Lecture Notes for MATH 137

Single Variable Differential Calculus

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Chapter 1. Exponential and Trigonometric Functions

1.1 Definition: Let X and Y be sets and let $f : X \to Y$. We say that f is **injective** (or **one-to-one**, written as $1\!:\!1$) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$. We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is bijective, we define the **inverse** of f to be the function $f^{-1} : Y \to X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1}, and in this case we have $(f^{-1})^{-1} = f$.

1.2 Example: Let $f(x) = \frac{1}{3}\sqrt{12x - x^2}$ for $0 \le x \le 6$. Show that f is injective and find a formula for its inverse function.

Solution: Note that when $0 \le x \le 6$ (indeed when $0 \le x \le 12$) we have $12x - x^2 = x(12 - x) \ge 0$, so that $\frac{1}{3}\sqrt{12x - x^2}$ exists, and we have $12x - x^2 = 36 - (x - 6)^2 \le 36$ so that $\frac{1}{3}\sqrt{12x - x^2} \le \frac{1}{3}\sqrt{36} = 2$. Thus if $0 \le x \le 6$ then $f(x) = \frac{1}{3}\sqrt{12x - x^2}$ exists and we have $0 \le f(x) \le 2$. Let $x, y \in \mathbf{R}$ with $0 \le x \le 6$ and $0 \le y \le 2$. Then we have

$$\begin{aligned}
y = f(x) &\iff y = \tfrac{1}{3}\sqrt{12x - x^2} \\
&\iff 3y = \sqrt{12x - x^2} \\
&\iff 9y^2 = 12x - x^2 \quad \text{, since } y \ge 0 \\
&\iff x^2 - 12x + 9y^2 = 0 \\
&\iff x = \frac{12 \pm \sqrt{144 - 36y^2}}{2} = 6 \pm 3\sqrt{4 - y^2} \quad \text{, by the Quadratic Formula} \\
&\iff x = 6 - 3\sqrt{4 - y^2} \quad \text{since } x \le 6.
\end{aligned}$$

Verify that when $0 \le y \le 2$ we have $0 \le 4 - y^2 \le 4$ so that $\sqrt{4 - y^2}$ exists and we have $0 \le 6 - 3\sqrt{4 - y^2} \le 6$. Thus when we consider f as a function $f : [0, 6] \to [0, 2]$, it is bisectve and its inverse $f^{-1} : [0, 2] \to [0, 6]$ is given by $f^{-1}(y) = 6 - 3\sqrt{4 - y^2}$.

1.3 Definition: Let $f : A \subseteq \mathbf{R} \to \mathbf{R}$. We say that f is **even** when $f(-x) = f(x)$ for all $x \in A$ and we say that f is **odd** when $f(-x) = -f(x)$ for all $x \in A$.

1.4 Definition: Let $f : A \subseteq \mathbf{R} \to \mathbf{R}$. We say that f is **increasing** (on A) when it has the property that for all $x, y \in A$, if $x < y$ then $f(x) < f(y)$, and we say f is decreasing (on A) when for all $x, y \in A$ with $x < y$ we have $f(x) > f(y)$. We say that f is **monotonic** when f is either increasing or decreasing. Note that every monotonic function is injective.

1.5 Remark: We assume familiarity with exponential, logarithmic, trigonometric and inverse trigonometric functions. These functions can be defined rigorously. We shall give a brief description of how one can define the exponential and logarithmic function rigorously, and we shall provide an informal (non-rigorous) description of the trigonometric and inverse trigonometric functions, and we shall summarize some of their properties (without giving rigorous proofs).

1.6 Definition: Let us outline one possible way to define the value of x^y for suitable real numbers $x, y \in \mathbf{R}$. First we define $x^0 = 1$ for all $x \in \mathbf{R}$. Then for $n \in \mathbf{Z}$ with $n \geq 1$ we define x^n recursively by $x^n = x \cdot x^{n-1}$ for all $x \in \mathbf{R}$. Also, for $n \in \mathbf{Z}$ with $n \geq 1$ we define $x^{-n} = \frac{1}{x^n}$ for all $x \neq 0$. At this stage we have defined x^y for $y \in \mathbf{Z}$.

When $0 < n \in \mathbf{Z}$ is odd, for all $x \in \mathbf{R}$ we define $x^{1/n} = y$ where y is the unique real number such that $y^n = x$ (to be rigorous, one must prove that this number y exists and is unique). When $0 < n \in \mathbf{Z}$ is even, for $x \geq 0$ we define $x^{1/n} = y$ where y is the unique nonnegative real number such that $y^n = x$ (again, to be rigorous a proof is required). Also, for $0 < n \in \mathbf{Z}$ we define $x^{-1/n} = \frac{1}{x^{1/n}}$, which is defined for $x \neq 0$ if n is odd, and is defined for $x > 0$ when n is even. When $n, m \in \mathbf{Z}$ with $n > 0$ and $m > 0$ and $\gcd(n, m) = 1$, we define $x^{n/m} = (x^n)^{1/m}$, which is defined for all $x \in \mathbf{R}$ when m is odd and for $x \geq 0$ when m is even, and we define $x^{-n/m} = \frac{1}{x^{n/m}}$, defined for $x \neq 0$ when m is odd and for $x > 0$ when m is even. At this stage, we have defined x^y for $y \in \mathbf{Q}$.

For $y \in \mathbf{R}$, when $x > 0$ and $y \in \mathbf{R}$, we define

$$x^y = \lim_{t \to y, t \in \mathbf{Q}} x^t$$

(to be rigorous, one needs to define this limit and prove that it exists and is unique). Finally, we define $1^y = 1$ for all $y \in \mathbf{R}$ and we define $0^y = 0$ for all $y > 0$.

1.7 Theorem: *(Properties of Exponentials) Let $a, b, x, y \in \mathbf{R}$ with $a, b > 0$. Then*
(1) $a^0 = 1$,
(2) $a^{x+y} = a^b \, a^c$,
(3) $a^{x-y} = a^x / a^y$,
(4) $(a^x)^y = a^{xy}$,
(5) $(ab)^x = a^x b^x$.

Proof: We omit the proof.

1.8 Theorem: *(Properties of Power Functions)*
(1) When $a > 0$, the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^a$ is increasing and bijective and its inverse function is given by $f^{-1}(x) = x^{1/a}$.
(2) When $a < 0$, the function $f : (0, \infty) \to (0, \infty)$ given by $f(x) = x^a$ is decreasing and bijective and its inverse is given by $f^{-1}(x) = a^{1/x}$.

Proof: We omit the proof.

1.9 Definition: A function of the form $f(x) = x^a$ is called a **power function**.

1.10 Theorem: *(Properties of Exponential Functions)*
(1) When $a > 1$ the function $f : \mathbf{R} \to (0, \infty)$ given by $f(x) = a^x$ is increasing and bijective.
(2) When $0 < a < 1$ the function $f : \mathbf{R} \to (0, \infty)$ given by $f(x) = a^x$ is decreasing and bijective.

Proof: We omit the proof.

1.11 Definition: For $a > 0$ with $a \neq 1$, the function $f : \mathbf{R} \to (0, \infty)$ given by $f(x) = a^x$ is called the base a **exponential function**, its inverse function $f^{-1} : (0, \infty) \to \mathbf{R}$ is called the base a **logarithmic function**, and we write $f^{-1}(x) = \log_a x$. By the definition of the inverse function, we have $\log_a(a^x) = x$ for all $x \in \mathbf{R}$ and $e^{\log_a y} = y$ for all $y > 0$, and for all $x, y \in \mathbf{R}$ with $y > 0$ we have $y = a^x \iff x = \log_a y$.

1.12 Theorem: *(Properties of Logarithms) Let* $a, b, x, y \in (0, \infty)$. *Then*
(1) $\log_a 1 = 0$,
(2) $\log_a(xy) = \log_a x + \log_a y$,
(3) $\log_a(x/y) = \log_a x - \log_a y$,
(4) $\log_a(x^y) = y \log_a x$, *and*
(5) $\log_b x = \log_a x / \log_a b$,
(6) if $a > 1$, *the function* $g : (0, \infty) \to \mathbf{R}$ *given by* $g(x) = \log_a x$ *is increasing and bijective.*

Proof: We leave it, as an exercise, to show that these properties follow from the properties of exponentials.

1.13 Definition: There is a number $e \in \mathbf{R}$ called the **natural base**, with $e \cong 2.71828$, which can be defined in such a way that the function $f(x) = e^x$ is equal to its own derivative. We define

$$e = \lim_{n \to \infty} \left(1 + \tfrac{1}{n}\right)^n$$

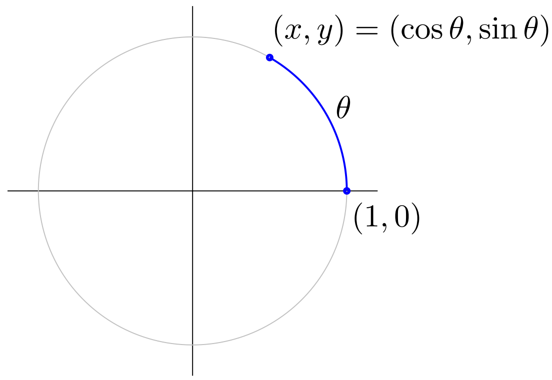
(to be rigorous, one must define this limit and prove that it exists and is unique). The logarithm to the base e is called the **natural logarithm**, and we write

$$\ln x = \log_e x \text{ for } x > 0.$$

1.14 Note: The properties of exponentials and logarithms in Theorems 1.7 and 1.12 give

$$e^0 = 1\ ,\ a^{x+y} = e^x e^y\ ,\ e^{x-y} = e^x / e^y\ ,\ (e^x)^y = e^{xy},$$
$$\ln 1 = 0\ ,\ \ln(xy) = \ln x + \ln y\ ,\ \ln(x/y) = \ln x - \ln y\ ,\ \ln x^y = y \ln x$$
$$\log_a x = \frac{\ln x}{\ln a} \quad \text{and} \quad a^x = e^{x \ln a}.$$

1.15 Definition: We define the trigonometric functions informally as follows. For $\theta \geq 0$, we define $\cos\theta$ and $\sin\theta$ to be the x- and y-coordinates of the point at which we arrive when we begin at the point $(1,0)$ and travel for a distance of θ units counterclockwise around the unit circle $x^2+y^2=1$. For $\theta \leq 0$, $\cos\theta$ and $\sin\theta$ are the x and y-coordinates of the point at which we arrive when we begin at $(1,0)$ and travel clockwise around the unit circle for a distance of $|\theta$ units. When $\cos\theta \neq 0$ we define $\sec\theta = 1/\cos\theta$ and $\tan\theta = \sin\theta/\cos\theta$, and when $\sin\theta \neq 0$ we define $\csc\theta = 1/\sin\theta$ and $\cot\theta = \cos\theta/\sin\theta$. (This definition is not rigorous because we did not define what it means to travel around the circle for a given distance).

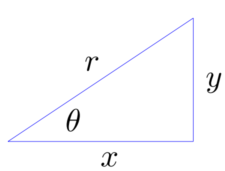


1.16 Definition: We define π, informally, to be the distance along the top half of the unit circle from $(1,0)$ to $(-1,0)$, and so we have $\cos\pi = -1$ and $\sin\pi = 0$. By symmetry, the distance from $(1,0)$ to $(0,1)$ along the circle is equal to $\frac{\pi}{2}$ so we also have $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

1.17 Theorem: *(Basic Trigonometric Properties) For $\theta \in \mathbf{R}$ we have*
(1) $\cos^2\theta + \sin^2\theta = 1$,
(2) $\cos(-\theta) = \cos\theta$ *and* $\sin(-\theta) = -\sin\theta$,
(3) $\cos(\theta+\pi) = -\cos\theta$ *and* $\sin(\theta+\pi) = -\sin\theta$,
(4) $\cos(\theta+2\pi) = \cos\theta$ *and* $\sin(\theta+2\pi) = \sin\theta$).

Proof: Informally, these properties can all be seen immediately from the above definitions. We omit a rigorous proof.

1.18 Theorem: *(Trigonometric Ratios) Let $\theta \in \left(0, \frac{\pi}{2}\right)$. For a right angle triangle with an angle of size θ and with sides of lengths x, y and r as shown, we have*



$$\cos\theta = \frac{x}{r}\ ,\ \sin\theta = \frac{y}{r} \text{ and } \tan\theta = \frac{y}{x}.$$

Proof: We can see this informally by scaling the picture in Definition 2.17 by a factor of r.

