MATH 137 Calculus 1, Solutions to Assignment 9

1: Let T be the triangle with vertices at $a=(-1,0)$, $b=(1,0)$ and $c=(0,2)$.

(a) A point p lies inside T along the y-axis. Find the smallest possible value for the sum $|p-a|+|p-b|+|p-c|$ (where $|p-a|$ denotes the distance between p and a).

Solution: Let $p=(0,y)$ where $0\leq y\leq 2$. We need to minimize the sum

$$S(y)=|p-a|+|p-b|+|p-c|=\sqrt{1+y^2}+\sqrt{1+y^2}+(2-y)=2\sqrt{1+y^2}+2-y\,.$$

We differentiate to get

$$S'(y)=\frac{2y}{\sqrt{1+y^2}}-1\,.$$

Since $y\geq 0$ we have

$$S'(y)=0\iff 2y=\sqrt{1+y^2}\iff 4y^2=1+y^2\iff 3y^2=1\iff y=\tfrac{1}{\sqrt{3}}\,.$$

Note that $S(0)=4$, $S\big(\frac{1}{\sqrt{3}}\big)=2\sqrt{\frac{4}{3}}+2-\frac{1}{\sqrt{3}}=2+\sqrt{3}$, and $S(2)=2\sqrt{5}$. Thus the minimum sum is

$$S\big(\tfrac{1}{\sqrt{3}}\big)=2+\sqrt{3}\,.$$

(b) A smaller triangle S, with its lower vertex at $(0,0)$ and its upper edge parallel to the x-axis, is inscribed in T. Determine the minimum possible perimeter of S.

Solution: Let $p=(x,y)$ be upper-right vertex of S. We have $0\leq x\leq 1$ (when $x=0$ or 1, S is a degenerate triangle with area zero). The point p lies on the right edge of T, which is the line through $(1,0)$ and $(0,2)$, and so we have $y=2-2x$. The perimeter of S is given by

$$P=2x+2\sqrt{x^2+y^2}=2\big(x+\sqrt{5x^2-8x+4}\big)\,.$$

Differentiate to get

$$P'(x)=2\left(1+\frac{5x-4}{\sqrt{5x^2-8x+4}}\right)\,.$$

We have

$$P'(x)=0\iff\sqrt{5x^2-8x+4}=4-5x\implies 5x^2-8x+4=16-40x+25x^2$$
$$\iff 20x^2-32x+12=0\iff 5x^2-8x+3=0\iff(5x-3)(x-1)=0\iff x=1\text{ or }\tfrac{3}{5}\,.$$

Since $P(0)=4$, $P\left(\frac{3}{5}\right)=\frac{16}{5}$, and $P(1)=4$, the minimum perimeter is $P\big(\frac{3}{5}\big)=\frac{16}{5}$.

2: Let $a = (1,5)$ and $b = (2,2)$ and let $c = (x,0)$ with $x \geq \frac{8}{3}$ be a point along the x-axis.

(a) Find the maximum possible angle at c in the triangle abc.

Solution: Let α be the angle between line ac and the x-axis, and let β be the angle between bc and the x-axis. Note that $\alpha = \tan^{-1}\frac{5}{x-1}$ and $\beta = \tan^{-1}\frac{2}{x-2}$, and the angle at c in the triangle abc is

$$\theta = \alpha - \beta = \tan^{-1}\frac{5}{x-1} - \tan^{-1}\frac{2}{x-2}\,.$$

Differentiate to get

$$\theta'(x) = \frac{\frac{-5}{(x-1)^2}}{1+\frac{25}{(x-1)^2}} - \frac{\frac{-2}{(x-2)^2}}{1+\frac{4}{(x-2)^2}} = -\frac{5}{x^2-2x+26} + \frac{2}{x^2-4x+8}\,.$$

We have

$$\theta'(x) = 0 \iff 5(x^2-4x+8) = 2(x^2-2x+26) \iff 3x^2-16x-12 = 0$$
$$\iff x = \tfrac{16\pm\sqrt{256+144}}{6} = \tfrac{8\pm 10}{3} \iff x = 6 \text{ or } -\tfrac{2}{3}\,.$$

Since $x > \frac{8}{3}$, $\theta'(x) = 0 \iff x = 6$. By the nature of the problem, it is clear that $x = 6$ will maximize the value of θ, and we have

$$\theta(6) = \tan^{-1}1 - \tan^{-1}\tfrac{1}{2} = \tfrac{\pi}{4} - \tan^{-1}\tfrac{1}{2}\,.$$

We remark that this can also be written as $\theta(6) = \tan^{-1}\frac{1}{3}$.

(b) Find the maximum possible length for the shadow along the y-axis cast by the line segment ab when a light is placed at point c.

