MATH 137 Calculus 1, Solutions to the Final Exam, Fall 2012

[2] **1:** (a) Find $\lim_{x\to 1} \dfrac{x^2-1}{\sqrt{x}-1}$.

Solution 1: We have

$$\lim_{x\to 1} \frac{x^2-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(x-1)(x+1)(\sqrt{x}+1)}{x-1} = \lim_{x\to 1} (x+1)(\sqrt{x}+1) = 2\cdot 2 = 4\,.$$

Solution 2: Since $x^2-1\to 0$ and $\sqrt{x}-1\to 0$ as $x\to 1$, we can apply l'Hôpital's Rule to get

$$\lim_{x\to 1} \frac{x^2-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{2x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 1} 4x^{3/2} = 4\,.$$

[3] (b) Let $f(x)=(x-1)^2$. Use the definition of the derivative to show that $f'(4)=6$.

Solution 1: We have

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \lim_{h\to 0} \frac{(h+3)^2-9}{h} = \lim_{h\to 0} \frac{h^2+6h}{h} = \lim_{h\to 0}(h+6) = 6\,.$$

Solution 2: We have

$$\begin{aligned} f'(4) &= \lim_{x\to 4} \frac{f(x)-f(4)}{x-4} = \lim_{x\to 4} \frac{(x-1)^2-9}{x-4} = \lim_{x\to 4} \frac{x^2-2x-8}{x-4} \\ &= \lim_{x\to 4} \frac{(x-4)(x+2)}{x-4} = \lim_{x\to 4}(x+2) = 6\,. \end{aligned}$$

[5] (c) Let $f(x)=\dfrac{6}{x}$. Use the definition of the limit to show that $\lim_{x\to 2} f(x) = 3$.

Solution 1: Note that

$$\left|\tfrac{6}{x}-3\right| = \left|\tfrac{6-3x}{x}\right| = \left|\tfrac{-3(x-2)}{x}\right| = \tfrac{3|x-2|}{|x|}$$

and note that

$$|x-2|<1 \Longrightarrow 1<x<3 \Longrightarrow |x|>1\,.$$

Let $\epsilon>0$. Choose $\delta=\min\left\{1,\frac{\epsilon}{3}\right\}$. Then for all x

$$\begin{aligned} 0<|x-2|<\delta &\Longrightarrow \left(|x-2|<1 \text{ and } |x-2|<\tfrac{\epsilon}{3}\right) \\ &\Longrightarrow \left(|x|>1 \text{ and } |x-2|<\tfrac{\epsilon}{3}\right) \\ &\Longrightarrow \left|\tfrac{6}{x}-3\right| = \tfrac{3|x-2|}{|x|} < \tfrac{3\,(\epsilon/3)}{1} = \epsilon\,. \end{aligned}$$

Solution 2: Let $\epsilon>0$. Choose $\delta=\frac{2\epsilon}{3+\epsilon}$. Note that $\delta<2$, so we have

$$0<|x-2|<\delta \Longrightarrow \left|\tfrac{6}{x}-3\right| = \left|\tfrac{6-3x}{x}\right| = \tfrac{3|x-2|}{|(x-2)+2|} < \tfrac{3\epsilon}{2-\delta} = \frac{3\cdot\frac{2\epsilon}{3+\epsilon}}{2-\frac{2\epsilon}{3+\epsilon}} = \tfrac{6\epsilon}{6+2\epsilon-2\epsilon} = \epsilon\,.$$

[5] **2:** (a) Approximate the value of $\sqrt{5}$ by finding the approximations x_2 and x_3 when Newton's Method is applied to the function $f(x) = x^2 - 5$ starting with $x_1 = 1$.

Solution: We have $f'(x) = 2x$ and so Newton's Method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^2 - 5}{2\,x_n} = \frac{{x_n}^2 + 5}{2\,x_n}\,.$$

Thus

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \tfrac{1^2+5}{2\cdot 1} = 3 \\ x_3 &= \tfrac{3^2+5}{2\cdot 3} = \tfrac{7}{3}\,. \end{aligned}$$

[5] (b) Let $y = g(x)$ be defined implicitly by the equation $y^3 + x^2 y = x + 3y^2$ with $g(2) = 1$. Use implicit differentiation to find $g'(2)$, then use the linearization of $g(x)$ at $x = 2$ to approximate the value of $g\left(\frac{5}{3}\right)$.