1.19 Theorem: *(Special Trigonometric Values) We have the following exact trigonometric values.*

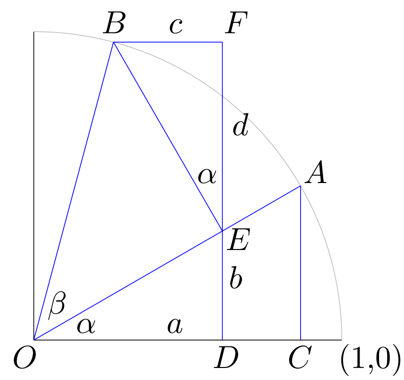
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Proof: This follows from the above theorem using certain particular right angled triangles.

1.20 Theorem: *(Trigonometric Sum Formulas) For $\alpha, \beta \in \mathbf{R}$ we have*

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ , and}$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Proof: Informally, we can prove this with the help of a picture. The picture below illustrates the situation when $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$.



In the picture, O is the origin, A is the point with coordinates $(\cos\alpha, \sin\alpha)$ and B is the point $(x,y) = \big(\cos(\alpha+\beta), \sin(\alpha+\beta)\big)$. In triangle ODE we see that $\cos\alpha = \frac{OD}{OE} = \frac{a}{\cos\beta}$ and $\sin\alpha = \frac{DE}{OE} = \frac{b}{\cos\beta}$, and so $a = \cos\alpha\cos\beta$, $b = \sin\alpha\cos\beta$. In triangle EFB, verify that the angle at E has size α, and so we have $\cos\alpha = \frac{EF}{EB} = \frac{d}{\sin\beta}$ and $\sin\alpha = \frac{BF}{BE} = \frac{c}{\sin\beta}$, and so $c = \sin\alpha\sin\beta$, $d = \cos\alpha\sin\beta$. The x and y-coordinates of the point B are $x = a - c$ and $y = b + d$, and so

$$\cos(\alpha+\beta) = x = a - c = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ , and}$$
$$\sin(\alpha+\beta) = y = b + d = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

This proves the theorem (informally) in the case that $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. One can then show that the theorem holds for all $\alpha, \beta \in \mathbf{R}$ by using the Basic Trigonometric Properties (2), (3) and (4).

1.21 Theorem: *(Double Angle Formulas) For all $x, y \in \mathbf{R}$ we have*

(1) $\sin 2x = 2\sin x\cos x$ *and* $\cos 2x = \cos^2 - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$*, and*

(2) $\cos^2 x = \dfrac{1+\cos 2x}{2}$ *and* $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

Proof: The proof is left as an exercise.

1.22 Theorem: *(Trigonometric Functions)*

(1) The function $f : [0,\pi] \to [-1,1]$ defined by $f(x) = \cos x$ is decreasing and bijective.
(2) The function $g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ given by $g(x) = \sin x$ is increasing and bijective.
(3) The function $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ given by $h(x) = \tan x$ is increasing and bijective.

Proof: We omit the proof.

1.23 Definition: The inverses of the functions f, g and h in the above theorem are called the **inverse cosine**, the **inverse sine**, and the **inverse tangent** functions. We write $f^{-1}(x) = \cos^{-1} x$, $g^{-1} = \sin^{-1} x$ and $h^{-1}(x) = \tan^{-1} x$. By the definition of the inverse function, we have

1.24 Definition: Let A and B be sets and let $c \in F$. Let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$. We define the functions cf , $f + g$, $f - g$, $f \cdot g$: $A \cap B \to \mathbf{R}$ by

$$\begin{aligned}(cf)(x) &= c\,f(x)\\(f+g)(x) &= f(x)+g(x)\\(f-g)(x) &= f(x)-g(x)\\(f\cdot g)(x) &= f(x)g(x)\end{aligned}$$

for all $x \in A \cap B$, and for $C = \{x \in A \cap B \,|\, g(x) \neq 0\}$ we define $f/g : C \to \mathbf{R}$ by

$$(f/g)(x) = f(x)/g(x)$$

for all $x \in C$.

1.25 Definition: A **polynomial function** (over $\mathbf{R}$) is a function $f : \mathbf{R} \to \mathbf{R}$ which can be obtained from the functions 1 and x using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$ and $f \circ g$. In other words, a polynomial is a function of the form

$$f(x) = \sum_{i=0}^{n} c_i x^i = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

for some $n \in \mathbf{N}$ and some $c_i \in F$. The numbers c_i are called the **coefficients** of the polynomial and when $c_n \neq 0$ the number n is called the **degree** of the polynomial.

1.26 Definition: A **rational function** (over $\mathbf{R}$) is a function $f : A \subseteq \mathbf{R} \to \mathbf{R}$ which can be obtained from the functions 1 and x using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$. In other words, a rational function is a function of the form

$$f(x) = p(x)/q(x)$$

for some polynomials p and q.

1.27 Definition: The functions 1, x, $x^{1/n}$ with $0 < n \in \mathbf{Z}$, e^x, $\ln x$, $\sin x$ and $\sin^{-1} x$, are called the **basic elementary functions**. An **elementary function** is any function $f : A \subseteq \mathbf{R} \to \mathbf{R}$ which can be obtained from the basic elementary functions using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$.

1.28 Example: The following functions are elementary

$$\begin{aligned}|x| &= \sqrt{x^2},\\ \cos x &= \sin\left(x + \tfrac{\pi}{2}\right),\\ \tan^{-1} x &= \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right),\\ f(x) &= \frac{e^{\sqrt{x}+\sin x}}{\tan^{-1}(\ln x)}\end{aligned}$$

We shall see later that every elementary function is continuous in its domain, so any function which is discontinuos at a point in its domain cannot be elementary.

Chapter 2. Limits of Sequences

2.1 Notation: We write $\mathbf{N} = \{0, 1, 2, \cdots\}$ for the set of **natural numbers** (which we take to include the number 0), $\mathbf{Z}^+ = \{1, 2, 3, \cdots\}$ for the set of **positive integers**, $\mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$ for the set of all **integers**, $\mathbf{Q}$ for the set of **rational numbers** and we write $\mathbf{R}$ for the set of **real numbers**. We assume familiarity with the sets $\mathbf{N}$, $\mathbf{Z}^+$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ and with the algebraic operations $+$, $-$, $\times$, $\div$ and the order relations $<$, $\leq$, $>$, $\geq$ on these sets.

2.2 Definition: For $p \in \mathbf{Z}$, let $\mathbf{Z}_{\geq p} = \{k \in \mathbf{Z} | k \geq p\}$. A **sequence** in a set A is a function of the form $x : \mathbf{Z}_{\geq p} \to A$ for some $p \in \mathbf{Z}$. Given a sequence $x : \mathbf{Z}_{\geq p} \to A$, the k^{th} **term** of the sequence is the element $x_k = x(k) \in A$, and we denote the sequence x by

$$(x_k)_{k \geq p} = (x_k | k \geq p) = (x_p, x_{p+1}, x_{p+2}, \cdots).$$

Note that the range of the sequence $(x_k)_{k \geq p}$ is the set $\{x_k\}_{k \geq p} = \{x_k | k \geq p\}$.

2.3 Definition: Let $(x_k)_{k \geq p}$ be a sequence in $\mathbf{R}$. For $a \in \mathbf{R}$ we say that the sequence $(x_k)_{k \geq p}$ **converges** to a (or that the **limit** of $(x_k)_{k \geq p}$ is equal to a), and we write $x_k \to a$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = a$, when

$$\forall\, \epsilon > 0\; \exists\, m \in \mathbf{Z}_{\geq p}\; \forall\, k \in \mathbf{Z}_{\geq p}\; \big(k \geq m \Longrightarrow |x_k - a| < \epsilon\big).$$

We say that the sequence $(x_k)_{k \geq p}$ **converges** (in $\mathbf{R}$) when there exists $a \in \mathbf{R}$ such that $(x_k)_{k \geq p}$ converges to a. We say that the sequence $(x_k)_{k \geq p}$ **diverges** (in $\mathbf{R}$) when it does not converge (to any $a \in \mathbf{R}$). We say that $(x_k)_{k \geq p}$ **diverges to infinity**, or that the limit of $(x_k)_{k \geq p}$ is equal to **infinity**, and we write $x_k \to \infty$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = \infty$, when

$$\forall\, r \in \mathbf{R}\; \exists\, m \in \mathbf{Z}_{\geq p}\; \forall\, k \in \mathbf{Z}_{\geq p}\; \big(k \geq m \Longrightarrow x_k > r\big).$$

Similarly we say that $(x_k)_{k \geq p}$ **diverges to** $-\infty$, or that the limit of $(x_k)_{k \geq p}$ is equal to **negative infinity**, and we write $x_k \to -\infty$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = -\infty$ when

$$\forall\, r \in \mathbf{R}\; \exists\, m \in \mathbf{Z}_{\geq p}\; \forall\, k \in \mathbf{Z}_{\geq p}\; \big(k \geq m \Longrightarrow x_k < r\big).$$

2.4 Example: Let $(x_k)_{k \geq 0}$ be the sequence in $\mathbf{R}$ given by $x_k = \frac{(-2)^k}{k!}$ for $k \geq 0$. Show that $\lim\limits_{k \to \infty} x_k = 0$.

Solution: Note that for $k \geq 2$ we have $|x_k| = \frac{2^k}{k!} = \left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right) \cdots \left(\frac{2}{k-1}\right)\left(\frac{2}{k}\right) \leq \frac{2}{1} \cdot \frac{2}{n} = \frac{4}{n}$. Given $\epsilon \in \mathbf{R}$ with $\epsilon > 0$, we can choose $m \in \mathbf{Z}_{\geq 2}$ with $m > \frac{4}{\epsilon}$ (by the Archimedean Property of $\mathbf{Z}$ in $\mathbf{R}$), and then for all $k \geq m$ we have $|x_k - 0| = |x_k| \leq \frac{4}{k} \leq \frac{4}{m} < \epsilon$. Thus $\lim\limits_{k \to \infty} x_k = 0$, by the definition of the limit.

2.5 Example: Let $(a_k)_{k \geq 0}$ be the **Fibonacci sequence** in $\mathbf{R}$, which is defined recursively by $a_0 = 0$, $a_1 = 1$ and by $a_k = a_{k-1} + a_{k-2}$ for $k \geq 2$. Show that $\lim\limits_{k \to \infty} a_k = \infty$.

Solution: We have $a_0 = 0$, $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. Note that $a_k \geq k - 1$ when $k \in \{0, 1, 2, 3\}$. Let $n \geq 4$ and suppose, inductively, that $a_k \geq k - 1$ for all $k \in \mathbf{Z}$ with $0 \leq k < n$. Then $a_n = a_{n-1} + a_{n-2} \geq (n-2) + (n-3) = n + n - 5 \geq n + 4 - 5 = n - 1$. By the Strong Principle of Induction, we have $a_n \geq n - 1$ for all $n \geq 0$. Given $r \in \mathbf{R}$ we can choose $m \in \mathbf{Z}_{\geq 0}$ with $m > r + 1$, and then for all $k \geq m$ we have $a_k \geq k - 1 \geq m - 1 > r$. Thus $\lim\limits_{k \to \infty} a_k = \infty$ by the definition of the limit.

2.6 Example: Let $x_k = (-1)^k$ for $k \geq 0$. Show that $(x_k)_{k\geq 0}$ diverges.

Solution: Suppose, for a contradiction, that $(x_k)_{k\geq 0}$ converges and let $a = \lim\limits_{k\to\infty} x_k$. By taking $\epsilon = 1$ in the definition of the limit, we can choose $m \in \mathbf{Z}$ so the for all $k \in \mathbf{N}$, if $k \geq m$ then $|x_k - a| < 1$. Choose $k \in \mathbf{N}$ with $2k \geq m$. Since $|x_{2k} - a| < 1$ and $x_{2k} = (-1)^{2k} = 1$, we have $|1 - a| < 1$ so that $0 < a < 2$. Since $|x_{2k+1} - a| < 1$ and $x_{2k+1} = (-1)^{2k+1} = -1$, we also have $|-1-a| < 1$ which implies that $-2 < a < 0$. But then we have $a < 0$ and $a > 0$, which is not possible.