Solution: To avoid confusion, we shall write $c = (t,0)$ $\big($instead of $c = (x,0)\big)$. The line ac has equation $y = \frac{-5}{t-1}(x-t)$, and it has y-intercept $y = u = \frac{5t}{t-1}$. The line bc has equation $y = \frac{-2}{t-2}(x-t)$, and it has y-intercepy $y = v = \frac{5t}{t-1}$. Thus the length of the shadow along the y-axis is

$$L(t) = u - v = \frac{5t}{t-1} - \frac{2t}{t-2}\,.$$

Differentiate to get

$$L'(t) = \frac{5(t-1)-5t}{(t-1)^2} - \frac{2(t-2)-2t}{(t-2)^2} = \frac{-5}{(t-1)^2} + \frac{4}{(t-2)^2}\,.$$

We have

$$L'(t) = 0 \iff 5(t-2)^2 = 4(t-1)^2 \iff 5(t^2-4t+4) = 4(t^2-2t+1) \iff t^2-12t+16 = 0$$
$$\iff t = \tfrac{12\pm 144-64}{2} = 6 \pm 2\sqrt{5}\,.$$

Since $x > \frac{8}{3}$, $L(t) = 0 \iff t = 6+2\sqrt{5}$. By the nature of the problem, it is fairly clear that $t = 6+2\sqrt{5}$ will maximize the value of L, and we have

$$L(6+2\sqrt{5}) = \tfrac{5(6+2\sqrt{5})}{5+2\sqrt{5}} - \tfrac{2(6+2\sqrt{5})}{4+2\sqrt{5}} = \tfrac{5(6+2\sqrt{5})(5-2\sqrt{5})}{25-20} - \tfrac{2(6+2\sqrt{5})(4-2\sqrt{5})}{16-20}$$
$$= (6+2\sqrt{5})(5-2\sqrt{5}) + (3+\sqrt{5})(4-2\sqrt{5}) = (10-2\sqrt{5}) + (2-2\sqrt{5})$$
$$= 12 - 4\sqrt{5}\,.$$

In case it is not clear that we maximized L, we note that $\lim\limits_{t\to\frac{8}{3}} L(t) = \frac{5\cdot 8/3}{5/3} - \frac{2\cdot 8/3}{2/3} = 0$ and $\lim\limits_{t\to\infty} L(t) = 5-2 = 3$, and we have $L(6+2\sqrt{5}) > 3$ because $12 - 4\sqrt{5} > 3 \iff 9 > 4\sqrt{5} \iff 81 > 80$.

3: (a) Find the maximum possible capacity of a conical cup which is made from a circular piece of paper, of radius 3, with a slit along a radius.

Solution: Let r be the radius of the rim of the cup and let h be the height (or depth) of the cup. Then we have $r^2 + h^2 = 3^2 = 9$. The volume of the cup is

$$V = \tfrac{1}{3}\,\pi r^2 h = \tfrac{1}{3}\,\pi r^2 \sqrt{9 - r^2}\,.$$

Differentiate to get

$$V'(r) = \frac{\pi}{3}\Big(2r\sqrt{9-r^2} - \frac{r^3}{\sqrt{9-r^2}}\Big) = \frac{\pi}{3}\Big(\frac{2r(9-r^2) - r^3}{\sqrt{9-r^2}}\Big) = \frac{-\pi r(r^2-6)}{\sqrt{9-r^2}}\,.$$

We have

$$V'(r) = 0 \iff r(r^2-6) = 0 \iff r = 0 \text{ or } \pm\sqrt{6}\,.$$

From the nature of the problem, it is clear that $r = \sqrt{6}$ must maximize the volume, and we have

$$V(\sqrt{6}) = \tfrac{1}{3}\,\pi\, 6\sqrt{3} = 2\pi\sqrt{3}\,.$$

(b) Find the volume of the largest cone which can be inscribed in a sphere of radius $\frac{3}{2}$ m.

Solution: Let x, r and θ be as shown and let $R = \frac{3}{2}$. Then the volume of the cone is

$$V = \tfrac{1}{3}\pi r^2(R+x) = \tfrac{1}{3}\pi(R\sin\theta)^2(R + R\cos\theta) = \tfrac{\pi R^3}{3}\,\sin^2\theta(1+\cos\theta) = \tfrac{\pi R^3}{3}(1-\cos^2\theta)(1+\cos\theta)$$

where $0 \le \theta \le \pi$. We introduce the variable $u = \cos\theta$ and we have

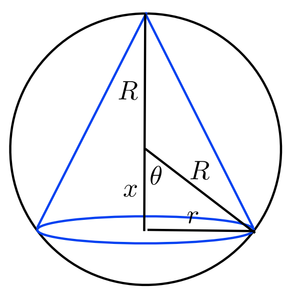
$$V(u) = \tfrac{\pi R^3}{3}\cdot(1-u^2)(u+1) = \tfrac{\pi R^3}{3}(-u^3 - u^2 + u + 1)$$

where $-1 \le u \le 1$. Differentiate to get

$$V'(u) = \tfrac{\pi R^3}{3}(-3u^2 - 2u + 1) = -\tfrac{\pi R^3}{3}(3u-1)(u+1)$$

so that $V'(u) = 0$ when $u = -\frac{1}{3}$ or 1. Since $V'(u) > 0$ for $-1 < u < -\frac{1}{3}$ and $V'(u) > 0$ for $-\frac{1}{3} < u < 1$, we see that the maximum volume is

$$V\big(-\tfrac{1}{3}\big) = \tfrac{\pi R^3}{3}\big(-\tfrac{1}{27} - \tfrac{1}{9} + \tfrac{1}{3} + 1\big) = \tfrac{\pi}{3}\big(\tfrac{3}{2}\big)^3\cdot\tfrac{32}{27} = \tfrac{4\pi}{3}\,.$$

