MATH 137 Calculus 1, Solutions to Assignment 8

1: Evaluate each of the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\ln(1-x^2)}$.

Solution: Since $(\cos x - 1) \to 0$ and $\ln(1-x^2) \to 0$ as $x \to 0$, we can use l'Hôpital's Rule. We obtain

$$\lim_{x\to 0} \frac{\cos x - 1}{\ln(1-x^2)} = \lim_{x\to 0} \frac{-\sin x}{\frac{-2x}{1-x^2}} = \lim_{x\to 0} \frac{1-x^2}{2} \cdot \frac{\sin x}{x} = \frac{1}{2} \cdot 1 = \tfrac{1}{2}\,,$$

since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

(b) $\lim\limits_{x\to 1^-} \dfrac{\sqrt{1-x}}{\cos^{-1} x}$.

Solution: Since $\lim\limits_{x\to 1^-} \sqrt{1-x} = 0$ and $\lim\limits_{x\to 1^-} \cos^{-1} = 0$, we can apply l'Hôpital's Rule. We obtain

$$\lim_{x\to 1^-} \frac{\sqrt{1-x}}{\cos^{-1} x} = \lim_{x\to 1^-} \frac{\frac{-1}{2\sqrt{1-x}}}{\frac{-1}{\sqrt{1-x^2}}} = \lim_{x\to 1^-} \frac{\sqrt{1-x^2}}{2\sqrt{1-x}} = \lim_{x\to 1^-} \frac{\sqrt{(1-x)(1+x)}}{2\sqrt{1-x}} = \lim_{x\to 1^-} \frac{\sqrt{1+x}}{2} = \frac{\sqrt{2}}{2}\,.$$

(c) $\lim\limits_{x\to \frac{1}{2}^-} (2x)^{\tan(\pi x)}$.

Solution: Note that $(2x)^{\tan(\pi x)} = e^{\tan(\pi x)\ln(2x)}$, and we have

$$\begin{aligned}
\lim_{x\to \frac{1}{2}^-} \tan(\pi x)\ln(2x) &= \lim_{x\to \frac{1}{2}^-} \frac{\sin(\pi x)\ln(2x)}{\cos(\pi x)} \\
&= \lim_{x\to \frac{1}{2}^-} \frac{\ln(2x)}{\cos(\pi x)}\ \text{, since } \lim_{x\to \frac{1}{2}^-} \sin x = 1 \\
&= \lim_{x\to \frac{1}{2}^-} \frac{\frac{2}{2x}}{-\pi\sin(\pi x)} \text{ by l'Hôpital's Rule, since } \lim_{x\to \frac{1}{2}^-} \ln(2x) = 0 = \lim_{x\to \frac{1}{2}^-} \cos(\pi x) \\
&= -\frac{2}{\pi},
\end{aligned}$$

and so $\lim\limits_{x\to \frac{1}{2}^-} (2x)^{\tan(\pi x)} = \lim\limits_{x\to \frac{1}{2}^-} e^{\tan(\pi x)\ln(2x)} = e^{-2/\pi}$.

2: (a) Let $f(x) = \dfrac{x+1}{x^2+3}$. Find all the local maximum and minimum values of f for $x \in \mathbf{R}$, and find the absolute maximum and minimum values of f for $x \in [0,5]$.

Solution: We have

$$f'(x) = \frac{(x^2+3)-(x+1)(2x)}{(x^2+3)^2} = \frac{-x^2-2x+3}{(x^2+3)^2} = -\frac{(x+3)(x-1)}{(x^2+3)^2}\,.$$

We indicate where $f'(x)$ is positive, negative and zero in the following a table.

x		-3		1	
$f'(x)$	$-$	0	$+$	0	$-$

From the above table together with the First Derivative Test, there is a local minimum at $x=-3$, where we have $f(-3) = -\frac{1}{6}$, and a local maximum at $x=1$ where $f(1)=\frac{1}{2}$. To find the absolute maximum and minimum values on $[0,5]$, we find the values of f at the endpoints and the critical numbers: $f(0)=\frac{1}{3}$, $f(1)=\frac{1}{2}$ and $f(5)=\frac{3}{14}$. Thus the absolute maximum value is $f(1)=\frac{1}{2}$, and the absolute minimum value is $f(5)=\frac{3}{14}$.

