

MATH 138 Calculus 2, Solutions to the Exercises for Chapter 2

1: Evaluate each of the following definite integrals.

(a) $\int_1^3 x \sqrt[3]{x^2 - 1} \, dx$

Solution: We use the substitution $u = x^2 - 1$ so that $du = 2x \, dx$ to get

$$\int_1^3 x (x^2 - 1)^{\frac{1}{3}} \, dx = \int_0^8 \frac{1}{2} u^{\frac{1}{3}} \, du = \left[\frac{3}{8} u^{4/3} \right]_0^8 = \frac{3}{8} (16 - 0) = 6.$$

(b) $\int_0^{\pi/2} \frac{\cos^3 x}{1 + \sin^2 x} \, dx$

Solution: We use the substitution $u = \sin x$ so $du = \cos x \, dx$ to get

$$\begin{aligned} \int_0^{\pi/2} \frac{\cos^3 x}{1 + \sin^2 x} \, dx &= \int_0^{\pi/2} \frac{(1 - \sin^2 x) \cos x}{1 + \sin^2 x} \, dx = \int_0^1 \frac{1 - u^2}{1 + u^2} \, du = \int_0^1 -1 + \frac{2}{1 + u^2} \, du \\ &= \left[-u + 2 \tan^{-1} u \right]_0^1 = (-1 + \frac{\pi}{2}) - (0) = \frac{\pi}{2} - 1. \end{aligned}$$

2: Evaluate each of the following definite integrals.

(a) $\int_1^4 \frac{\ln x}{\sqrt{x}} \, dx$

Solution: We integrate by parts using $u = \ln x$, $du = \frac{1}{x} \, dx$, $v = 2x^{1/2}$ and $dv = x^{-1/2} \, dx$ to get

$$\int_1^4 \frac{\ln x}{\sqrt{x}} \, dx = \left[2x^{1/2} \ln x - \int 2x^{-1/2} \, dx \right]_1^4 = \left[2x^{1/2} \ln x - 4x^{1/2} \right]_1^4 = (4 \ln 4 - 8) - (-4) = 4 \ln 4 - 4.$$

(b) $\int_0^2 x^2 e^{x/2} \, dx$

Solution: We integrate by parts twice, the first time using $u_1 = x^2$, $du_1 = 2x \, dx$, $v_1 = 2e^{x/2}$, $dv_1 = e^{x/2} \, dx$, and the second time using $u_2 = 4x$, $du_2 = 4 \, dx$, $v_2 = v_1 = 2e^{x/2}$, $dv_2 = dv_1 = e^{x/2} \, dx$, to get

$$\begin{aligned} \int_0^2 x^2 e^{x/2} \, dx &= \left[2x^2 e^{x/2} - \int 4x e^{x/2} \, dx \right]_0^2 = \left[2x^2 e^{x/2} - \left(8x e^{x/2} - \int 8 e^{x/2} \, dx \right) \right]_0^2 \\ &= \left[2x^2 e^{x/2} - 8x e^{x/2} + 16 e^{x/2} \right]_0^2 = \left[2(x^2 - 4x + 8) e^{x/2} \right]_0^2 = 2(4e - 8) = 8e - 16. \end{aligned}$$

3: Find the following indefinite integrals.

(a) $\int (x^2 + 1) e^x dx$

Solution: Let $u = x^2 + 1$ and $v = e^x$. Then $\int (x^2 + 1) e^x dx = \int u dv = uv - \int v du = (x^2 + 1) e^x - \int 2x e^x dx$.

Let $u_1 = 2x$ and $v_1 = e^x$. Then $\int 2x e^x dx = \int u_1 dv_1 = u_1 v_1 - \int v_1 du_1 = 2x e^x - \int 2 e^x = 2x e^x - 2 e^x + a$.

Thus $\int (x^2 + 1) e^x dx = (x^2 + 1) e^x - \int 2x e^x dx = (x^2 + 1) e^x - 2x e^x + 2 e^x - a = (x^2 - x + 3) e^x + c$

(b) $\int \sin^3 x \cos^2 x dx$

Solution: Let $u = \cos x$ so $du = -\sin x dx$. Then $\int \sin^3 x \cos^2 x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx = \int (u^2 - 1) u^2 du = \int u^4 - u^2 du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + c = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + c$.

4: Evaluate each of the following definite integrals.

(a) $\int_0^{\sqrt{3}} \frac{x^3 dx}{\sqrt{x^2 + 1}}$

Solution: We use the substitution $u = \sqrt{x^2 + 1}$ so $du = \frac{x}{\sqrt{x^2 + 1}} dx$ and $x^2 = u^2 - 1$ to get

$$\int_0^{\sqrt{3}} \frac{x^3 dx}{\sqrt{x^2 + 1}} = \int_1^2 u^2 - 1 du = \left[\frac{1}{3} u^3 - u \right]_1^2 = \left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - 1 \right) = \frac{4}{3}.$$

(b) $\int_0^1 x \tan^{-1} x dx$

Solution: We integrate by parts using $u = \tan^{-1} x$, $du = \frac{1}{1+x^2} dx$, $v = \frac{1}{2} x^2 dx$ and $dv = x dx$ to get

$$\begin{aligned} \int_0^1 x \tan^{-1} x dx &= \left[\frac{1}{2} x^2 \tan^{-1} x - \int \frac{1}{2} \frac{x^2}{1+x^2} dx \right]_0^1 = \left[\frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int 1 - \frac{1}{1+x^2} dx \right]_0^1 \\ &= \left[\frac{1}{2} x^2 \tan^{-1} x + \frac{1}{2} \tan^{-1} x - \frac{1}{2} x \right]_0^1 = \left(\frac{\pi}{8} + \frac{\pi}{8} - \frac{1}{2} \right) - (0) = \frac{\pi}{4} - \frac{1}{2}. \end{aligned}$$

5: Evaluate the following definite integrals.

