

MATH 138 Calculus 2, Solutions to the Exercises for Chapter 3

- 1:** A tank initially contains 500 L of water. Water drains out at a rate $r(t)$ L/min. Some values of $r(t)$ are tabulated below.

t	0	10	20	30	40	50	60
r	1.0	1.4	2.0	2.8	4.0	5.8	8.0

- (a) Estimate the amount of water remaining in the tank after one hour by approximating the definite integral $\int_0^{60} r(t) dt$ using the midpoint rule on 3 subintervals.

Solution: The amount of water that drained out is

$$\int_0^{60} r(t) dt \cong (r(10) + r(30) + r(50))(20) = (1.4 + 2.8 + 5.8)(20) = (10)(20) = 200.$$

The number of liters remaining in the tank is approximately $500 - 200 = 300$.

- (b) Estimate the amount of water remaining in the tank after one hour by approximating the same integral using the Trapezoidal Rule on 6 subintervals.

Solution: Using the Trapezoid Rule T_6 , the amount of water that drained out is

$$\begin{aligned} \int_0^{60} r(t) dt &\cong \frac{1}{2}(r(10) + 2r(20) + 2r(30) + 2r(40) + 2r(50) + r(60))(10) \\ &= \frac{1}{2}(1.0 + 2.8 + 4.0 + 5.6 + 8.0 + 11.6 + 8.0)(10) \\ &= \frac{1}{2}(41)(10) = 205 \end{aligned}$$

so the number of litres remaining is approximately $500 - 205 = 295$.

- 2:** Suppose that $|f''(x)| \leq \frac{1}{2}$ and $|f'''(x)| \leq 2$ for all $x \in [0, 24]$, and that $f(x)$ has the following table of values.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$f(x)$	4	5	4	3	2	2	2	3	4	5	6	3	2	1	1	2	4	4	4	3	71	1	3	4	7

- (a) Use S_8 to approximate $\int_0^{24} f(x) dx$.

Solution: We have

$$\begin{aligned} \int_0^{24} f(x) dx &\cong \frac{24}{3 \cdot 8} \cdot (f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + 2f(18) + 4f(21) + 2f(24)) \\ &= 1 \cdot (4 + 12 + 4 + 20 + 4 + 8 + 8 + 4 + 7) = 71. \end{aligned}$$

- (b) Find a value of n such that if we estimate $\int_0^{24} f(x) dx$ using T_n the error is $E \leq .01$.

Solution: The error is $E \leq \frac{(b-a)^3}{12n^2} \max_{0 \leq x \leq 24} |f''(x)| = \frac{24^3}{12n^2} \frac{1}{2} = \frac{24^2}{n^2}$. To get $E \leq \frac{1}{100}$ we need $\left(\frac{24}{n}\right)^2 \leq \frac{1}{100}$, that is $\frac{24}{n} \leq \frac{1}{10}$, so we can take $n \geq 240$.

3: (a) Approximate $\int_0^{2\pi} 4^{\cos x} dx$ using R_6 .

Solution: The subintervals all have size $\frac{\pi}{3}$ and the endpoints of the subintervals are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$, so $\int_0^{2\pi} f(x) dx \cong \frac{\pi}{3} \left(f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right) + f(\pi) + f\left(\frac{4\pi}{3}\right) + f\left(\frac{5\pi}{3}\right) + f(2\pi) \right) = \frac{\pi}{3} (4^{1/2} + 4^{-1/2} + 4^{-1} + 4^{-1/2} + 4^{1/2} + 4^1) = \frac{\pi}{3} (2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + 2 + 4) = \frac{37\pi}{12}$.

(b) Find a value of n such that if we approximate $\int_0^4 3\sqrt{2x+1} dx$ using S_n then the error is $E \leq .0001$.

Solution: Let $f(x) = 3\sqrt{2x+1}$. Then $f'(x) = 3(2x+1)^{-1/2}$, $f''(x) = -3(2x+1)^{-3/2}$, $f'''(x) = 9(2x+1)^{-5/2}$ and $f''''(x) = -45(2x+1)^{-7/2}$, so for $0 \leq x \leq 4$ we have $|f''''(x)| \leq 45$. If E_n denotes the error, then $E_n = \frac{(b-a)^5}{180n^4} \max |f''''(x)| = \frac{4^5}{180n^4} 45 = \frac{4^4}{n^4}$. To get $\frac{4^4}{n^4} \leq 10^{-4}$ we need $n^4 \geq 4^4 \cdot 10^4$, so we take $n = 40$.

