}{243}$ by the Alternating Series Test.

For the second solution, note that for all $|x|<1$ we have

$$\begin{aligned} \frac{1}{1-x} &= 1+x+x^2+x^3+\cdots \\ -\ln(1-x) &= x+\tfrac12x^2+\tfrac13x^3+\tfrac14x^4+\cdots \end{aligned}$$

and hence, by taking $x=\frac14$, we have

$$\ln\tfrac34 = \ln\left(1-\tfrac14\right) = -\left(\tfrac14+\tfrac{1}{2\cdot4^2}+\tfrac{1}{3\cdot4^3}+\cdots\right) \cong -\left(\tfrac14+\tfrac{1}{2\cdot4^2}\right) = -\tfrac{9}{32}$$

with absolute error

$$E = \tfrac{1}{3\cdot4^3}+\tfrac{1}{4\cdot4^4}+\tfrac{1}{5\cdot4^5}+\cdots \le \tfrac{1}{3\cdot4^3}+\tfrac{1}{3\cdot4^4}+\tfrac{1}{3\cdot4^5}+\cdots = \frac{\frac{1}{3\cdot4^3}}{1-\frac14} = \tfrac{1}{3\cdot4^3}\cdot\tfrac43 = \tfrac{1}{144}$$

by the Comparison Test and the formula for the sum of a geometric series.

(c) Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{n^2\,2^n}{n!}$.

Solution: For all $x\in\mathbb{R}$ we have $e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$. Differente to get $e^x = \sum_{n=1}^{\infty}\frac{n}{n!}x^{n-1}$. Multiply by x to get $x\,e^x = \sum_{n=1}^{\infty}\frac{n}{n!}x^n$. Differentiate again to get $(x+1)\,e^x = \sum_{n=1}^{\infty}\frac{n^2}{n!}x^{n-1}$. Multiply by x again to get $x(x+1)\,e^x = \sum_{n=1}^{\infty}\frac{n^2}{n!}x^n$. In particular, taking $x=2$ gives $\sum_{n=1}^{\infty}\frac{n^2 2^n}{n!} = 2\cdot(2+1)\,e^2 = 6\,e^2$.