(b) Let $f(x) = \dfrac{(2x-1)}{e^{x^2}}$. Find all the local maximum and minimum values of f for $x \in \mathbf{R}$, and find the absolute maximum and minimum values of f for $x \in [-1,2]$.

Solution: We can also write $f(x) = (2x-1)e^{-x^2}$, so we have

$$f'(x) = 2\,e^{-x^2} + (2x-1)(-2x)e^{-x^2} = (-4x^2+2x+2)e^{-x^2} = -2(2x+1)(x-1)e^{-x^2}\,.$$

Since e^{-x^2} is never zero, $f'(x)$ is positive, negative and zero as indicated in the following table.

x		$-\frac{1}{2}$		1	
$f'(x)$	$-$	0	$+$	0	$-$

From the above table, together with the First Derivative Test, we see that $f(x)$ has a local minimum value at $x=-\frac{1}{2}$ where we have $f\big(-\frac{1}{2}\big) = -\frac{2}{e^{1/4}}$ and a local maximum value at $x=1$ where we have $f(1)=\frac{1}{e}$. To find the absolute maximum and minimum values on $[-1,2]$, we find the values of f at the endpoints and the critical numbers: $f(-1) = -\frac{3}{e}$, $f\big(-\frac{1}{2}\big) = -\frac{2}{e^{1/4}}$, $f(1)=\frac{1}{e}$ and $f(2)=\frac{3}{e^4}$. The absolute maximum value is $f(1)=\frac{1}{e}$ and the absolute minimum value is $f\big(-\frac{1}{2}\big) = -\frac{2}{e^{1/4}}$. $\big($We remark that a calculator is not needed here, for example we know that $f(-1) > f\big(-\frac{1}{2}\big)$ because f is decreasing for $x < -\frac{1}{2}\big)$.

3: (a) Let $f(x) = 2 - \dfrac{3}{x} + \dfrac{1}{x^3}$. Sketch the graph $y = f(x)$, showing all x-intercepts, all asymptotes, all local maxima and minima, and all points of inflection.

Solution: We have

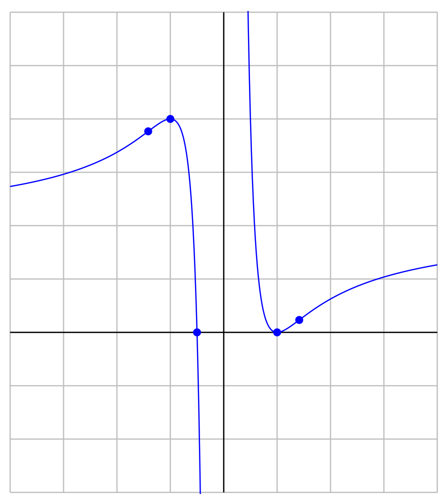
$$\begin{aligned} f(x) &= 2 - \frac{3}{x} + \frac{1}{x^3} = \frac{2x^3 - 3x^2 + 1}{x^3} = \frac{(x-1)(2x - x - 1)}{x^3} = \frac{2(x-1)^2(2x+1)}{x^3} \\ f'(x) &= \frac{3}{x^2} - \frac{3}{x^4} = \frac{3x^2 - 3}{x^4} = \frac{3(x-1)(x+1)}{x^4} \\ f''(x) &= -\frac{6}{x^3} + \frac{12}{x^5} = \frac{-6x^2 + 12}{x^5} = \frac{-6(x-\sqrt{2})(x+\sqrt{2})}{x^5} \end{aligned} .$$