Solution: We differentiate implicitly to get

$$3y^2\,y' + 2xy + x^2 y' = 1 + 6y\,y'\,.$$

Putting in $x = 2$ and $y = 1$ gives $3y' + 4 + 4y' = 1 + 6y'$, that is $y' = -3$, so we have $g'(2) = -3$. The linearization of $g(x)$ at $x = 2$ is

$$l(x) = g(2) + g'(2)(x-2) = 1 - 3(x-2)$$

and we make the approximation

$$g\left(\tfrac{5}{3}\right) \cong l\left(\tfrac{5}{3}\right) = 1 - 3\left(\tfrac{5}{3} - 2\right) = 2\,.$$

3: Let $f(x) = \ln\left(\dfrac{x^2+1}{4}\right)$.

[5] (a) Determine where each of $f(x)$, $f'(x)$ and $f''(x)$ is positive, negative, and zero.

Solution: We have

$$\begin{aligned} f(x) &= \ln(x^2+1) - \ln 4 \\ f'(x) &= \frac{2x}{x^2+1} \\ f''(x) &= 2 \cdot \frac{(x^2+1) - x \cdot 2x^2}{(x^2+1)^2} = \frac{2\,(-x^2+1)}{(x^2+1)^2} = \frac{-2\,(x-1)(x+1)}{(x^2+1)} \end{aligned}$$

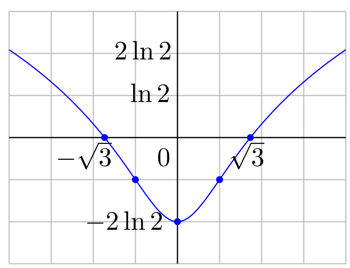
Note that $f(x) < 0 \iff \ln(x^2+1) < \ln 4 \iff x^2+1 < 4 \iff x^2 < 3 \iff |x| < \sqrt{3}$. We indicate where $f(x)$, $f'(x)$ and $f''(x)$ are positive, negative and zero in the following table.

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$f(x)$	$+$	0	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0	$-$

[5] (b) Sketch the curve $y = f(x)$ showing all x and y-intercepts, all local maxima and minima, and all points of inflection. (Note that $\ln 2 \cong 0.7$).

Solution: We make a table of values and sketch the curve.

x	$f(x)$	
$\to -\infty$	∞	
$-\sqrt{3}$	0	intercept
-1	$-\ln 2$	inflection
0	$-2\ln 2$	minimum
1	$-\ln 2$	inflection
$\sqrt{3}$	0	intercept
$\to \infty$	∞	



[5] **4:** (a) Let L be a line with negative slope which passes through the point $(2,1)$. Find the minimum possible area for the triangle bounded by L and the x and y-axes.

Solution: Let u be the x-intercept and let v be the y-intercept of the line L. Using similar triangles, we see that $\frac{v}{u} = \frac{1}{u-2}$ and so we have $v = \frac{u}{u-2}$. The area of the triangle is

$$A = \tfrac{1}{2}\, uv = \tfrac{1}{2} \cdot \frac{u^2}{u-2}\,.$$

Differentiate, with respect to u, to get

$$A' = \tfrac{1}{2} \cdot \frac{2u(u-2) - u^2}{(u-2)^2} = \frac{u^2 - 4u}{2\,(u-2)^2} = \frac{u(u-4)}{2\,(u-2)^2}\,.$$

We see that $A'(u) < 0$ for $u \in (2,4)$ and $A' > 0$ for $u \in (4,\infty)$ and so the minimum value of A occurs when $u = 4$ and then $A = \frac{u^2}{2\,(u-2)} = \frac{16}{2\cdot 2} = 4$.

[5] (b) Let $a = (0,0)$, $b = (3,0)$ and $c = (2,y)$. Let θ be the angle at c in the triangle abc. The point c moves downwards with $y' = -1$. Find θ' when $y = 1$.