2.7 Theorem: *(Independence of the Limit on the Initial Terms) Let $(x_k)_{k\geq p}$ be a sequence in $\mathbf{R}$.*

(1) If $q \geq p$ and $y_k = x_k$ for all $k \geq q$, then $(x_k)_{k\geq p}$ converges if and only if $(y_k)_{k\geq q}$ converges, and in this case $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} y_k$.

(2) If $l \geq 0$ and $y_k = x_{k+l}$ for all $k \geq p$, then $(x_k)_{k\geq p}$ converges if and only if $(y_k)_{k\geq p}$ converges, and in this case $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} y_k$.

Proof: We prove Part 1 and leave the proof of Part 2 as an exercise. Let $q \geq p$ and let $y_k = x_k$ for $k \geq q$. Suppose $(x_k)_{k\geq p}$ converges and let $a = \lim\limits_{k\to\infty} x_k$. Let $\epsilon > 0$. Choose $m \in \mathbf{Z}$ so that for all $k \in \mathbf{Z}_{\geq p}$, if $k \geq m$ then $|x_k - a| < \epsilon$. Let $k \in \mathbf{Z}_{\geq q}$ with $k \geq m$. Since $q \geq p$ we also have $k \in \mathbf{Z}_{\geq p}$ and so $|y_k - a| = |x_k - a| < \epsilon$. Thus $(y_k)_{k\geq q}$ converges with $\lim\limits_{k\to\infty} y_k = a$. Conversely, suppose that $(y_k)_{k\geq q}$ converges and let $a = \lim\limits_{k\to\infty} y_k$. Let $\epsilon > 0$. Choose $m_1 \in \mathbf{Z}$ so that for all $k \in \mathbf{Z}_{\geq q}$, if $k \geq m_1$ then $|y_k - a| < \epsilon$. Choose $m = \max\{m_1, q\}$. Let $k \in \mathbf{Z}_{\geq p}$ with $k \geq m$. Since $k \geq m$, we have $k \geq q$ and $k \geq m_1$ and so $|x_k - a| = |y_k - a| < \epsilon$. Thus $(x_k)_{k\geq p}$ converges with $\lim\limits_{k\to\infty} x_k = a$.

2.8 Remark: Because of the above theorem, we often denote the sequence $(x_k)_{k\geq p}$ simply as (x_k), omitting the initial index p from our notation. Also, in the statements of some theorems and in some proofs we select a particular starting point, often $p = 1$, with the understanding that any other starting value would work just as well.

2.9 Theorem: *(Uniqueness of the Limit) Let (x_k) be a sequence in $\mathbf{R}$. If (x_k) has a limit (finite or infinite) then the limit is unique.*

Proof: Suppose, for a contradiction, that $x_k \to \infty$ and $x_k \to -\infty$. Since $x_k \to \infty$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \implies x_k > 0$. Since $x_k \to -\infty$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \implies x_k < 0$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then $x_k > 0$ and $x_k < 0$, which is not possible.

Suppose, for a contradiction, that $x_k \to \infty$ and $x_k \to a \in F$. Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \implies |x_k - a| < 1$. Since $x_k \to \infty$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \implies x_k > a + 1$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then we have $|x_k - a| < 1$ so that $x < a + 1$ and we have $x_k > a + 1$, which is not possible. Similarly, it is not possible to have $x_k \to -\infty$ and $x_k \to a \in F$.

Finally suppose, for a contradiction, that $x_k \to a$ and $x_k \to b$ where $a, b \in F$ with $a \neq b$. Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \implies |x_k - a| < \frac{|a-b|}{2}$. Since $x_k \to b$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \implies |x_k - b| < \frac{|a-b|}{2}$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then we have $|x_k - a| < \frac{b-a}{2}$ and $|x_k - b| < \frac{b-a}{2}$ and so, using the Triangle Inequality, we have

$$|a - b| = |a - x_k + x_k - b| \leq |x_k - a| + |x_k - b| < \tfrac{|a-b|}{2} + \tfrac{|a-b|}{2} = |a - b|,$$

which is not possible.

2.10 Theorem: *(Basic Limits) For $a \in \mathbf{R}$ we have*

$$\lim_{k\to\infty} a = a\ ,\ \lim_{k\to\infty} k = \infty \ \text{ and } \ \lim_{k\to\infty} \frac{1}{k} = 0.$$

Proof: The proof is left as an exercise.

2.11 Theorem: *(Operations on Limits) Let (x_k) and (y_k) be sequences in $\mathbf{R}$ and let $c \in \mathbf{R}$. Suppose that (x_k) and (y_k) both converge with $x_k \to a$ and $y_k \to b$. Then*
(1) $(c\,x_k)$ converges with $c\,x_k \to ca$,
(2) $(x_k + y_k)$ converges with $(x_k + y_k) \to a + b$,
(3) $(x_k - y_k)$ converges with $(x_k - y_k) \to a - b$,
(4) $(x_k y_k)$ converges with $x_k y_k \to ab$, and
(5) if $b \neq 0$ then (x_k/y_k) converges with $x_k/y_k \to a/b$.

Proof: We prove Parts 4 and 5 leaving the proofs of the other parts as an exercise. First we prove Part 4. Note that for all k we have

$$|x_k y_k - ab| = |x_k y_k - x_k b + x_k b - ab| \leq |x_k y_k - x_k b| + |x_k b - ab| = |x_k|\,|y_k - b| + |b||x_k - a|.$$

Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| < 1$ and we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |x_k - a| < \frac{\epsilon}{2(1+|b|)}$. Since $y_k \to b$ we can choose $m_3 \in \mathbf{Z}$ so that $k \geq m_3 \Longrightarrow |y_k - b| < \frac{\epsilon}{2(1+|a|)}$. Let $m = \max\{m_1, m_2, m_3\}$ and let $k \geq m$. Then we have $|x_k - a| < 1$, $|x_k - a| < \frac{\epsilon}{2(1+|b|)}$ and $|x_k - b| < \frac{\epsilon}{2(1+|a|)}$. Since $|x_k - a| < 1$, we have $|x_k| = |x_k - a + a| \leq |x_k - a| + |a| < 1 + |a|$. By our above calculation (where we found a bound for $|x_k y_k - ab|$) we have

$$\begin{aligned} |x_k y_k - ab| &\leq |x_k||y_k - b| + |b||x_k - a| \leq (1 + |a|)|y_k - b| + (1 + |b|)|x_k - a| \\ &< (1 + |a|)\tfrac{\epsilon}{2(1+|a|)} + (1 + |b|)\tfrac{\epsilon}{2(1+|b|)} = \epsilon. \end{aligned}$$

Thus we have $x_k y_k \to ab$, by the definition of the limit.

To prove Part 5, suppose that $b \neq 0$. Since $y_k \to b \neq 0$, we can choose $m_1 \in \mathbf{Z}$ so that that $k \geq m_1 \Longrightarrow |y_k - b| < \frac{|b|}{2}$. Then for $k \geq m_1$ we have

$$|b| = |b - y_k + y_k| \leq |b - y_k| + |y_k| < \tfrac{|b|}{2} + |y_k|$$

so that

$$|y_k| > |b| - \tfrac{|b|}{2} = \tfrac{|b|}{2} > 0.$$

In particular, we remark that when $k \geq m_1$ we have $y_k \neq 0$ so that $\frac{1}{y_k}$ is defined. Note that for all $k \geq m_1$ we have

$$\left|\frac{1}{y_k} - \frac{1}{b}\right| = \frac{|b - y_k|}{|y_k|\,|b|} \leq \frac{|b - y_k|}{\frac{|b|}{2} \cdot |b|} = \frac{2}{|b|^2} \cdot |y_k - b|.$$

Let $\epsilon > 0$. Choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |y_k - b| < \frac{|b|^2\epsilon}{2}$. Let $m = \max\{m_1, m_2\}$. For $k \geq m$ we have $k \geq m_1$ and $k \geq m_2$ and so $|y_k| > \frac{|b|^2}{2}$ and $|y_k - b| < \frac{|b|^2\epsilon}{2}$ and so

$$\left|\frac{1}{y_k} - \frac{1}{b}\right| \leq \tfrac{2}{|b|^2} \cdot |y_k - b| < \tfrac{2}{|b|^2} \cdot \tfrac{|b|^2\epsilon}{2} = \epsilon.$$

This proves that $\lim\limits_{k\to\infty} \frac{1}{y_k} = \frac{1}{b}$. Using Part 4, we have $\lim\limits_{k\to\infty} \frac{x_k}{y_k} = \lim\limits_{k\to\infty} \left(x_k \cdot \frac{1}{y_k}\right) = a \cdot \frac{1}{b} = \frac{a}{b}$.

2.12 Example: Let $x_k = \frac{k^2+1}{2k^2+k+3}$ for $k \geq 0$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: We have $x_k = \frac{k^2+1}{2k^2+k+2} = \frac{1+\left(\frac{1}{k}\right)^2}{2+\frac{1}{k}+3\cdot\left(\frac{1}{k}\right)^2} \longrightarrow \frac{1+0^2}{2+0+3\cdot 0^2} = \frac{1}{2}$ where we used the Basic Limits $1 \to 1$, $2 \to 2$ and $\frac{1}{k} \to 0$ together with Operations on Limits.

2.13 Definition: The above theorem can be extended to include many situations involving infinite limits. To deal with these cases, we define the set of **extended real numbers** to be the set

$$\widehat{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}.$$

We extend the order relation $<$ on $\mathbf{R}$ to an order relation on $\widehat{\mathbf{R}}$ by defining $-\infty < \infty$ and $-\infty < a$ and $a < \infty$ for all $a \in \mathbf{R}$. We partially extend the operations $+$ and $\times$ to $\widehat{\mathbf{R}}$ as follows: for $a \in \mathbf{R}$ we define

$$\infty + \infty = \infty\ ,\ \infty + a = \infty\ ,\ (-\infty) + (-\infty) = -\infty\ ,\ (-\infty) + a\ ,$$
$$\infty \cdot \infty = \infty\ ,(\infty)(-\infty) = -\infty\ ,\ (-\infty)(-\infty) = \infty\ ,$$
$$\infty \cdot a = \begin{cases} \infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases} \text{ and } (-\infty)(a) = \begin{cases} -\infty \text{ if } a > 0, \\ \infty \text{ if } a < 0, \end{cases}$$

but other values, including $\infty + (-\infty)$, $\infty \cdot 0$ and $-\infty \cdot 0$ are left undefined in $\widehat{\mathbf{R}}$. In a similar way, we partially extend the inverse operations $-$ and $\div$ to $\widehat{\mathbf{R}}$. For example, for $a \in \mathbf{R}$ we define

$$\infty - (-\infty) = \infty\,, -\infty - \infty = -\infty\,,\ \infty - a = \infty\,, -\infty - a = -\infty\,,\ a - \infty = -\infty\,,\ a - (-\infty) = \infty\,,$$
$$\frac{a}{\infty} = 0\ ,\ \frac{\infty}{a} = \begin{cases} \infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases} \text{ and } \frac{-\infty}{a} = \begin{cases} -\infty \text{ if } a > 0 \\ \infty \text{ if } a < 0 \end{cases}$$

with other values, including $\infty - \infty$, $\frac{\infty}{\infty}$ and $\frac{\infty}{0}$, left undefined. The expressions which are left undefined in $\widehat{\mathbf{R}}$, including

$$\infty - \infty\ ,\ \infty \cdot 0\ ,\ \frac{\infty}{\infty}\ ,\ \frac{\infty}{0}\ ,\ \frac{a}{0}\,,$$

are known as **indeterminate forms**.

2.14 Theorem: *(Extended Operations on Limits) Let (x_k) and (y_k) be sequences in $\mathbf{R}$. Suppose that $\lim\limits_{k\to\infty} x_k = u$ and $\lim\limits_{k\to\infty} y_k = v$ where $u, v \in \widehat{\mathbf{R}}$.*

(1) if $u + v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k + y_k) = u + v$,

(2) if $u - v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k - y_k) = u - v$,

(3) if $u \cdot v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k \cdot y_k) = u \cdot v$, and

(4) if u/v is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k / y_k) = u/v$.

Proof: The proof is left as an exercise.