4: Evaluate the following improper integrals.

(a) $\int_0^2 x^3 \ln(x/2) dx$

Solution: Let $u = \ln(x/2)$ so $du = \frac{1}{x} dx$, and let $v = \frac{1}{4}x^4$ so $dv = x^3 dx$. Then $\int_0^2 x^3 \ln(x/2) dx = \int_0^2 u dv = \left[uv - \int v du \right]_0^2 = \left[\frac{1}{4}x^4 \ln(x/2) - \int \frac{1}{4}x^3 dx \right]_{0+}^2 = \left[\frac{1}{4}x^4 \ln(x/2) - \frac{1}{16}x^4 \right]_{0+}^2 = -1$, since $\lim_{x \rightarrow 0+} x^4 \ln(x/2) = 0$.

(b) $F(s) = \int_0^\infty e^{-st} \sin t dt$, where $s > 0$.

Solution: Let $I(s) = \int e^{-st} \sin t dt$. Let $u = e^{-st}$ and $v = -\cos t$. Then $I(s) = \int u dv = uv - \int v du = -e^{-st} \cos t - sJ(s)$, where $J(s) = \int e^{-st} \cos t dt$. Now let $u_1 = e^{-st}$ and $v_1 = \sin t$. Then $J(s) = \int u_1 dv_1 = u_1 v_1 - \int v_1 du_1 = e^{-st} \sin t + s \int e^{-st} \sin t dt = e^{-st} \sin t + sI$. So we have $I(s) = -e^{-st} \cos t - sJ(s) = -e^{-st} \cos t - s(e^{-st} \sin t + sI(s))$. We solve this for $I(s)$ to get $I(s) = \frac{-e^{-st}}{1+s^2} (\cos t + s \sin t)$. For fixed $s > 0$ we have $\lim_{t \rightarrow \infty} e^{-st} (\cos t + s \sin t) = 0$ and so $F(s) = \left[I(s) \right]_0^\infty = \frac{1}{1+s^2}$.

5: Evaluate the following improper integrals.

(a) $\int_2^{\infty} \frac{dx}{x^4 \sqrt{x^2 - 4}}$

Solution: Write $I = \int_2^{\infty} \frac{dx}{x^4 \sqrt{x^2 - 4}}$. Let $2 \sec \theta = x$ so $2 \tan \theta = \sqrt{x^2 - 4}$ and $2 \sec \theta \tan \theta d\theta = dx$. Then $I = \int_0^{\pi/2} \frac{2 \sec \theta \tan \theta d\theta}{16 \sec^4 \theta 2 \tan \theta} = \frac{1}{16} \int_0^{\pi/2} \cos^3 \theta d\theta = \frac{1}{16} \int_0^{\pi/2} (1 - \sin^2 \theta) \cos \theta d\theta = \frac{1}{16} \int_0^1 1 - u^2 du$, where $u = \sin \theta$, and so $I = \frac{1}{16} \left[u - \frac{1}{3} u^3 \right]_0^1 = \frac{1}{24}$.

(b) $\int_{-\infty}^{\infty} \frac{x(x+1)}{(x^2+1)^2} dx$.

Solution: Write $I = \int_{-\infty}^{\infty} \frac{x(x+1)}{(x^2+1)^2} dx$. To get $\frac{x(x+1)}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$ we need to have $A(x^3+x) + B(x^2+1) + Cx+D = x^2+x$. Equate coefficients to get $A=0$, $B=1$, $A+C=1$ and $B+D=0$. Solve these to get $A=0$, $B=1$, $C=1$ and $D=-1$. Thus $I = \int_{-\infty}^{\infty} \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} - \frac{1}{(x^2+1)^2} dx$. We have $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \left[\tan^{-1} x \right]_{-\infty}^{\infty} = \pi$, and $\int_{-\infty}^{\infty} \frac{x dx}{(x^2+1)^2} = \left[-\frac{1}{2(x^2+1)} \right]_{-\infty}^{\infty} = 0$, and to get $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$ we let $\tan \theta = x$ so $\sec \theta = \sqrt{1+x^2}$ and $\sec^2 \theta d\theta = dx$, and then $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \int_{-\pi/2}^{\pi/2} \frac{\sec^2 \theta d\theta}{\sec^4 \theta} = \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = \frac{\pi}{2}$. Thus $I = \pi + 0 - \frac{\pi}{2} = \frac{\pi}{2}$.