MATH 137 Calculus 1, Solutions to the Midterm Test, Fall 2012

[3] **1:** (a) Find the exact value of $\cos\left(-\frac{8\pi}{3}\right)$.

Solution: $\cos\left(-\frac{8\pi}{3}\right) = -\cos\left(-\frac{8\pi}{3}+3\pi\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$.

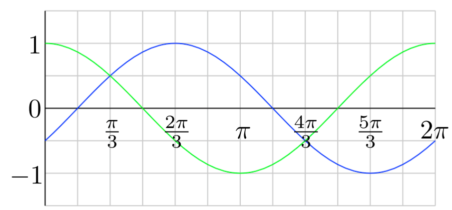
[3] (b) Use the formula $\sin(2\theta) = 2\sin\theta\cos\theta$ to find the exact value of $\sin\left(2\tan^{-1}\frac{1}{3}\right)$.

Solution: Consider the right-angled triangle with vertices at $a = (0,0)$, $b = (3,0)$ and $c = (3,1)$ (draw a picture of this triangle). Let θ be the angle at a. Since side ab has length 3 and side bc has length 1, we see that $\theta = \tan^{-1}\frac{1}{3}$. Since side ac has length $\sqrt{10}$ we see that $\sin\theta = \frac{1}{\sqrt{10}}$ and $\cos\theta = \frac{3}{\sqrt{10}}$. Thus

$$\sin\left(2\tan^{-1}\tfrac{1}{3}\right) = \sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\tfrac{1}{\sqrt{10}}\cdot\tfrac{3}{\sqrt{10}} = \tfrac{6}{10} = \tfrac{3}{5}.$$

[4] (c) Find all values of $x \in [0, 2\pi]$ such that $\sin\left(x - \frac{\pi}{6}\right) = \cos x$.

Solution: We provide two solutions. The first solution is graphical. We plot the graphs $y = \sin\left(x - \frac{\pi}{6}\right)$ and $y = \cos x$ on the same grid (the graph of $y = \sin\left(x - \frac{\pi}{6}\right)$ is shown in blue, and it is obtained from the graph of $y = \sin x$ by translating $\frac{\pi}{6}$ units to the right).



From the graph, we see that $\sin\left(x - \frac{\pi}{6}\right) = \cos x$ when $x = \frac{\pi}{3}, \frac{4\pi}{3}$. The second solution is algebraic. For $x \in [0, 2\pi]$ we have

$$\begin{aligned}
\sin\left(x - \tfrac{\pi}{6}\right) = \cos x \iff& \sin x\cos\left(-\tfrac{\pi}{6}\right) + \cos x\sin\left(-\tfrac{\pi}{6}\right) = \cos x \\
\iff& \tfrac{\sqrt{3}}{2}\sin x - \tfrac{1}{2}\cos x = \cos x \iff \sqrt{3}\sin x = 3\cos x \\
\iff& \tan x = \sqrt{3} \iff x = \tfrac{\pi}{3}, \tfrac{4\pi}{3}.
\end{aligned}$$

2: Let $f(x) = \dfrac{2e^x + 1}{e^x - 1}$.

[2] (a) Find the domain of f.

Solution: The domain of f is

$$\text{Domain}(f) = \left\{x \in \mathbf{R} \middle| e^x \neq 1\right\} = \left\{x \in \mathbf{R} \middle| x \neq 0\right\} = (-\infty, 0) \cup (0, \infty)\,.$$

[4] (b) Find a formula for the inverse function f^{-1}.

Solution: To find a formula for f^{-1}, we solve $y = f(x)$ for x in terms of y. We have

$$\begin{aligned} y = f(x) &\iff y = \frac{2e^x + 1}{e^x - 1} \iff e^x y - y = 2e^x + 1 \iff e^x(y-2) = y + 1 \\ &\iff e^x = \frac{y+1}{y-2} \iff x = \ln\left(\frac{y+1}{y-2}\right)\,. \end{aligned}$$

We see that $f^{-1}(y) = \ln\left(\frac{y+1}{y-2}\right)$.

[4] (c) Find the range of f (which is equal to the domain of f^{-1}).