2.15 Theorem: *(Comparison) Let* (x_k) *and* (y_k) *be sequences in* $\mathbf{R}$*. Suppose that* $x_k \leq y_k$ *for all* k*. Then*

(1) if $x_k \to a$ *and* $y_k \to b$ *then* $a \leq b$*,*
(2) if $x_k \to \infty$ *then* $y_k \to \infty$*, and*
(3) if $y_k \to -\infty$ *then* $x_k \to -\infty$*.*

Proof: We prove Part 1. Suppose that $x_k \to a$ and $y_k \to b$. Suppose, for a contradiction, that $a > b$. Choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \implies |x_k - a| < \frac{a-b}{2}$. Choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \implies |y_k - b| < \frac{a-b}{2}$. Let $k = \max\{m_1, m_2\}$. Since $|x_k - a| < \frac{a-b}{2}$, we have $x_k > a - \frac{a-b}{2} = \frac{a+b}{2}$. Since $|y_k - b| < \frac{a-b}{2}$, we have $y_k < b + \frac{a-b}{2} = \frac{a+b}{2}$. This is not possible since $x_k \leq y_k$.

2.16 Example: Let $x_k = (\frac{3}{2} + \sin k) \ln k$ for $k \geq 1$. Find $\lim\limits_{k \to \infty} x_k$.

Solution: For all $k \geq 1$ we have $\sin k \geq -1$ so $(\frac{3}{2} + \sin k) \geq \frac{1}{2}$ and hence $x_k \geq \frac{1}{2} \ln k$. Since $x_k \geq \frac{1}{2} \ln k$ for all $k \geq 1$ and $\frac{1}{2} \ln k \longrightarrow \frac{1}{2} \cdot \infty = \infty$, it follows that $x_k \to \infty$ by the Comparison Theorem.

2.17 Theorem: *(Squeeze) Let* (x_k)*,* (y_k) *and* (z_k) *be sequences in* $\mathbf{R}$ *and let* $a \in \mathbf{R}$*.*

(1) If $x_k \leq y_k \leq z_k$ *for all* k *and* $x_k \to a$ *and* $z_k \to a$ *then* $y_k \to a$*.*
(2) If $|x_k| \leq y_k$ *for all* k *and* $y_k \to 0$ *then* $x_k \to 0$*.*

Proof: We prove Part 1. Suppose that $x_k \leq y_k \leq z_k$ for all k, and suppose that $x_k \to a$ and $z_k \to a$. Let $\epsilon > 0$. Choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \implies |x_k - a| < \epsilon$, choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \implies |z_k - a| < \epsilon$ and let $m = \max\{m_1, m_2\}$. Then for $k \geq m$ we have $a - \epsilon < x_k \leq y_k \leq z_k < a + \epsilon$ and so $|y_k - a| < \epsilon$. Thus $y_k \to a$, as required.

2.18 Example: Let $x_k = \frac{k + \tan^{-1} k}{2k + \sin k}$ for $k \geq 1$. Find $\lim\limits_{k \to \infty} x_k$.

Solution: For all $k \geq 1$ we have $-\frac{\pi}{2} < \tan^{-1} k < \frac{\pi}{2}$ and $-1 \leq \sin k \leq 1$ and so

$$\frac{k - \frac{\pi}{2}}{2k + 1} \leq \frac{k + \tan^{-1} k}{2k + \sin k} \leq \frac{k + \frac{\pi}{2}}{2k - 1}.$$

As in previous examples, we have $\frac{k - \frac{\pi}{2}}{2k+1} \to \frac{1}{2}$ and $\frac{k + \frac{\pi}{2}}{2k-1} \to \frac{1}{2}$, and so $x_k = \frac{k + \tan^{-1} k}{2k + \sin k} \to \frac{1}{2}$ by the Squeeze Theorem.

2.19 Definition: Let (x_k) be a sequence in $\mathbf{R}$. For $a, b \in \mathbf{R}$, we say that the sequence (x_k) is **bounded above** by b when $x_k \leq b$ for all k, and we say that the sequence (x_k) is **bounded below** by a when $a \leq x_k$ for all k. We say (x_k) is **bounded above** when it is bounded above by some element $b \in \mathbf{R}$, we say that (x_k) is **bounded below** when it is bounded below by some $a \in \mathbf{R}$, and we say that (x_k) is **bounded** when it is bounded above and bounded below.

2.20 Definition: Let (x_k) be a sequence in $\mathbf{R}$. We say that (x_k) is **increasing** (for $k \geq p$) when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k \leq l$ then $x_k \leq x_l$. We say that (x_k) is **strictly increasing** (for $k \geq p$) when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k < l$ then $x_k < x_l$. Similarly, we say that (x_k) is **decreasing** when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k \leq l$ the $x_k \geq x_l$ and we say that (x_k) is **strictly decreasing** when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k < l$ the $x_k > x_l$. We say that (x_k) is **monotonic** when it is either increasing or decreasing.

2.21 Theorem: *(Monotonic Convergence) Let (x_k) be a sequence in $\mathbf{R}$.*

(1) Suppose (x_k) is increasing. If (x_k) is bounded above then it converges, and if (x_k) is not bounded above then $x_k \to \infty$.
(2) Suppose (x_k) is decreasing. If (x_k) is bounded below then it converges, and if (x_k) is not bounded below then $x_k \to -\infty$.

Proof: The statement of this theorem is intuitively reasonable, but it is quite difficult to prove. In most calculus courses this theorem is accepted axiomatically, without proof. A rigorous proof is often provided in analysis courses.

2.22 Example: Let $x_1 = \frac{4}{3}$ and let $x_{k+1} = 5 - \frac{4}{x_n}$ for $k \geq 1$. Determine whether (x_k) converges, and if so then find the limit.

Solution: Suppose, for now, that (x_k) does converge, say $x_k \to a$. By Independence of Converge on Initial Terms, we also have $x_{k+1} \to a$. Using Operations on Limits, we have $a = \lim\limits_{k\to\infty} x_{k+1} = \lim\limits_{k\to\infty} \left(5 - \frac{4}{x_k}\right) = 5 - \frac{4}{a}$. Since $a = 5 - \frac{4}{a}$, we have $a^2 = 5a - 4$ or equivalently $(a-1)(a-4) = 0$. We have proven that if the sequence converges then its limit must be equal to 1 or 4.

The first few terms of the sequence are $x_1 = \frac{4}{3}$, $x_2 = 2$ and $x_3 = 3$. Since the terms appear to be increasing, we shall try to prove that $1 \leq x_n \leq x_{n+1} \leq 4$ for all $n \geq 1$. This is true when $n = 1$. Suppose it is true when $n = k$. Then we have

$$1 \leq x_k \leq x_{k+1} \leq 4 \implies 1 \geq \frac{1}{x_k} \geq \frac{1}{x_{k+1}} \geq \frac{1}{4} \implies -4 \leq -\frac{4}{x_k} \leq -\frac{4}{x_{k+1}} \leq -1$$
$$\implies 1 \leq 5 - \frac{4}{x_k} \leq 5 - \frac{4}{x_{k+1}} \leq 4 \implies 1 \leq x_{k+1} \leq x_{k+2} \leq 4.$$

Thus, by the Principle of Induction, we have $1 \leq x_n \leq x_{n+1} \leq 4$ for all $n \geq 1$.

Since $x_n \leq x_{n+1}$ for all $n \geq 1$, the sequence is increasing, and since $x_n \leq 4$ for all $n \geq 1$, the sequence is bounded above by 4. By the Monotone Convergence Theorem, the sequence does converge. By the first paragraph, we know the limit must be either 1 or 4, and since the sequence starts at $x_1 = 2$ and increases, the limit must be 4.

Chapter 3. Limits of Functions and Continuity

3.1 Definition: Let $A \subseteq \mathbf{R}$ and let $a \in \mathbf{R}$. We say that a is a **limit point** of A when

$$\forall\,\delta>0\ \exists\,x\in A\ \ 0<|x-a|<\delta.$$

We say that a is a **limit point of A from below** (or **from the left**) when

$$\forall\delta>0\ \exists x\in A\quad a-\delta<x<a\,.$$

We say that a is a **limit point of A from above** (or **from the right**) when

$$\forall\delta>0\ \exists x\in A\quad a<x<a+\delta\,.$$

We say that A is **not bounded above** when $\forall m\in\mathbf{R}\ \exists x\in A\ \ x\geq m$, and we say that A is **not bounded below** when $\forall m\in\mathbf{R}\ \exists x\in A\ \ x\leq m$.

3.2 Example: Let A be a finite union of non-degenerate intervals in $\mathbf{R}$ (a non-degenerate interval is an interval which contains more than one point). The limit points of A are the points $a \in \mathbf{R}$ such that either $a\in A$ or a is an endpoint of one of the intervals. The limit points of A from below are the points $a\in\mathbf{R}$ such that either $a\in A$ or a is the right endpoint of one of the intervals. The set A is not bounded above when one of the intervals is of one of the forms (a,∞), $[a,\infty)$ or $(-\infty,\infty)=\mathbf{R}$.

3.3 Definition: Let $A\subseteq\mathbf{R}$ and let $f:A\to\mathbf{R}$. When $a\in\mathbf{R}$ is a limit point of A, we make the following definitions.

(1) For $b\in\mathbf{R}$, we say that the **limit** of $f(x)$ as x tends to a is equal to b, and we write $\lim\limits_{x\to a} f(x)=b$ or we write $f(x)\to b$ as $x\to a$, when

$$\forall\epsilon>0\ \exists\delta>0\ \forall x\in A\ \big(0<|x-a|<\delta\Longrightarrow|f(x)-b|<\epsilon\big).$$

(2) We say the limit of $f(x)$ as x tends to a is equal to **infinity**, and we write $\lim\limits_{x\to a} f(x)=\infty$, or we write $f(x)\to\infty$ as $x\to a$, when

$$\forall r\in\mathbf{R}\ \exists\delta>0\ \forall x\in A\ \big(0<|x-a|<\delta\Longrightarrow f(x)>r\big).$$

(3) We say that the limit of $f(x)$ as x tends to a is equal to **negative infinity**, and we write $\lim\limits_{x\to a} f(x)=-\infty$, or we write $f(x)\to-\infty$ as $x\to a$, when

$$\forall r\in\mathbf{R}\ \exists\delta>0\ \forall x\in A\ \big(0<|x-a|<\delta\Longrightarrow f(x)<r\big).$$

When a is a limit point of A from below and $b\in\mathbf{R}$, we make the following definitions.

(4) $\lim\limits_{x\to a^-} f(x)=b \iff \forall\epsilon>0\ \exists\delta>0\ \forall x\in A\ \big(a-\delta<x<a\Longrightarrow|f(x)-b|<\epsilon\big).$

(5) $\lim\limits_{x\to a^-} f(x)=\infty \iff \forall r\in\mathbf{R}\ \exists\delta>0\ \forall x\in A\ \big(a-\delta<x<a\Longrightarrow f(x)>r\big).$

(6) $\lim\limits_{x\to a^-} f(x)=-\infty \iff \forall r\in\mathbf{R}\ \exists\delta>0\ \forall x\in A\ \big(a-\delta<x<a\Longrightarrow f(x)<r\big).$

When a is a limit point of A from above and $b\in\mathbf{R}$, we make the following definitions.

(7) $\lim\limits_{x\to a^+} f(x)=b \iff \forall\epsilon>0\ \exists\delta>0\ \forall x\in A\ \big(a<x<a+\delta\Longrightarrow|f(x)-b|<\epsilon\big).$

(8) $\lim\limits_{x\to a^+} f(x)=\infty \iff \forall r\in\mathbf{R}\ \exists\delta>0\ \forall x\in A\ \big(a<x<a+\delta\Longrightarrow f(x)>r\big).$

(9) $\lim\limits_{x\to a^+} f(x)=-\infty \iff \forall r\in\mathbf{R}\ \exists\delta>0\ \forall x\in A\ \big(a<x<a+\delta\Longrightarrow f(x)<r\big).$

When A is not bounded above and $b \in \mathbf{R}$, we make the following definitions.

(10) $\lim\limits_{x\to\infty} f(x) = b \iff \forall \epsilon > 0\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \geq m \Longrightarrow |f(x) - b| < \epsilon\big)$.

(11) $\lim\limits_{x\to\infty} f(x) = \infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \geq m \Longrightarrow f(x) > r\big)$.

(12) $\lim\limits_{x\to\infty} f(x) = -\infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \geq m \Longrightarrow f(x) < r\big)$.

When A is not bounded below and $b \in \mathbf{R}$, we make the following definitions.

(13) $\lim\limits_{x\to-\infty} f(x) = b \iff \forall \epsilon > 0\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \leq m \Longrightarrow |f(x) - b| < \epsilon\big)$.