We indicate where each of these is positive, negative, zero and undefined (indicated by #) in the following table.

x		$-\sqrt{2}$		-1		$-\frac{1}{2}$		0		1		$\sqrt{2}$	
$f(x)$	+	+	+	+	+	0	−	#	+	0	+	+	+
$f'(x)$	+	+	+	0	−	−	−	#	−	0	+	+	+
$f''(x)$	+	0	−	−	−	−	−	#	+	+	+	0	−

The table gives a lot of information about the graph. The graph lies above that x-axis when $f(x) > 0$ and below the x-axis when $f(x) < 0$, the graph is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$, and the graph is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$. The table also indicates that the x-intercepts are at $x = -\frac{1}{2}$ and $x = 1$, that there is a local maximum when $x = -1$ and a local minimum when $x = 1$, and that there are points of inflection at $x = \pm\sqrt{2}$. To help sketch the graph, we make a table of values and limits. The limits in the table indicate that f has a vertical asymptote along $x = 0$ (the y-axis) and horizontal asymptotes, both to the left and to the right, along $y = 2$.

x	y
$\to -\infty$	2
$-\sqrt{2}$	$2 + \frac{3}{\sqrt{2}} - \frac{1}{2\sqrt{2}} = 2 + \frac{5\sqrt{2}}{4}$
-1	4
$-\frac{1}{2}$	0
$\to 0^-$	$-\infty$
$\to 0^+$	∞
1	0
$\sqrt{2}$	$2 - \frac{3}{\sqrt{2}} + \frac{1}{2\sqrt{2}} = 2 - \frac{5\sqrt{2}}{4}$
$\to \infty$	2



(b) Let $f(x) = \dfrac{x}{\sqrt{x^4+1}}$. Sketch the graph $y = f(x)$, showing all x-intercepts, all asymptotes, all local maxima and minima, and all points of inflection.

Solution: We have

$$f'(x) = \frac{\sqrt{x^4+1} - 2x^4/\sqrt{x^4+1}}{(x^2+1)} = \frac{(x^4+1)-(2x^4)}{(x^4+1)^{3/2}} = -\frac{x^4-1}{(x^4+1)^{3/2}} = -\frac{(x+1)(x-1)(x^2+1)}{(x^4+1)^{3/2}}$$

$$f''(x) = -\frac{(4x^3)(x^4+1)^{3/2} - (x^4-1)\frac{3}{2}(x^4+1)^{1/2}(4x^3)}{(x^4+1)^{5/2}} = -\frac{4(x^3)(x^4+1) - 6(x^4-1)(x^3)}{(x^4+1)^{5/2}}$$

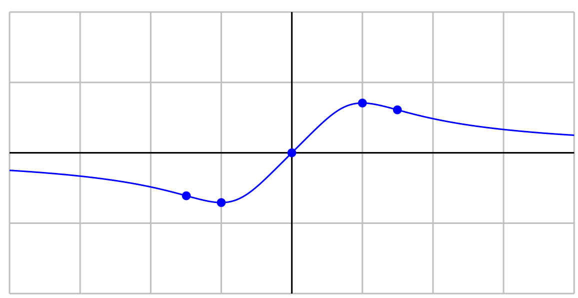
$$= \frac{2x^7 - 10x^3}{(x^4+1)^{5/2}} = \frac{2x^3(x+\sqrt[4]{5})(x-\sqrt[4]{5})(x^2+\sqrt{5})}{(x^4+1)^{5/2}}$$

We make a table indicating where $f(x)$, $f'(x)$ and $f''(x)$ are positive, negative and zero.

x		$-\sqrt[4]{5}$		-1		0		1		$\sqrt[4]{5}$	
y	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
y'	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$