Solution: The range of f is

$$\text{Range}(f) = \text{Domain}\big(f^{-1}\big) = \left\{y \in \mathbf{R} \middle| \tfrac{y+1}{y-2} > 0\right\}.$$

When $y < -1$, we have $y + 1 < 0$ and $y - 2 < 0$ so $\frac{y+1}{y-2} > 0$. When $y = -1$, we have $\frac{y+1}{y-2} = 0$. When $-1 < y < 2$, we have $y + 1 > 0$ and $y - 2 < 0$ so $\frac{y+1}{y-2} < 0$. When $y = 2$, $\frac{y+1}{y-2}$ is undefined. When $y > 2$, we have $y + 1 > 0$ and $y - 2 > 0$ so $\frac{y+1}{y-2} > 0$. Thus $\frac{y+1}{y-2} > 0$ when $y < -1$ or $y > 2$ and we have

$$\text{Range}(f) = \left\{y \in \mathbf{R} \middle| y < -1 \text{ or } y > 2\right\} = (-\infty, -1) \cup (2, \infty)\,.$$

3: Evaluate each of the following limits, if they exist or are infinite.

[3] (a) $\lim\limits_{x\to 1} \dfrac{\sqrt{3x+1}-2}{x-1}$.

Solution: We have

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt{3x+1}-2}{x-1} &= \lim_{x\to 1} \frac{\sqrt{3x+1}-2}{x-1}\cdot\frac{\sqrt{3x+1}+2}{\sqrt{3x+1}+2} = \lim_{x\to 1}\frac{(3x+1)-4}{(x-1)\big(\sqrt{3x+1}+2\big)}\\ &= \lim_{x\to 1}\frac{3(x-1)}{(x-1)\big(\sqrt{3x+1}+2\big)} = \lim_{x\to 1}\frac{3}{\sqrt{3x+1}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4}\,.\end{aligned}$$

[3] (b) $\lim\limits_{x\to 2^-} \dfrac{|x^2-3x+2|}{x-2}$.

Solution: For $1<x<2$ we have $x-2<0$ and $x-1>0$ so $|x-2| = -(x-2)$ and $|x-1| = (x-1)$ and so

$$\begin{aligned}\lim_{x\to 2^-}\frac{\big|x^2-3x+2\big|}{x-2} &= \lim_{x\to 2^-}\frac{\big|(x-2)(x-1)\big|}{x-2} = \lim_{x\to 2^-}\frac{|x-2|\,|x-1|}{x-2}\\ &= \lim_{x\to 2^-}\frac{-(x-2)(x-1)}{x-2} = \lim_{x\to 2^-} -(x-1) = -1\,.\end{aligned}$$

[4] (c) $\lim\limits_{x\to 0^-} \sin^{-1}\left(\dfrac{1}{2+e^{1/x}}\right)$.

Solution: As $x\to 0^-$ we have $\frac{1}{x}\to-\infty$ so $e^{1/x}\to 0$ and so

$$\lim_{x\to 0^-}\sin^{-1}\left(\frac{1}{2+e^{1/x}}\right) = \sin^{-1}\left(\frac{1}{2+0}\right) = \sin^{-1}\tfrac{1}{2} = \tfrac{\pi}{6}\,.$$

[5] **4:** (a) Use the definition of the limit to show that $\lim_{x\to 2}\left(x^2-x-3\right)=-1$.

Solution: We must show that

$$\forall\epsilon>0\ \ \exists\delta>0\ \ \forall x\in\mathbf{R}\ \left(0<|x-2|<\delta\Longrightarrow\left|(x^2-x-3)+1\right|<\epsilon\right).$$

Note that

$$\left|(x^2-x-3)+1\right|=\left|x^2-x-2\right|=\left|(x+1)(x-2)\right|=|x+1|\,|x-2|$$

and that

$$0<|x-2|<1\Longrightarrow 1<x<3\Longrightarrow 2<x+1<4\Longrightarrow |x+1|<4\,.$$

Let $\epsilon>0$. Choose $\delta=\min\left(1,\frac{\epsilon}{4}\right)$.Let $x\in\mathbf{R}$.Then

$$\begin{aligned}0<|x-2|<\delta &\Longrightarrow\left(0<|x-2|<1\text{ and }0<|x-2|<\tfrac{\epsilon}{4}\right)\\ &\Longrightarrow\left(|x+1|<4\text{ and }|x-2|<\tfrac{\epsilon}{4}\right)\\ &\Longrightarrow\left|(x^2-x-3)+1\right|=|x+1|\,|x-2|<4\cdot\tfrac{\epsilon}{4}=\epsilon\,.\end{aligned}$$

[5] (b) Suppose that $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}g(x)=M$. Prove that $\lim_{x\to a}\big(f(x)+g(x)\big)=L+M$.