(14) $\lim\limits_{x\to-\infty} f(x) = \infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \leq m \Longrightarrow f(x) > r\big)$.

(15) $\lim\limits_{x\to-\infty} f(x) = -\infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \leq m \Longrightarrow f(x) < r\big)$.

3.4 Example: Let $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 1}$. Show that $\lim\limits_{x\to 1} f(x) = 2$.

Solution: Note that for $x \neq 1$ we have

$$|f(x) - 2| = \left|\tfrac{x^2+2x-3}{x^2-1} - 2\right| = \left|\tfrac{(x+3)(x-1)}{(x+1)(x-1)} - 2\right| = \left|\tfrac{x+3}{x+1} - 2\right| = \left|\tfrac{x+3-2x-2}{x+1}\right| = \left|\tfrac{-x+1}{x+1}\right| = \tfrac{|x-1|}{|x+1|}.$$

Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon\}$. Let $0 < |x - 1| < \delta$. Since $0 < |x - 1|$ we have $x \neq 1$ so, as shown above, $|f(x) - 2| = \frac{|x-1|}{|x+1|}$. Since $|x - 1| < \delta \leq 1$ we have $0 < x < 3$ so that $1 < x + 1$, and hence $|f(x) - 2| = \frac{|x-1|}{|x+1|} < |x - 1|$. Finally, since $|x - a| < \delta \leq \epsilon$ we have $|f(x) - 2| \leq |x - 1| < \epsilon$. Thus $\lim\limits_{x\to 1} f(x) = 2$.

3.5 Theorem: *(Two Sided Limits) Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and let $a \in \mathbf{R}$. Suppose that a is a limit point of A both from the left and from the right. Then for $u \in \widehat{\mathbf{R}}$ we have $\lim\limits_{x\to a} f(x) = u$ if and only if $\lim\limits_{x\to a^-} f(x) = u = \lim\limits_{x\to a^+} f(x)$.*

Proof: We prove the theorem in the case that $u = b \in \mathbf{R}$. Suppose that $\lim\limits_{x\to a} f(x) = b \in \mathbf{R}$. Let $\epsilon > 0$. Choose $\delta > 0$ so that for all $x \in A$, if $0 < |x - a| < \delta$ then $|f(x) - b| < \epsilon$. For $x \in A$ with $a - \delta < x < a$ we have $0 < |x - a| < \delta$ and so $|f(x) - b| < \epsilon$. This shows that $\lim\limits_{x\to a^-} f(x) = b$. For $x \in A$ with $a < x < x + \delta$ we have $0 < |x - a| < \delta$ and so $|f(x) - b| < \epsilon$. This show that $\lim\limits_{x\to a^+} f(x) = b$.

Conversely, suppose that $\lim\limits_{x\to a^-} f(x) = b = \lim\limits_{x\to a^+} f(x)$. Let $\epsilon > 0$. Since $f(x) \to b$ as $x \to a^-$, we can choose $\delta_1 > 0$ so that for all $x \in A$ with $a - \delta < a < a$ we have $|f(x) - b| < \epsilon$. Since $f(x) \to b$ as $x \to a^+$ we can choose $\delta_2 > 0$ so that for all $x \in A$ with $a < x < a + \delta_2$ we have $|f(x) - b| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x \in A$ with $0 < |x - a| < \delta$. Either we have $x < a$ or we have $x > a$. In the case that $x < a$ we have $a - \delta_1 \leq a - \delta < x < a$ and so $|f(x) - b| < \epsilon$ (by the choice of δ_1). In the case that $x > a$ we have $a < x < a + \delta \leq a + \delta_2$ and so $|f(x) - b| < \epsilon$ (by the choice of δ_2). In either case we have $|f(x) - b| < \epsilon$, and so $\lim\limits_{x\to a} f(x) = b$, as required.

3.6 Remark: For the sequence $(x_k)_{k\geq p}$ in $\mathbf{R}$ given by $x_k = f(k)$ where $f : \mathbf{Z}_{\geq p} \to \mathbf{R}$, the definitions (10), (11) and (12) agree with our definitions for limits of sequences. Thus limits of sequences are a special case of limits of functions. The following theorem shows that limits of functions are determined by limits of sequences.

3.7 Theorem: *(The Sequential Characterization of Limits of Functions) Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$, and let $u \in \widehat{\mathbf{R}}$.*
(1) When $a \in \mathbf{R}$ is a limit point of A, $\lim\limits_{x\to a} f(x) = u$ if and only if for every sequence (x_k) in $A \setminus \{a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(2) When a is a limit point of A from below, $\lim\limits_{x\to a^-} f(x) = u$ if and only if for every sequence (x_k) in $\{x \in A|x < a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(3) When a is a limit point of A from above, $\lim\limits_{x\to a^+} f(x) = u$ if and only if for every sequence (x_k) in $\{x \in A|x > a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(4) When A is not bounded above, $\lim\limits_{x\to\infty} f(x) = u$ if and only if for every sequence (x_k) in A with $x_k \to \infty$ we have $f(x_k) \to u$.
(5) When A is not bounded below, $\lim\limits_{x\to-\infty} f(x) = u$ if and only if for every sequence (x_k) in A with $x_k \to -\infty$ we have $f(x_k) \to u$.

Proof: We prove Part 1 in the case that $u = b \in \mathbf{R}$. Let $a \in \mathbf{R}$ be a limit point of A. Suppose that $\lim\limits_{x\to a} f(x) = b \in \mathbf{R}$. Let (x_k) be a sequence in $A \setminus \{a\}$ with $x_k \to a$. Let $\epsilon > 0$. Since $\lim\limits_{x\to a} f(x) = b$, we can choose $\delta > 0$ so that $0 < |x-a| < \delta \Longrightarrow |f(x) = b| < \epsilon$. Since $x_k \to a$ we can choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |x_k - a| < \delta$. Then for $k \geq m$, we have $|x_k - a| < \delta$ and we have $x_k \neq a$ (since the sequence (x_k) is in the set $A \setminus \{a\}$) so that $0 < |x-a| < \delta$ and hence $|f(x_k) - b| < \epsilon$. This shows that $f(x_k) \to b$.

Conversely, suppose that $\lim\limits_{x\to a} f(x) \neq b$. Choose $\epsilon_0 > 0$ so that for all $\delta > 0$ there exists $x \in A$ with $0 < |x-a| < \delta$ and $|f(x) - b| \geq \epsilon_0$. For each $k \in \mathbf{Z}^+$, choose $x_k \in A$ with $0 < |x_k - a| \leq \frac{1}{k}$ and $|f(x_k) - b| \geq \epsilon_0$. In this way we obtain a sequence $(x_k)_{k\geq 1}$ in $A \setminus \{a\}$. Since $|x_k - a| \leq \frac{1}{k}$ for all $k \in \mathbf{Z}^+$, it follows that $x_k \to a$ (indeed, given $\epsilon > 0$ we can choose $m \in \mathbf{Z}$ with $m > \frac{1}{\epsilon}$ and then $k \geq m \Longrightarrow |x_k - a| \leq \frac{1}{k} \leq \frac{1}{m} < \epsilon$). Since $|f(x_k) - b| \geq \epsilon_0$ for all k, it follows that $f(x_k) \not\to b$ (indeed if we had $f(x_k) \to b$ we could choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |f(x_k) - b| < \epsilon_0$ and then we could choose $k = m$ to get $|f(x_k) - b| < \epsilon_0$).

3.8 Remark: It follows from the Sequential Characterization of Limits of Functions that all of our theorems about limits of sequences imply analogous theorems in the more general setting of limits of functions. We list several of those theorems and give one sample proof.

3.9 Theorem: *(Local Determination of Limits) Let $A \subseteq B \subseteq \mathbf{R}$, let a be a limit point of A (hence also of B) and let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ with $f(x) = g(x)$ for all $x \in A$. Then if $\lim\limits_{x\to a} g(x) = u \in \widehat{\mathbf{R}}$ then $\lim\limits_{x\to a} f(x) = u$.*
Similar results holds for limits $x \to a^\pm$ and $x \to \pm\infty$.

3.10 Theorem: *(Uniqueness of Limits) Let $A \subseteq \mathbf{R}$, let a be a limit point of A, and let $f : A \to \mathbf{R}$. For $u, v \in \widehat{\mathbf{R}}$, if $\lim\limits_{x\to a} f(x) = u$ and $\lim\limits_{x\to a} f(x) = v$ then $u = v$. Similar results hold for limits $x \to a^\pm$ and $x \to \pm\infty$.*

3.11 Theorem: *(Basic Limits) Let F be a subfield of $\mathbf{R}$, and let $A \subseteq F$. For the constant function $f : A \to F$ given by $f(x) = b$ for all $x \in A$, we have*

$$\lim_{x \to a} f(x) = b \ , \ \lim_{x \to a^+} f(x) = b \ , \ \lim_{x \to a^-} f(x) = b \ , \ \lim_{x \to \infty} f(x) = b \ \text{ and } \ \lim_{x \to -\infty} f(x) = b,$$

and for the identity function $f : A \to F$ given by $f(x) = x$ for all $x \in A$ we have

$$\lim_{x \to a} f(x) = a \ , \ \lim_{x \to a^+} f(x) = a \ , \ \lim_{x \to a^-} f(x) = a \ , \ \lim_{x \to \infty} f(x) = \infty \ \text{ and } \ \lim_{x \to -\infty} f(x) = -\infty$$

whenever the limits are defined.

3.12 Theorem: *(Extended Operations on Limits) Let $A \subseteq \mathbf{R}$, let $f, g : A \to \mathbf{R}$ and let a be a limit point of A. Let $u, v \in \widehat{\mathbf{R}}$ and suppose that $\lim\limits_{x \to a} f(x) = u$ and $\lim\limits_{x \to a} g(x) = v$. Then*

(1) if $u + v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x \to a} (f + g)(x) = u + v$,

(2) if $u - v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x \to a} (f - g)(x) = u - v$,

(3) if $u \cdot v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x \to a} (fg)(x) = u \cdot v$, and

(4) if u/v is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x \to a} (f/g)(x) = u/v$.

Similar results hold for limits $x \to a^\pm$ and $x \to \pm\infty$.

Proof: We prove Part 4. Suppose that u/v is defined in $\widehat{\mathbf{R}}$. Let (x_k) be any sequence in $A \setminus \{a\}$ with $x_k \to a$. By the Sequential Characterization of Limits, since $\lim\limits_{x \to a} f(x) = u$ we have $f(x_k) \to u$, and since $\lim\limits_{x \to a} g(x) = v$ we have $f(x_k) \to v$. By Extended Operations on Limits of Sequences (Theorem 1.14), since $f(x_k) \to u$ and $g(x_k) \to v$ and u/v is defined in $\widehat{\mathbf{R}}$, we have $(f/g)(x_k) = \frac{f(x_k)}{g(x_k)} \to u/v$. Thus $(f/g)(x_k) \to u/v$ for every sequence (x_k) in $A \setminus \{a\}$ with $x_k \to a$. By the Sequential Characterization of Limits, it follows that $\lim\limits_{x \to a} (f/g)(x) = u/v$.

3.13 Theorem: *(The Comparison Theorem) Let $A \subseteq F$, let $f, g : A \to \mathbf{R}$ and let $a \in \mathbf{R}$ be a limit point of A. Suppose that $f(x) \le g(x)$ for all $x \in A$. Then*

(1) if $\lim\limits_{x \to a} f(x) = u$ and $\lim\limits_{x \to a} f(x) = v$ with $u, v \in \widehat{\mathbf{R}}$, then $u \le v$,

(2) if $\lim\limits_{x \to a} f(x) = \infty$ then $\lim\limits_{x \to a} g(x) = \infty$, and

(3) if $\lim\limits_{x \to a} g(x) = -\infty$ then $\lim\limits_{x \to a} g(x) = -\infty$.

Similar results hold for limits $x \to a^\pm$ and $x \to \pm\infty$.

3.14 Theorem: *(The Squeeze Theorem) Let $A \subseteq \mathbf{R}$, let $f, g, h : A \to \mathbf{R}$, and let $a \in \mathbf{R}$ be a limit point of A.*

(1) If $f(x) \le g(x) \le h(x)$ for all $x \in A$ and $\lim\limits_{x \to a} f(x) = b = \lim\limits_{x \to a} h(x)$, then $\lim\limits_{x \to a} g(x) = b$.