Solution 1: Let α be the angle at a and let β be the angle at b in triangle abc. Note that $\tan\alpha = \frac{y}{2}$, $\tan\beta = \frac{y}{1}$ and $\alpha + \beta + \theta = \pi$. Thus

$$\theta = \pi - \alpha - \beta = \pi - \tan^{-1}\tfrac{y}{2} - \tan^{-1} y$$
$$\theta' = -\frac{\frac{1}{2}\,y'}{1 + \frac{y^2}{4}} - \frac{y'}{1+y^2} = -\Big(\frac{2}{4+y^2} + \frac{1}{1+y^2}\Big)\,y'\,.$$

Put in $y = 1$ and $y' = -1$ to get $\theta' = -\big(\frac{2}{5} + \frac{1}{2}\big)(-1) = \frac{9}{10}$.

Solution 2: By the Law of Cosines we have

$$9 = (4+y^2) + (1+y^2) - 2\sqrt{4+y^2}\sqrt{1+y^2}\cos\theta$$
$$2\sqrt{(y^2+4)(y^2+1)}\cos\theta = 2y^2 - 4$$
$$\cos\theta = \frac{y^2-2}{\sqrt{y^4+5y^2+4}}\,.$$

Differentiate to get

$$-\sin\theta\cdot\theta' = \frac{2y\sqrt{y^4+5y^2+4} - (y^2-2)\frac{4y^3+10y)}{2\sqrt{y^4+5y^2+4}}}{y^4+5y^2+4}\cdot y'\,.$$

Note that when $y = 1$ we have $\cos\theta = \frac{y^2-2}{\sqrt{y^4+5y^2+4}} = \frac{-1}{\sqrt{10}}$ and so $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1 - \frac{1}{10}} = \frac{3}{\sqrt{10}}$. We put $y = 1$, $y' = -1$ and $\sin\theta = \frac{3}{\sqrt{10}}$ into the above equation to get

$$-\tfrac{3}{\sqrt{10}}\cdot\theta' = \frac{2\sqrt{10} + \frac{14}{2\sqrt{10}}}{10}\cdot(-1)$$

and so

$$\theta' = \tfrac{\sqrt{10}}{3}\cdot\frac{2\sqrt{10} + \frac{7}{\sqrt{10}}}{10} = \tfrac{20+7}{30} = \tfrac{9}{10}\,.$$

[3] **5:** (a) Find $\displaystyle\int_0^{\pi/6} \frac{\cos x\; dx}{\sqrt{1+6\sin x}}$.

Solution: Let $u = 1+6\sin x$ so $du = 6\cos x\,dx$. Then

$$\int_{x=0}^{\pi/6} \frac{\cos x\; dx}{\sqrt{1+6\sin x}} = \int_{u=1}^{4} \frac{\frac{1}{6}\,du}{\sqrt{u}} = \int_1^4 \tfrac{1}{6}\,u^{-1/2}\,du = \Big[\tfrac{1}{3}\,u^{1/2}\Big]_1^4 = \tfrac{2}{3} - \tfrac{1}{3} = \tfrac{1}{3}\,.$$

[3] (b) Let $\displaystyle g(x) = \int_1^{\sqrt{x}} \sqrt{5+t^2}\,dt$. Find $g'(4)$.

Solution: Let $u(x) = \sqrt{x}$ and let $\displaystyle F(u) = \int_1^u f(t)\,dt$ where $f(t) = \sqrt{5+t^2}$, so that we have $g(x) = F\big(u(x)\big)$. By the FTC we have $F'(u) = f(u)$ and by the Chain Rule we have

$$g'(x) = F'\big(u(x)\big)u'(x) = f\big(u(x)\big)u'(x) = \sqrt{5+u(x)^2}\cdot\frac{1}{2\sqrt{x}} = \frac{\sqrt{5+x}}{2\sqrt{x}}\,.$$

In particular $g'(4) = \frac{3}{4}$.

[4] (c) Evaluate $\displaystyle\int_{-1}^{2} x^2+1\;dx$ by finding a limit of Riemann sums using the right endpoints of n equal-sized subintervals.