(a) $\int_0^4 \frac{x+2}{\sqrt{2x+1}} dx$

Solution: Let $u = 2x + 1$ so $x = \frac{u-1}{2}$ and $du = 2 dx$. Then $\int_0^4 \frac{x+2}{\sqrt{2x+1}} dx = \int_1^9 \frac{\frac{u-1}{2} + 2}{\sqrt{u}} \frac{1}{2} du = \frac{1}{4} \int_1^9 \frac{u+3}{\sqrt{u}} du = \frac{1}{4} \int_1^9 u^{1/2} + 3u^{-1/2} du = \frac{1}{4} \left[\frac{2}{3} u^{3/2} + 6u^{1/2} \right]_1^9 = \frac{1}{4} \left[(18 + 18) - \left(\frac{2}{3} + 6 \right) \right] = \frac{22}{3}.$

(b) $\int_0^{\pi/6} \sec^4 2x dx$

Solution: Let $u = \tan 2x$ so $du = 2 \sec^2 2x dx$. Then $\int_0^{\pi/6} \sec^4 2x dx = \int_0^{\pi/6} (1 + \tan^2 2x) \sec^2 2x dx =$

$$\frac{1}{2} \int_0^{\sqrt{3}} (1 + u^2) du = \frac{1}{2} \left[u + \frac{1}{3} u^3 \right]_0^{\sqrt{3}} = \sqrt{3}.$$

6: Find the following indefinite integrals.

$$(a) \int \frac{4x}{(x+1)(x^2+4x+5)} dx$$

Solution: Use partial fractions. To get $\frac{4x}{(x+1)(x^2+4x+5)} = \frac{A}{x+1} + \frac{B(2x+4)+C}{x^2+4x+5}$, we need to have $A(x^2+4x+5) + B(2x+4)(x+1) + C(x+1) = 4x$. We equate coefficients to get $A+2B=0$, $4A+6B+C=4$ and $5A+4B+C=0$. Solve these three equations to get $A=-2$, $B=1$ and $C=6$. So $\int \frac{4x}{(x+1)(x^2+4x+5)} dx = \int \frac{-2}{x+1} + \frac{2x+4}{x^2+4x+5} + \frac{6}{(x+2)^2+1} dx = -2\ln|x+1| + \ln(x^2+4x+5) + 6\tan^{-1}(x+2) + c$.

$$(b) \int \frac{x^2}{\sqrt{1-4x^2}} dx$$

Solution: Let $\sin \theta = 2x$, so $\cos \theta = \sqrt{1-4x^2}$ and $\cos \theta d\theta = 2 dx$. Then $\int \frac{x^2 dx}{\sqrt{1-4x^2}} = \int \frac{\frac{1}{4} \sin^2 \theta \frac{1}{2} \cos \theta d\theta}{\cos \theta} = \frac{1}{8} \int \sin^2 \theta d\theta = \frac{1}{16} \int (1 - \cos 2\theta) d\theta = \frac{1}{16} (\theta - \frac{1}{2} \sin 2\theta) + c = \frac{1}{16} (\theta - \sin \theta \cos \theta) + c = \frac{1}{16} \sin^{-1} 2x - \frac{1}{8} x \sqrt{1-4x^2} + c$.

7: Evaluate the following definite integrals.

$$(a) \int_0^1 (3+2x-x^2)^{-3/2} dx$$

Solution: Let $2\sin \theta = x-1$ so $2\cos \theta = \sqrt{4-(x-1)^2} = \sqrt{3+2x-x^2}$ and $2\cos \theta d\theta = dx$. Then $\int_0^1 \frac{dx}{(3+2x-x^2)^{3/2}} = \int_{x=0}^1 \frac{2\cos \theta d\theta}{(2\cos \theta)^3} = \frac{1}{4} \int_{x=0}^1 \sec^2 \theta d\theta = \frac{1}{4} [\tan \theta]_{x=0}^1 = \frac{1}{4} \left[\frac{x-1}{\sqrt{3+2x-x^2}} \right]_0^1 = \frac{1}{4\sqrt{3}}$.

$$(b) \int_1^2 \frac{5x^2+9}{x^4-9x^2} dx$$

Solution: To get $\frac{5x^2+9}{x^4-9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} + \frac{D}{x+3}$, we need $A(x^3-9x) + B(x^2-9) + C(x^3+3x^2) + D(x^3-3x) = 5x^2+9$. Equate coefficients to get $A+C+D=0$, $B+3C-3D=5$, $-9A=0$ and $-9B=0$. Solve these 4 equations to get $A=0$, $B=-1$, $C=1$ and $D=-1$. So $\int_1^2 \frac{5x^2+9}{x^4-9x^2} dx = \int_1^2 -\frac{1}{x^2} + \frac{1}{x-3} - \frac{1}{x+3} dx = \left[\frac{1}{x} + \ln(x-3) - \ln|x+3| \right]_1^2 = (\frac{1}{2} + \ln 1 - \ln 5) - (1 + \ln 2 - \ln 4) = -\frac{1}{2} - \ln \frac{5}{2}$