(2) If $|f(x)| \le g(x)$ for all $x \in A$ and $\lim\limits_{x \to a} g(x) = 0$ then $\lim\limits_{x \to a} f(x) = 0$.

Similar results hold for limits $x \to a^\pm$ and $x \to \pm\infty$.

3.15 Definition: Let $A \subseteq \mathbf{R}$, and let $f : A \to \mathbf{R}$. For $a \in A$, we say that f is **continuous** at a when

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in A \;\; \big(|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon\big).$$

We say that f is **continuous** (on A) when f is continuous at every point $a \in A$.

3.16 Theorem: *Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and let $a \in A$. Then*
(1) if a is not a limit point of A then f is continuous at a, and
(2) if a is a limit point of A then f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

Proof: The proof is left as an exercise.

3.17 Theorem: *(The Sequential Characterization of Continuity) Let $A \subseteq \mathbf{R}$, let $a \in A$, and let $f : A \to \mathbf{R}$. Then f is continuous at a if and only if for every sequence (x_k) in A with $x_k \to a$ we have $f(x_k) \to f(a)$.*

Proof: Suppose that f is continuous at a. Let (x_k) be a sequence in A with $x_k \to a$. Let $\epsilon > 0$. Choose $\delta > 0$ so that for all $x \in A$ we have $|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon$. Choose $m \in \mathbf{Z}$ so that for all indices k we have $k \geq m \Longrightarrow |x_k - a| < \delta$. Then when $k \geq m$ we have $|x_k - a| < \delta$ and hence $|f(x_k) - f(a)| < \epsilon$. Thus we have $f(x_k) \to f(a)$.

Conversely, suppose that f is not continuous at a. Choose $\epsilon_0 > 0$ so that for all $\delta > 0$ there exists $x \in A$ with $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon_0$. For each $k \in \mathbf{Z}^+$, choose $x_k \in A$ with $|x_k - a| \leq \frac{1}{k}$ and $|f(x_k) - f(a)| \geq \epsilon_0$. Consider the sequence (x_k) in A. Since $|x_k - a| \leq \frac{1}{k}$ for all $k \in \mathbf{Z}^+$, it follows that $x_k \to a$. Since $|f(x_k) - f(a)| \geq \epsilon_0$ for all $k \in \mathbf{Z}^+$, it follows that $f(x_k) \not\to f(a)$.

3.18 Theorem: *(Operations on Continuous Functions) Let $A \subseteq \mathbf{R}$, let $f, g : A \to \mathbf{R}$, let $a \in A$ and let $c \in \mathbf{R}$. Suppose that f and g are continuous at a. Then the functions cf, $f + g$, $f - g$ and fg are all continuous at a, and if $g(a) \neq 0$ then the function f/g is continuous at a.*

Proof: The proof is left as an exercise.

3.19 Theorem: *(Composition of Continuous Functions) Let $A, B \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, and let $h = g \circ f : C \to \mathbf{R}$ where $C = A \cap f^{-1}(B)$.*

(1) If f is continuous at $a \in C$ and g is continuous at $f(a)$, then h is continuous at a.
(2) If f is continuous (on A) and g is continuous (on B) then h is continuous (on C).

Proof: Note that Part 2 follows immediately from Part 1, so it suffices to prove Part 1. Suppose that f is continuous at $a \in A$ and g is continuous at $b = f(a) \in B$. Let (x_k) be a sequence in C with $x_k \to a$. Since f is continuous at a, we have $f(x_k) \to f(a) = b$ by the Sequential Characterization of Continuity. Since $\big(f(x_k)\big)$ is a sequence in B with $f(x_k) \to b$ and since g is continuous at b, we have $g(f(x_k)) \to g(b)$ by the Sequential Characterization of Continuity. Thus we have $h(x_k) = g\big(f(x_k)\big) \to g(b) = g\big(f(a)\big) = h(a)$. We have shown that for every sequence (x_k) in C with $x_k \to a$ we have $h(x_k) \to h(a)$. Thus h is continuous at a by the Sequential Characterization of Continuity.

3.20 Theorem: *(Functions Acting on Limits) Let $A, B \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$, let $g : B \to \mathbf{R}$ and let $h = g \circ f : C \to F$ where $C = A \cap f^{-1}(B)$. Let a be a limit point of C (hence also of A) and let b be a limit point of B. Suppose that $\lim\limits_{x \to a} f(x) = a$ and $\lim\limits_{y \to b} g(y) = c$. Suppose either that $f(x) \neq b$ for all $x \in C \setminus \{a\}$ or that g is continuous at $b \in B$. Then $\lim\limits_{x \to a} h(x) = c$.*

Analogous results hold, dealing with limits $x \to a^{\pm}$, $x \to \pm\infty$, $y \to b^{\pm}$ and $y \to \pm\infty$.

Proof: The proof is left as an exercise. It is similar to the proof of the Composition of Continuous Functions Theorem.

3.21 Definition: The functions 1, x, $\sqrt[n]{x}$ with $n \in \mathbf{Z}^+$, e^x, $\ln x$, $\sin x$ and $\sin^{-1} x$, are called the **basic elementary functions**. An **elementary function** is any function $f : A \subseteq \mathbf{R} \to \mathbf{R}$ which can be obtained from the basic elementary functions using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$.

3.22 Example: Each of the following functions $f(x)$ is elementary: $f(x) = |x| = \sqrt{x^2}$, $f(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$, $f(x) = \tan x = \frac{\sin x}{\cos x}$, $f(x) = \tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$, $f(x) = x^a = e^{a \ln x}$ where $a \in \mathbf{R}$, $f(x) = a^x = e^{x \ln a}$ where $a > 0$, and $f(x) = \frac{e^{\sqrt{x} + \sin x}}{\tan^{-1}(\ln x)}$.

3.23 Note: We shall assume familiarity with exponential, logarithmic, trigonometric and inverse trigonometric functions. In particular, we shall assume that they are known to be continuous in their domains, (and it follows that every elementary function is continuous in its domain). We shall also assume that their asymptotic behaviour, the intervals on which they are increasing and decreasing, and all of their usual algebraic identities are known. A review of this material can be found in Chapter 1.

A rigorous proof that these basic elementary functions are continuous, and that they satisfy their usual well-known properties, is quite long and difficult (and we shall not give a proof in this course). The main difficulty lies in giving a rigorous definition for each of the basic elementary functions. In most calculus courses, we define exponential and trigonometric functions informally. We might define the function $f(x) = e^x$ to be the function with $f(0) = 1$ which is equal to its own derivative, but we do not ever prove rigorously that such a function actually exists. We might define the sine and cosine functions by saying that for $\theta > 0$, when we start at $(1, 0)$ and travel a distance θ units counterclockwise around the unit circle $x^2 + y^2 = 1$, the point at which we arrive is (by definition) the point $(x, y) = (\cos\theta, \sin\theta)$, but we have not yet rigorously defined the meaning of distance along a curve. We use these informal definitions to argue, informally, that $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$ and then we argue that because e^x, $\sin x$ and $\cos x$ are differentiable, therefore they must be continuous.

There are various possible ways to define exponential and trigonometric functions rigorously. One way is to wait until one has rigorously defined power series and then define

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n \;,\; \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^n \;,\; \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^n.$$

If we define e^x, $\sin x$ and $\cos x$ using these formulas, then one can prove (rigorously) that they are differential and continuous, and one can verify (although it is quite time consuming to do so) that they satisfy all of their usual well-known properties.

3.24 Example: For each of the following sequences $(x_k)_{k\geq 0}$, evaluate $\lim\limits_{k\to\infty} x_k$ if it exists.

(a) $x_k = \frac{\sqrt{3k^2+1}}{k+2}$ (b) $x_k = \frac{1+3k}{\sqrt[3]{2+k-k^2}}$ (c) $x_k = \sin^{-1}\left(k - \sqrt{k^2+k}\right)$

Solution: For Part (a), we have $x_k = \frac{\sqrt{3k^2+1}}{k+2} = \frac{\sqrt{3+\frac{1}{k^2}}}{1+2\cdot\frac{1}{k}} \longrightarrow \frac{\sqrt{3+0}}{1+2\cdot 0} = \sqrt{3}$ where we used Basic Limits, Extended Operations on Limits, the fact that $\sqrt{x}$ is continuous, and the Sequential Characterization of Limits $\big($since $\sqrt{x}$ is continuous at 3 we have $\lim\limits_{x\to 3}\sqrt{x} = \sqrt{3}$, and since $3 + \frac{1}{k^2} \to 3$ we have $\lim\limits_{k\to\infty}\sqrt{3+\frac{1}{k^2}} = \lim\limits_{x\to 3}\sqrt{x} = \sqrt{3}$ by the Sequential Characterization of Limits$\big)$.

For Part (b), $x_k = \frac{1+3k}{\sqrt[3]{2+k-k^2}} = \frac{\frac{1}{k}+3}{\sqrt[3]{\frac{2}{k^2}+\frac{1}{k}-1}}\cdot k^{1/3} \longrightarrow \frac{0+3}{\sqrt[3]{0+0-1}}\cdot\infty = -1\cdot\infty = -\infty$ where we used Basic Limits, Extended Operations, the continuity of $\sqrt[3]{x}$, and the Sequential Characterization of Limits

For Part (c), note that $k - \sqrt{k^2+k} = \frac{k^2-(k^2+k)}{k+\sqrt{k^2+k}} = \frac{-k}{k+\sqrt{k^2+k}} = \frac{-1}{1+\sqrt{1+\frac{1}{k}}} \longrightarrow \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}$, and so $x_k = \sin^{-1}\left(k - \sqrt{k^2-k}\right) \longrightarrow \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

3.25 Exercise: Evaluate each of the following limits, if they exist.

(a) $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-2}{3-x}$ (b) $\lim\limits_{x\to 1}\sin^{-1}\left(\dfrac{2}{x-1} - \dfrac{x+3}{x^2-1}\right)$ (c) $\lim\limits_{x\to 0} e^{-1/x^2}$

(d) $\lim\limits_{x\to\infty}\dfrac{(3x+1)\sqrt{x}}{\sqrt{4x^3-2x+1}}$ (e) $\lim\limits_{x\to 1^-}\dfrac{\sqrt{x^3-2x^2+x}}{x^2+2x-3}$ (f) $\lim\limits_{x\to -1^+}\dfrac{x^2-2x-3}{x^3+4x^2+5x+2}$

3.26 Theorem: *(Intermediate Value Theorem) Let I be an interval in $\mathbf{R}$ and let $f : I \to \mathbf{R}$ be continuous. Let $a, b \in I$ with $a \leq b$ and let $y \in \mathbf{R}$. Suppose that either $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$. Then there exists $x \in [a,b]$ with $f(x) = y$.*

Proof: Like the Monotone Convergence Theorem, the statement of this theorem is intuitively reasonable, but it is quite difficult to prove, and in most calculus courses this theorem is accepted axiomatically, without proof.

3.27 Example: Prove that there exists $x \in [0,1]$ such that $3x - x^3 = 1$.

Solution: Let $f(x) = 3x - x^3$. Note that f is continuous (it is an elementary function) with $f(0) = 0$ and $f(1) = 2$ and so, by the Intermediate Value Theorem, there exists $x \in [0,1]$ such that $f(x) = 1$. We remark that in fact $f(x) = 1$ when $x = 2\cos\left(\frac{2\pi}{9}\right)$.

3.28 Definition: Let $A \subseteq \mathbf{R}$, and let $f : A \to \mathbf{R}$. For $a \in A$, if $f(a) \geq f(x)$ for every $x \in A$, then we say that $f(a)$ is the **maximum value** of f and that f attains its maximum value at a. Similarly for $b \in A$, if $f(b) \leq f(x)$ for every $x \in A$ then we say that $f(b)$ is the **minimum value** of f (in A) and that f attains its minimum value at b. We say that f attains its **extreme values** in A when f attains its maximum value at some point $a \in A$ and f attains its minimum value at some point $b \in A$.

3.29 Theorem: *(Extreme Value Theorem) Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : [a,b] \to \mathbf{R}$ be continuous. Then f attains its extreme values in $[a,b]$.*

Proof: Like the Monotone Convergence Theorem and the Intermediate Value Theorem, the statement of this theorem seems reasonable, but it is difficult to prove.