Solution: Let $a=-1$, $b=2$, $\Delta_n x = \frac{b-a}{n} = \frac{3}{n}$, and $x_{n,i} = a + i\,\Delta_n x = -1+\frac{3i}{n}$. Then

$$\begin{aligned}\int_{-1}^{2} x^2+1\;dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\big(x_{n,i}\big)\Delta_n x = \lim_{n\to\infty}\sum_{i=1}^{n} f\big(-1+\tfrac{3i}{n}\big)\cdot\tfrac{3}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\Big(\big(-1+\tfrac{3i}{n}\big)^2+1\Big)\cdot\tfrac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\big(2-\tfrac{6i}{n}+\tfrac{9i^2}{n^2}\big)\cdot\tfrac{3}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\big(\tfrac{6}{n}-\tfrac{18\,i}{n}+\tfrac{27\,i^2}{n^3}\big) = \lim_{n\to\infty}\Big(\tfrac{6}{n}\sum_{i=1}^{n}1-\tfrac{18}{n^2}\sum_{i=1}^{n}i+\tfrac{27}{n^3}\sum_{i=1}^{n}i^2\Big)\\
&= \lim_{n\to\infty}\Big(\tfrac{6}{n}\cdot n-\tfrac{18}{n^2}\cdot\tfrac{n(n+1)}{2}+\tfrac{27}{n^3}\cdot\tfrac{n(n+\frac{1}{2})(n+1)}{3}\Big) = 6-\tfrac{18}{2}+\tfrac{27}{3} = 6\,.\end{aligned}$$

[5] **6:** (a) An object moves along the x-axis with acceleration at time t given by $a(t) = \dfrac{2}{\sqrt{t+1}} - 1$ for $0 \le t \le 8$. Given that $x(0) = v(0) = 0$, find $x(8)$.

Solution: We have

$$v(t) = \int a(t)\,dt = \int 2(t+1)^{-1/2} - 1\;dt = 4(t+1)^{1/2} - t + c$$

for some constant c. Since $v(0) = 0$ we have $4 + c = 0$ so $c = -4$.Thus

$$v(t) = 4(t+1)^{1/2} - t - 4\,.$$

Since $x(0) = 0$ we have

$$\begin{aligned} x(8) = x(8) - x(0) &= \int_0^8 v(t)\,dt = \int_0^8 4(t+1)^{1/2} - t - 4\;dt \\ &= \Big[\tfrac{8}{3}\,(t+1)^{3/2} - \tfrac{1}{2}\,t^2 - 4t\Big]_0^8 = \big(72 - 32 - 32\big) - \big(\tfrac{8}{3}\big) = \tfrac{16}{3}\,. \end{aligned}$$

[5] (b) Find the area of the region bounded by the curves $y = x(x-4)$ and $y = \dfrac{2x}{x-3}$.

Solution: It helps to sketch the two curves. The parabola $y = x(x-4)$ is shown in cyan, the hyperbola $y = \frac{2x}{x-3} = 2 + \frac{6}{x-3}$ is shown in green, and the region bounded by the two curves is outlined in blue. If your sketch is sufficiently accurate, then it will show the exact coordinates of the points of intersection of the two curves. Alternatively, we can find the x-coordinates of the points of intersection as follows:

$$\begin{aligned} x(x-4) = \frac{2x}{x-3} &\iff x(x-4)(x-3) = 2x \iff x^3 - 7x^2 + 12x = 2x \\ &\iff x^3 - 7x^2 + 10x = 0 \iff x(x-2)(x-5) = 0 \iff x \in \{0, 2, 5\}\,. \end{aligned}$$

The area of the bounded region is

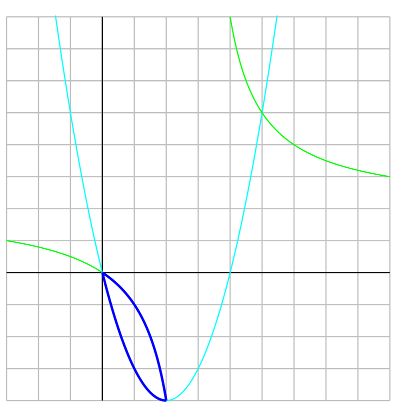
$$A = \int_0^2 \frac{2x}{x-3} - x(x-4)\;dx\,.$$