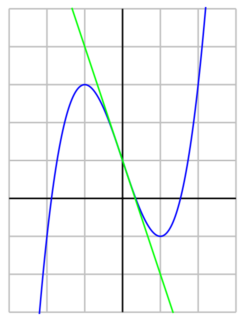


4: (a) Let $f(x) = x^3 - 3x + 1$ and let $x_1 = 0$. Apply Newton's method to find the approximations x_2 and x_3 to one of the roots of f. Sketch the graph of f and indicate which root is being approximated.

Solution: We have $f(x) = x^3 - 3x + 1$ and $f'(x) = 3x^2 - 3$ and $x_1 = 0$, and so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{1}{-3} = \tfrac{1}{3}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{1}{3} - \frac{\frac{1}{27} - 1 + 1}{\frac{1}{3} - 3} = \frac{1}{3} - \frac{\frac{1}{27}}{-\frac{8}{3}} = \tfrac{1}{3} + \tfrac{1}{72} = \tfrac{25}{72}\,.$$

From the graph of $y = f(x)$ and its tangent line at $x = 0$, it is clear that it is the middle root which is being approximated. (We remark that the exact value of this root is $2\sin(10°)$).



(b) Let $f(x) = x^3 - 4x$. Find $x_1 > 0$ such that when Newton's method is applied, we obtain $x_n = (-1)^{n+1} x_1$. Draw a sketch which explains the situation.

Solution: We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^3 - 4x_n}{3{x_n}^2 - 4} = \frac{3{x_n}^3 - 4x_n - {x_n}^3 + 4x_n}{3{x_n}^2 - 4} = \frac{2{x_n}^3}{3{x_n}^2 - 4}\,,$$

and so

$$x_{n+1} = -x_n \iff -x_n = \frac{2{x_n}^3}{3{x_n}^2 - 4} \iff -3{x_n}^3 + 4x_1 = 2{x_n}^3$$
$$\iff x_n(5{x_n}^2 - 4) = 0 \iff x_n = 0 \text{ or } x_n = \pm\tfrac{2}{\sqrt{5}}\,.$$

Thus we take $x_1 = \frac{2}{\sqrt{5}}$. Here is a picture showing the graph $y = f(x)$ together with the tangent lines at $x = \pm\frac{2}{\sqrt{5}}$. The sequence $x_1, x_2, x_3, \cdots$ follows the circuit indicated by the red parallelogram.

5: Use MAPLE to apply Newton's Method on a suitable function to find the approximate distance (accurate to about 10 decimal places) from the point $(1,2)$ to the curve $y = \ln x$.

Solution: For $x > 0$, let $f(x)$ be the square of the distance from $(1,2)$ to the point $(x, \ln x)$, that is

$$f(x) = (x-1)^2 + (\ln x - 2)^2 .$$

To find the point on the graph $y = \ln x$ which is nearest to $(1,2)$, we need to minimize $f(x)$. Differentiate to get

$$f'(x) = 2(x-1) + \frac{2(\ln x - 2)}{x} = \frac{2(x^2 - x + \ln x - 2)}{x} .$$

We see that $f'(x) = 0$ when x is a root of

$$g(x) = x^2 - x + \ln x - 2 .$$

By the nature of the problem, we expect that there is only one root and that this root minimizes $f(x)$ (and indeed, if you want you can verify that $g'(x) > 0$ for all $x > 0$, so $g(x)$ is increasing). We apply Newton's Method, starting with $x_1 = 1$, to approximate the root of $g(x)$. We have $g'(x) = 2x - 1 + \frac{1}{x}$, so we use the recursion

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{{x_n}^2 - x_n + \ln x_n - 2}{2x_n - 1 + \frac{1}{x_n}} .$$

We instruct MAPLE to perform the calculations using the following commands.

```
x[1]:=1.0;
for n from 1 to 10 do;
x[n+1]:=x[n]-(x[n]^2-x[n]+ln(x[n])-2)/(2*x[n]-1+1/x[n]);
end do;
```

We see that the values of x_n stabilize at $n = 6$ with $x_6 \cong 1.791173500$. Thus the distance from $(1,2)$ to the curve $y = \ln x$ is approximately equal to $\sqrt{f(x_6)}$, which we can find with the following MAPLE command.

```
sqrt((x[6]-1)^2+(ln(x[6])-2)^2);
```

We find that the distance is approximately

$$\sqrt{f(x_5)} \cong 1.623024994 .$$