Chapter 4. Differentiation

4.1 Definition: For a subset $A \subseteq \mathbf{R}$, we say that A is **open** when it is a union of open intervals. Let $A \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$. For $a \in A$, we say that f is **differentiable** at a when the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in $\mathbf{R}$. In this case we call the limit the **derivative** of f at a, and we denote to by $f'(a)$, so we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We say that f is **differentiable** (on A) when f is differentiable at every point $a \in A$. In this case, the **derivative** of f is the function $f' : A \to \mathbf{R}$ defined by

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}.$$

When f' is differentiable at a, denote the derivative of f' at a by $f''(a)$, and we call $f''(a)$ the **second derivative** of f at a. When $f''(a)$ exists for every $a \in A$, we say that f is **twice differentiable** (on A), and the function $f'' : A \to \mathbf{R}$ is called the **second derivative** of f. Similarly, $f'''(a)$ is the derivative of f'' at a and so on. More generally, for any function $f : A \to \mathbf{R}$, we define its **derivative** to be the function $f' : B \to \mathbf{R}$ where $B = \big\{a \in A \big| f \text{ is differentiable at } a\big\}$, and we define its **second derivative** to be the function $f'' : C \to \mathbf{R}$ where $C = \big\{a \in B \big| f' \text{ is differentiable at } a\big\}$ and so on.

4.2 Remark: Note that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

To be precise, the limit on the left exists in $\mathbf{R}$ if and only if the limit on the right exists in $\mathbf{R}$, and in this case the two limits are equal.

4.3 Note: Let $A \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$, and let $a \in A$. Then

$$\begin{aligned}
&f \text{ is differentiable at } a \text{ with derivative } f'(a) \iff \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x - a| < \delta \implies \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon\right) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x - a| < \delta \implies \left|\frac{f(x) - f(a) - f'(a)(x - a)}{x - a}\right| < \epsilon\right) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \Big(0 < |x - a| < \delta \implies \big|f(x) - f(a) - f'(a)(x - a)\big| < \epsilon\,|x - a|\Big)
\end{aligned}$$

We can also simplify this last expression a little bit by noting that when $x = a$ we have $\big|f(x) - f(a) - f'(a)(x - a)\big| = 0 = \epsilon\,|x - a|$, so we can replace inequalities by equalities and say that f is differentiable at a if and only if

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \Big(|x - a| \le \delta \implies \big|f(x) - l(x)\big| \le \epsilon\,|x - a|\Big)$$

where $l : \mathbf{R} \to \mathbf{R}$ is given by $l(x) = f(a) + f'(a)(x - a)$.

4.4 Definition: When $f : A \to \mathbf{R}$ is differentiable at a with derivative $f'(a)$, the function

$$l(x) = f(a) + f'(a)(x-a)$$

is called the **linearization** of f at a. Note that the graph $y = l(x)$ of the linearization is the line through the point $(a, f(a))$ with slope $f'(a)$. This line is called the **tangent line** to the graph $y = f(x)$ at the point $(a, f(a))$.

4.5 Theorem: *(Differentiability Implies Continuity) Let $A \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$ and let $a \in A$. If f is differentiable at a then f is continuous at a.*

Proof: We have

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a) \longrightarrow f'(a) \cdot 0 = 0 \ \text{ as } x \to a$$

and so

$$f(x) = \big(f(x) - f(a)\big) + f(a) \longrightarrow 0 + f(a) = f(a) \ \text{ as } x \to a.$$

This proves that f is continuous at a.

4.6 Theorem: *(Local Determination of the Derivative) Let $A, B \subseteq \mathbf{R}$ be open with $A \subseteq B$, let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ wih $f(x) = g(x)$ for all $x \in A$. and let $a \in A$. Then f is differentiable at a if and only if g is differentiable at a and, in this case, $f'(a) = g'(a)$.*

Proof: The proof is left as an exercise.

4.7 Theorem: *(Operations on Derivatives) Let $A \subseteq \mathbf{R}$ be open, let $f, g : A \to \mathbf{R}$, let $a \in A$, and let $c \in \mathbf{R}$. Suppose that f and g are differentiable at a. Then*

(1) (Linearity) the functions cf, $f+g$ and $f-g$ are differentiable at a with

$$(cf)'(a) = c\, f'(a)\ ,\ (f+g)'(a) = f'(a) + g'(a)\ ,\ (f-g)'(a) - f'(a) - g'(a),$$

(2) (Product Rule) the function fg is differentiable at a with

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a),$$

(3) (Reciprocal Rule) if $g(a) \neq 0$ then the function $1/g$ is differentiable at a with

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2},$$

(4) (Quotient Rule) if $g(a) \neq 0$ then the function f/g is differentiable at a with

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof: We prove Parts (2), (3) and (4). For $x \in A$ with $x \neq a$, we have

$$\begin{aligned}
\frac{(fg)(x) - (fg)(a)}{x-a} &= \frac{f(x)g(x) - f(a)g(a)}{x-a} \\
&= \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} \\
&= f(x) \cdot \frac{g(x)-g(a)}{x-a} + g(a) \cdot \frac{f(x)-f(a)}{x-a} \\
&\longrightarrow f(a) \cdot g'(a) + g(a) \cdot f'(a) \ \text{ as } x \to a.
\end{aligned}$$

Note that $f(x) \to f(a)$ as $x \to a$ because f is continuous at a since differentiability implies continuity. This proves the Product Rule.

Suppose that $g(a) \neq 0$. Since g is continuous at a (because differentiability implies continuity) we can choose $\delta > 0$ so that $|x-a| \leq \delta \Longrightarrow |g(x)-g(a)| \leq \frac{|g(a)|}{2}$ and then when $|x-a| \leq \delta$ we have $|g(x)| \geq \frac{|g(a)|}{2}$ so that $g(x) \neq 0$. For $x \in A$ with $|x-a| \leq \delta$ we have

$$\frac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(a)}{x - -a} = \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} = \frac{-1}{g(x)g(a)} \cdot \frac{g(x)-g(a)}{x-a} \longrightarrow \frac{-1}{g(a)^2} \cdot g'(a)$$

as $x \to a$. This Proves the Reciprocal Rule.

Finally, note that Part (4) follows from Parts (2) and (3). Indeed when $g(a) \neq 0$, we have

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a) = f'(a) \cdot \left(\frac{1}{g}\right)(a) + f(a) \cdot \left(\frac{1}{g}\right)'(a) \\ &= f'(a) \cdot \frac{1}{g(a)} + f(a) \cdot \frac{-g'(a)}{g(a)^2} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.\end{aligned}$$

4.8 Theorem: *(Chain Rule) Let $A, B \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$, let $g : B \to \mathbf{R}$ and let $h = g \circ f : C \to \mathbf{R}$ where $C = A \cap f^{-1}(B)$. Let $a \in C$ and let $b = f(a) \in B$. Suppose that f is differentiable at a and g is differentiable at b. Then h is differentiable at a with*

$$h'(a) = g'\big(f(a)\big)\, f'(a).$$

Proof: We provide an explanation which can be converted (with a bit of trouble) into a rigorous proof. When $x \in A$ with $x \neq a$ and $y = f(x) \in B$ wih $y \neq b$ we have

$$\begin{aligned}\frac{h(x)-h(a)}{x-a} &= \frac{g\big(f(x)\big) - g\big(f(a)\big)}{x-a} = \frac{g(y)-g(b)}{x-a} \\ &= \frac{g(y)-g(b)}{y-b} \cdot \frac{y-b}{x-a} = \frac{g(y)-g(b)}{y-b} \cdot \frac{f(x)-f(a)}{x-a} \\ &\longrightarrow g'(b) \cdot f'(a) = g'\big(f(a)\big) \cdot f'(a) \text{ as } x \to a\end{aligned}$$

because as $x \to a$, since f is continuous at a we also have $f(x) \to f(a)$, that is $y \to b$.

We remark that when one tries to make this argument rigorous, using the ϵ-δ definition of limits, a difficulty arises because $x \neq a$ does not imply that $y \neq b$.

4.9 Definition: Recall that when $f : A \subseteq \mathbf{R} \to \mathbf{R}$, we say that f is **nondecreasing** (on A when for all $x, y \in A$, if $x \leq y$ then $f(x) \leq f(y)$, we say that f is (strictly) **increasing** (on A) when for all $x, y \in A$, if $x < y$ then $f(x) < f(y)$, we say that f is (strictly) **decreasing** (on A) when for all $x, y \in A$, if $x < y$ then $f(x) > f(y)$, and we say that f is (strictly) **monotonic** (on A) when either f is strictly increasing on A or f is strictly decreasing on A.

4.10 Theorem: *(The Inverse Function Theorem) Let I be an interval in $\mathbf{R}$, let $f : I \to \mathbf{R}$, let $J = f(I)$, and let a be a point in I which is not an endpoint.*

(1) If f is continuous then its range $J = f(I)$ is an interval in $\mathbf{R}$.
(2) If f is injective and continuous then f is strictly monotonic.
(3) If $f : I \to J$ is strictly monotonic, then so is its inverse $g : J \to I$.
(4) If f is bijective and continuous then its inverse g is continuous.

(5) If f is bijective and continuous, and f is differentiable at a with $f'(a) \neq 0$, then its inverse g is differentiable at $b = f(a)$ with $g'(b) = \frac{1}{f'(a)}$.

Proof: This theorem is quite difficult to prove and we omit the proof.

4.11 Theorem: *(Derivatives of the Basic Elementary Functions) The basic elementary functions have the following derivatives.*

(1) $(x^a)' = a\,x^{a-1}$ *where* $a \in \mathbf{R}$ *and* $x \in \mathbf{R}$ *is such that* x^{a-1} *is defined,*

(2) $(a^x)' = \ln a \cdot a^x$ *where* $a > 0$ *and* $x \in \mathbf{R}$ *and*
$(\log_a x)' = \frac{1}{\ln a} \cdot \frac{1}{x}$ *where* $0 < a \neq 1$ *and* $x > 0$, *and in particular*
$(e^x)' = e^x$ *for all* $x \in \mathbf{R}$ *and* $(\ln x)' = \frac{1}{x}$ *for all* $x > 0$,

(3) $(\sin x)' = \cos x$ *and* $(\cos x)' = -\sin x$ *for all* $x \in \mathbf{R}$, *and*
$(\tan x)' = \sec^2 x$ *and* $(\sec x)' = \sec x \tan x$ *for all* $x \in \mathbf{R}$ *with* $x \neq \frac{\pi}{2} + k\pi, k \in \mathbf{Z}$,
$(\cot x)' = -\csc^2 x$ *and* $(\csc x)' = -\cot x \csc x$ *for all* $x \in \mathbf{R}$ *with* $x \neq \pi + k\pi, k \in \mathbf{Z}$,

(4) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ *and* $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$ *for* $|x| < 1$,
$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$ *and* $(\csc^{-1} x)' = \frac{-1}{x\sqrt{x^2-1}}$ *for* $|x| > 1$, *and*
$(\tan^{-1} x)' = \frac{1}{1+x^2}$ *and* $(\cot^{-1} x)' = \frac{-1}{1+x^2}$ *for all* $x \in \mathbf{R}$.

Proof: First we prove Part 1 in the case that $a \in \mathbf{Q}$. When $n \in \mathbf{Z}^+$ and $f(x) = x^n$ we have

$$\frac{f(u)-f(x)}{u-x} = \frac{u^n - x^n}{u-x} = \frac{(u-x)(u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1})}{u-x}$$
$$= u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1} \longrightarrow n\,x^{n-1} \quad \text{as } u \to x.$$

This shows that $(x^n)' = n\,x^{n-1}$ for all $x \in \mathbf{R}$ when $n \in \mathbf{Z}^+$. By the Reciprocal Rule, for $x \neq 0$ we have

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{(x^n)'}{(x^n)^2} = -\frac{n\,x^{n-1}}{x^{2n}} = -n\,x^{-n-1}.$$

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$ (when n is odd, $x^{1/n}$ is defined for all $x \in \mathbf{R}$, and when n is even, $x^{1/n}$ is defined only for $x \geq 0$). Since $f'(x) = (x^n)' = n\,x^{n-1}$ we have $f'(x) = 0$ when $x = 0$. By the Inverse Function Theorem, when $x \neq 0$ we have

$$(x^{1/n})' = g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n\,g(x)^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n\,x^{1-\frac{1}{n}}} = \tfrac{1}{n}\,x^{\frac{1}{n}-1}.$$

Finally, when $n \in \mathbf{Z}^+$ and $k \in \mathbf{Z}$ with $\gcd(k,n) = 1$, by the Chain Rule we have

$$(x^{k/n})' = \big((x^{1/n})^k\big)' = k(x^{1/n})^{k-1}(x^{1/n})' = k\,x^{\frac{k-1}{n}} \cdot \tfrac{1}{n}\,x^{\frac{1-n}{n}} = \tfrac{k}{n}\,x^{\frac{k}{n}-1}.$$

We have proven Part 1 in the case that $a \in \mathbf{Q}$.