We solve the integral in two ways. For the first solution, we write $\frac{2x}{x-3} = \frac{6}{x-3} + 2$ to get

$$\begin{aligned} A &= \int_0^2 \frac{6}{x-3} + 2 + 4x - x^2\;dx = \Big[6\ln|x-3| + 2x + 2x^2 - \tfrac{1}{3}x^3\Big]_0^2 \\ &= \big(6\ln 1 + 4 + 8 - \tfrac{8}{3}\big) - \big(6\ln 3\big) = \tfrac{28}{3} - 6\ln 3\,. \end{aligned}$$

For the second solution we make the substitution $u = x - 3$ so that $x = u + 3$ and $dx = du$ to get

$$\begin{aligned} A &= \int_{u=-3}^{-1} \frac{2(u+3)}{u} - (u+3)(u-1)\;du = \int_{-3}^{-1} 2 + \frac{6}{u} - (u^2 + 2u - 3)\;du = \int_{-3}^{-1} \frac{6}{u} + 5 - 2u + u^2\;du \\ &= \Big[6\ln|u| + 5u - u^2 - \tfrac{1}{3}\,u^3\Big]_{-3}^{-1} = \big(-5 - 1 + \tfrac{1}{3}\big) - \big(6\ln 3 - 15 - 9 + 9\big) = \tfrac{28}{3} - 6\ln 3\,. \end{aligned}$$



7: Suppose that $f(x)$ is defined for all x in an open interval I with $a \in I$.

[4] (a) Prove the Decreasing Test: if $f'(x) < 0$ for all $x \in I$ then f is decreasing in I.

Solution: Suppose that $f'(x) < 0$ for all $x \in I$. Let $a, b \in I$ with $a < b$. We must show that $f(a) > f(b)$. Since $f(x)$ is differentiable in (a, b) and $f(x)$ is continuous on $[a, b]$ (indeed $f(x)$ is differentiable and hence continuous at every point $x \in I$), by the Mean Value Theorem we can choose $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f'(c) < 0$ and $b - a > 0$ we have $f(b) - f(a) = f'(c)(b - a) < 0$ and so $f(b) < f(a)$, as required.

[6] (b) Prove Fermat's Theorem: if $f'(a)$ exists and f has a local maximum or minimum at $x = a$, then $f'(a) = 0$.

Solution 1: We shall prove the equivalent statement that if $f'(a)$ exists but $f'(a) \neq 0$ then $f(x)$ does not have a maximum or minimum value at $x = a$. Suppose that $f'(a)$ exists but $f'(a) \neq 0$. Say $f'(a) > 0$ (the case that $f'(a) < 0$ is similar). By the definition of the derivative, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Since $f'(a) > 0$, by the definition of the limit we can choose $\delta > 0$ so that for all $x \in I$

$$0 < |x - a| < \delta \Longrightarrow \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < f'(a).$$

Note that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < f'(a) \Longrightarrow \frac{f(x) - f(a)}{x - a} - f'(a) > -f'(a) \Longrightarrow \frac{f(x) - f(a)}{x - a} > 0.$$

If $x \in (a, a + \delta)$ so that $x - a > 0$ then we have $f(x) - f(a) > 0$ so that $f(x) > f(a)$, and if $x \in (a - \delta, a)$ so that $x - a < 0$ then we have $f(x) - f(a) < 0$ so that $f(x) < f(a)$. Since $f(x) > f(a)$ for all $x \in (a, a + \delta)$ it follows that $f(x)$ cannot have a local maximum at $x = a$, and since $f(x) < f(a)$ for all $x \in (a - \delta, a)$ it follows that $f(x)$ cannot have a local minimum at $x = a$.

Solution 2: Suppose that $f'(a)$ exists and that f has a local maximum at $x = a$ (the case that f has a local minimum at $x = a$ is similar). Then for h sufficiently close to zero we have $f(a + h) \leq f(a)$, that is

$$f(a + h) - f(a) \leq 0 \qquad (1).$$

When $h > 0$, dividing both sides of (1) by h gives $\dfrac{f(a + h) - f(a)}{h} \leq 0$ and so we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a + h) = f(a)}{h} \leq 0.$$

When $h < 0$, dividing both sides of (1) by h gives $\dfrac{f(a + h) - f(a)}{h} \geq 0$ and so we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0^-} \frac{f(a + h) = f(a)}{h} \geq 0.$$

Since $f'(a) \leq 0$ and $f'(a) \geq 0$, we have $f'(a) = 0$, as required.