Solution: This was one of the three required proofs. We must show that

$$\forall\,\epsilon{>}0\ \ \exists\,\delta{>}0\ \ \forall\,x{\in}\text{Domain}(f+g)\ \ \left(0<|x-a|<\delta\Longrightarrow\left|(f+g)(x)-(L+M)\right|<\epsilon\right).$$

Note that $\left|(f+g)(x)-(L+M)\right|=\left|(f(x)-L)+(g(x)-M)\right|\le\left|f(x)-L\right|+\left|g(x)-M\right|$ by the Triangle Inequality. Let $\epsilon>0$. Since $\lim_{x\to a}F(x)=L$, we can choose $\delta_1>0$ so that for all $x\in\text{Domain}(f)$ we have

$$0<|x-a|<\delta_1\Longrightarrow|f(x)-L|<\tfrac{\epsilon}{2}\,.$$

Since $\lim_{x\to a}g(x)=M$, we can choose $\delta_2>0$ so that for all $x\in\text{Domain}(g)$ we have

$$0<|x-a|<\delta_2\Longrightarrow|g(x)-M|<\tfrac{\epsilon}{2}\,.$$

Let $\delta=\min(\delta_1,\delta_2)$. Let $x\in\text{Domain}(f+g)$, so we have $x\in\text{Domain}(f)$ and $x\in\text{Domain}(g)$. Then

$$\begin{aligned}0<|x-a|<\delta &\Longrightarrow\left(0<|x-a|<\delta_1\text{ and }0<|x-a|<\delta_2\right)\\ &\Longrightarrow\left(|f(x)-L|<\tfrac{\epsilon}{2}\text{ and }|g(x)-M|<\tfrac{\epsilon}{2}\right)\\ &\Longrightarrow\left|(f+g)(x)-(L+M)\right|\le\left|f(x)-L\right|+\left|g(x)-M\right|<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon\,.\end{aligned}$$

5: Let $f(x) = \dfrac{1}{\sqrt{x}}$ for $x > 0$.

[5] (a) Use the definition of the derivative to find $f'(x)$.

Solution: We have

$$\begin{aligned} f'(x) &= \lim_{u\to x} \frac{f(u)-f(x)}{u-x} = \lim_{u\to x} \frac{\frac{1}{\sqrt{u}} - \frac{1}{\sqrt{x}}}{u-x} = \lim_{u\to x} \frac{\sqrt{x}-\sqrt{u}}{\sqrt{u}\sqrt{x}\,(u-x)} \\ &= \lim_{u\to x} \frac{\sqrt{x}-\sqrt{u}}{\sqrt{u}\sqrt{x}\,(u-x)} \cdot \frac{\sqrt{x}+\sqrt{u}}{\sqrt{x}+\sqrt{u}} = \lim_{u\to x} \frac{x-u}{\sqrt{u}\sqrt{x}\,(u-x)\big(\sqrt{x}+\sqrt{u}\big)} \\ &= \lim_{u\to x} \frac{-1}{\sqrt{u}\sqrt{x}\,\big(\sqrt{x}+\sqrt{u}\big)} = \frac{-1}{\sqrt{x}\sqrt{x}\,\big(\sqrt{x}+\sqrt{x}\big)} = \frac{-1}{2\,x^{3/2}}\,. \end{aligned}$$

Alternatively, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\,\sqrt{x}\;h} \\ &= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\,\sqrt{x}\;h} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} = \lim_{h\to 0} \frac{x-(x+h)}{\sqrt{x+h}\,\sqrt{x}\;h\,\big(\sqrt{x}+\sqrt{x+h}\big)} \\ &= \lim_{h\to 0} \frac{-h}{\sqrt{x+h}\,\sqrt{x}\;h\,\big(\sqrt{x}+\sqrt{x+h}\big)} = \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\,\sqrt{x}\,\big(\sqrt{x}+\sqrt{x+h}\big)} \\ &= \frac{-1}{\sqrt{x}\sqrt{x}\big(\sqrt{x}+\sqrt{x}\big)} = \frac{-1}{2\,x^{3/2}}\,. \end{aligned}$$

[5] (b) Find the equation of the tangent line to the curve $y = f(x)$ at the point where the tangent line has slope -4.

Solution: The slope of the tangent at the point $\big(a, f(a)\big)$ is equal to $f'(a) = \frac{-1}{2\,a^{3/2}}$, and we have

$$f'(a) = -4 \iff \frac{-1}{2\,a^{3/2}} = -4 \iff a^{3/2} = \tfrac{1}{8} \iff a = \big(\tfrac{1}{8}\big)^{2/3} = \tfrac{1}{4}\,.$$

Since $f\big(\frac{1}{4}\big) = 2$ and $f'\big(\frac{1}{4}\big) = -4$, the equation of the tangent line with slope -4 is $y = 2 - 4\big(x - \frac{1}{4}\big)$, or equivalently $y = -4x + 3$.