Next we shall prove Part 2. For $f(x) = a^x$ where $a > 0$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h} - a^x}{h} = \frac{a^x a^h - a^x}{h} = a^x \cdot \frac{a^h - 1}{h}$$

and so we have $f'(x) = a^x \left(\lim\limits_{h \to 0} \frac{a^h - 1}{h} \right)$ provided that the limit exists and is finite. For $g(x) = \log_a x$, where $0 < a \neq 1$ and $x > 0$, we have

$$\frac{g(x+h)-g(x)}{h} = \frac{\log_a(x+h) - \log_a x}{h} = \frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \frac{\log_a\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}} = \tfrac{1}{x} \cdot \log_a\left(1 + \tfrac{h}{x}\right)^{x/h}$$

and so we have $g'(x) = \frac{1}{x} \cdot \log_a \left(\lim\limits_{h \to 0} \left(1 + \frac{h}{x}\right)^{x/h} \right)$ provided the limit exists and is finite.

By letting $u = \frac{h}{x}$ we see that

$$\lim_{h \to 0^+} \left(1 + \tfrac{h}{x}\right)^{x/h} = \lim_{u \to \infty} \left(1 + \tfrac{1}{u}\right)^u = e$$

by the definition of the number e. By letting $u = -\frac{h}{x}$, a similar argument shows that

$$\lim_{h \to 0^-} \left(1 + \tfrac{h}{x}\right)^{x/h} = \lim_{u \to \infty} \left(1 - \tfrac{1}{u}\right)^{-u} = e.$$

Thus the derivative $g'(x)$ does exist and we have

$$(\log_a x)' = g'(x) = \tfrac{1}{x} \log_a \left(\lim_{h \to 0} \left(1 + \tfrac{h}{x}\right)^{x/h} \right) = \tfrac{1}{x} \log_a e = \tfrac{1}{x} \cdot \tfrac{\ln e}{\ln a} = \tfrac{1}{x \ln a}.$$

Since $g(x) = \log_a x$ is differentiable with $g'(x) \neq 0$ it follows from the Inverse Function Theorem that $f(x) = a^x$ is differentiable with derivative

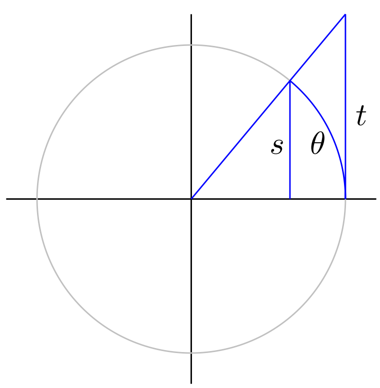
$$(a^x)' = f'(x) = \frac{1}{g'(f(x))} = \frac{1}{\frac{1}{f(x)\ln a}} = \ln a \cdot f(x) = \ln a \cdot a^x.$$

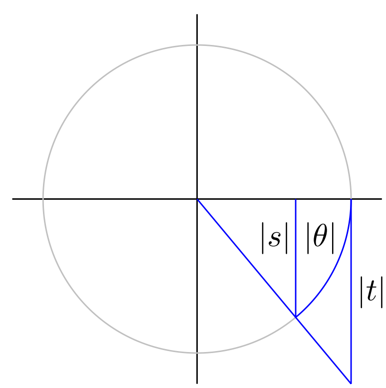
This proves Part 2.

Now we return to complete the proof of Part 1, in the case that $a \notin \mathbf{Q}$. When $a > 0$ we have $a^x = e^{x \ln a}$ for all $x > 0$ and so by the Chain Rule

$$(x^a)' = \left(e^{a \ln x}\right)' = e^{a \ln x}(a \ln x)' = x^a \cdot \tfrac{a}{x} = a\, x^{a-1}.$$

Let us move on to the proof of Part 3. We shall need two trigonometric limits which we shall explain informally (non-rigorously) with the help of pictures. Consider the following two pictures, the first showing an angle θ with $0 < \theta < \frac{\pi}{2}$ and the second with $-\frac{\pi}{2} < \theta < 0$. In both diagrams, the circle has radius 1 and $s = \sin\theta$ and $t = \tan\theta$.





In the first diagram, where $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta < \theta < \tan\theta$, and dividing by $\sin\theta$ (which is positive) gives $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. In the second diagram, where $-\frac{\pi}{2} < \theta < 0$, we have $-\sin\theta < -\theta < -\tan\theta$, and dividing by $-\sin\theta$ (which is positive) gives $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. In either case, taking the reciprocal gives $\cos\theta < \frac{\sin\theta}{\theta} < 1$. Since $\lim\limits_{\theta \to 0} \cos\theta = \cos(0) = 1$, it follows from the Squeeze Theorem that

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

From this limit we obtain the second trigonometric limit,

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta\,(1 + \cos\theta)} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta} = 1 \cdot \tfrac{0}{2} = 0.$$

Using the above two trigonometric limits, we have

$$\begin{aligned}(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h-\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\Big(\cos x\cdot\frac{\sin h}{h}-\sin x\cdot\frac{1-\cos h}{h}\Big)\\ &= \cos x\cdot 1-\sin x\cdot 0=\cos x\\ (\cos x)' &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\ &= \lim_{h\to 0}\Big(-\sin x\cdot\frac{\sin h}{h}-\cos x\cdot\frac{1-\cos h}{h}\Big)\\ &= -\sin x\cdot 1-\cos x\cdot 0=-\sin x.\end{aligned}$$

By the Quotient Rule, we have

$$(\tan x)' = \Big(\frac{\sin x}{\cos x}\Big)' = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

We leave it as an exercise to complete the proof of Part 3 by calculating the derivatives of $\sec x$ and $\csc x$.

Finally, we shall derive the formula for $(\sin^{-1} x)'$ and leave the rest of the proof of Part 4 as an exercise. Note that by the Inverse Function Theorem (which we did not prove), the function $\sin^{-1} x$ is differentiable in $(-1,1)$. Since $\sin(\sin^{-1} x) = x$ for all $x\in(-1,1)$, we can take the derivative on both sides (using the Chain Rule on the left) to get $\cos(\sin^{-1} x)\cdot(\sin^{-1} x)' = 1$ and hence

$$(\sin^{-1} x)' = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-\sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1-x^2}}.$$

4.12 Definition: Let $A\subseteq\mathbf{R}$, let $f:A\to\mathbf{R}$ and let $a\in A$. We say that f has a **local maximum** value at a when

$$\exists\delta>0\ \forall x\in A\ \Big(|x-a|\le\delta\implies f(x)\le f(a)\Big).$$

Similarly, we say that f has a **local minimum** value at a when

$$\exists\delta>0\ \forall x\in A\ \Big(|x-a|\le\delta\implies f(x)\ge f(a)\Big).$$

4.13 Theorem: *(Fermat's Theorem) Let $A\subseteq\mathbf{R}$ be open, let $f:A\to\mathbf{R}$, and let $a\in A$. Suppose that f is differentiable at a and that f has a local maximum or minimum value at a. Then $f'(a)=0$.*

Proof: We suppose that f has a local maximum value at a (the case that f has a local minimum value at a is similar). Choose $\delta>0$ so that $|x-a|\le\delta\implies f(x)\le f(a)$. For $x\in A$ with $a<x<a+\delta$, since $x>a$ and $f(x)\ge f(a)$ we have $\frac{f(x)-f(a)}{x-a}\ge 0$, and so

$$f'(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}\ge 0$$

by the Comparison Theorem. Similarly, for $x\in A$ with $a-\delta\le x<a$, since $x<a$ and $f(x)\ge f(a)$ we have $\frac{f(x)-f(a)}{x-a}\le 0$, and so

$$f'(a)=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}\le 0.$$

4.14 Theorem: *(Rolle's Theorem and the Mean Value Theorem) Let $a, b \in \mathbf{R}$ with $a < b$.*

(1) (Rolle's Theorem) If $f : [a, b] \to \mathbf{R}$ differentiable in (a, b) and continuous at a and b with $f(a) = 0 = f(b)$ then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

(2) (The Mean Value Theorem) If $f : [a, b] \to \mathbf{R}$ is differentiable in (a, b) and continuous at a and b then there exists a point $c \in (a, b)$ with $f'(c)\,(b - a) = f(b) - f(a)$.

Proof: To Prove Rolle's Theorem, let $f : [a, b] \to \mathbf{R}$ be differentiable in (a, b) and continuous at a and b with $f(a) = 0 = f(b)$. If f is constant, then $f'(x) = 0$ for all $x \in [a, b]$. Suppose that f is not constant. Either $f(x) > 0$ for some $x \in (a, b)$ or $f(x) < 0$ for some $x \in (a, b)$. Suppose that $f(x) > 0$ for some $x \in (a, b)$ (the case that $f(x) < 0$ for some $x \in (a, b)$ is similar). By the Extreme Value Theorem, f attains its maximum value at some point, say $c \in [a, b]$. Since $f(x) > 0$ for some $x \in (a, b)$, we must have $f(c) > 0$. Since $f(a) = f(b) = 0$ and $f(c) > 0$, we have $c \in (a, b)$. By Fermat's Theorem, we have $f'(c) = 0$. This completes the proof of Rolle's Theorem.

To prove the Mean Value Theorem, suppose that $f : [a, b] \to \mathbf{R}$ is differentiable in (a, b) and continuous at a and b. Let $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}\,(x - a)$. Then g is differentiable in (a, b) with $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ and g is continuous at a and b with $g(a) = 0 = g(b)$. By Rolle's Theorem, we can choose $c \in (a, b)$ so that $f'(c) = 0$, and then $g'(c) = \frac{f(b)-f(a)}{b-a}$, as required.

4.15 Corollary: *Let $a, b \in \mathbf{R}$ with $a < b$. Let $f : [a, b] \to \mathbf{R}$. Suppose that f is differentiable in (a, b) and continuous at a and b.*

(1) If $f'(x) \geq 0$ for all $x \in (a, b)$ then f is nondecreasing on $[a, b]$.
(2) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$.
(3) If $f'(x) \leq 0$ for all $x \in (a, b)$ then f is nonincreasing on $[a, b]$.
(4) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.
(5) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.
(6) If $g : [a, b] \to \mathbf{R}$ is continuous at a and b and differentiable in (a, b) with $g'(x) = f'(x)$ for all $x \in (a, b)$, then for some $c \in \mathbf{R}$ we have $g(x) = f(x) + c$ for all $x \in (a, b)$.

Proof: We prove Part 1 and leave the rest of the proofs as exercises. Suppose that $f'(x) \geq 0$ for all $x \in (a, b)$. Let $a \leq x < y \leq b$. Choose $c \in (x, y)$ so that $f'(c) = \frac{f(y)-f(x)}{y-x}$. Then $f(y) - f(x) = f'(c)(y - x) \geq 0$ and so $f(y) \geq f(x)$. Thus f is nondecreasing on $[a, b]$.

4.16 Corollary: *(The Second Derivative Test) Let I be an interval in $\mathbf{R}$, let $f : I \to \mathbf{R}$ and let $a \in I$. Suppose that f is differentiable in I with $f'(a) = 0$.*

(1) If $f''(a) > 0$ then f has a local minimum at a.
(2) If $f''(a) < 0$ then f has a local maximum at a.

Proof: The proof is left as an exercise.

4.17 Theorem: *(l'Hôpital's Rule) Let I be a non degenerate interval in $\mathbf{R}$. Let $a \in I$, or let a be an endpoint of I. Let $f, g : I \setminus \{a\} \to \mathbf{R}$. Suppose that f and g are differentiable in $I \setminus \{a\}$ with $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. Suppose either that $\lim\limits_{x \to a} f(x) = 0 = \lim\limits_{x \to a} g(x)$ or that $\lim\limits_{x \to a} g(x) = \pm\infty$. Suppose that $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = u \in \hat{\mathbf{R}}$. Then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = u$.*

Similar results hold for limits $x \to a^+$, $x \to a^-$, $x \to \infty$ and $x \to -\infty$.

Proof: We omit the proof, which is fairly difficult.