As in part (a), the table indicates where the graph lies above the x-axis and where it lies below, and where the graph is increasing and where it is decreasing, and where the graph is concave up and where it is concave down, and it indicates that the graph has an x-intercept at $x = 0$, it has a local maximum at $x = 1$ and a local minimum at $x = -1$, and it has points of inflection when $x = \pm\sqrt[4]{5}$ and when $x = 0$. To help sketch the curve, we make a table of values and limits. The limits indicate that the graph has horizontal asymptotes, both to the left and to the right, along $y = 0$ (the x-axis).

x	y	significance
$\to -\infty$	0	assyptote
$-\sqrt[4]{5}$	$-\sqrt[4]{5}/\sqrt{6}$	inflection
-1	$-1/\sqrt{2}$	min
0	0	inflextion
1	$1/\sqrt{2}$	max
$\sqrt[4]{5}$	$\sqrt[4]{5}/\sqrt{6}$	inflection
$\to \infty$	0	assymptote



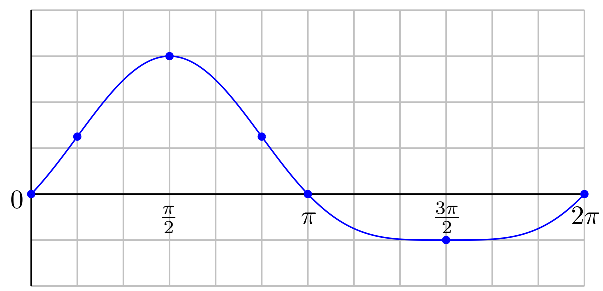
4: (a) Let $f(x) = 2\sin x + \sin^2 x$ for $0 \le x \le 2\pi$. Sketch the graph $y = f(x)$ showing all x-intercepts, all local maxima and minima, and all points of inflection.

Solution: We have $f(x) = 2\sin x + \sin^2 x = \sin x(\sin x + 2)$, and we note that $(\sin x + 2) > 0$ for all x, so $f(x) = 0$ when $x = 0$ and π, and $f(x)$ is positive when $\sin x$ is positive. Also, $f'(x) = 2\cos x + 2\sin x\cos x = 2\cos x(\sin x + 1)$, and note that $(\sin x + 1) \ge 0$ for all x, so $f'(x) = 0$ when $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ and $f'(x)$ is positive when $\cos x$ is positive. Finally, $f''(x) = -2\sin x + 2(\cos^2 x - \sin^2 x) = -2\sin x + 2(1 - 2\sin^2 x) = -2(2\sin^2 x - \sin x - 1) = 2(2\sin x - 1)(\sin x + 1)$, and note that $(\sin x + 1) \ge 0$ for all x, so $f''(x) = 0$ when $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$ and $\frac{3\pi}{2}$, and $f''(x)$ is positive when $\sin x < \frac{1}{2}$. We summarize:

x	0		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$		π		$\frac{3\pi}{2}$		2π
$f(x)$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0	$-$	$-$	$-$	0
$f'(x)$	$+$	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$
$f''(x)$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$+$	$+$

The graph has x-intercepts at $x = 0$, $x = \pi$ and $x = 2\pi$, it has a local maximum at $x = \frac{\pi}{2}$ and a local minimum at $x = \frac{3\pi}{2}$, and the points of inflection are at $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. We make a table of values and draw the sketch:

x	y	significance
0	0	intercept
$\frac{\pi}{6}$	$5/4$	inflection
$\frac{\pi}{2}$	3	maximum
$\frac{5\pi}{6}$	$5/4$	inflection
π	0	intercept
$\frac{3\pi}{2}$	-1	minimum
2π	0	intercept



(b) Let $f(x) = \tan^{-1}\left(\dfrac{(x-1)^2}{(x+1)^2}\right)$. Sketch the graph of $y = f(x)$ showing all intercepts, all asymptotes, all local maxima and minima, and find the x-value of each point of inflection.

Solution: We have

$$f'(x) = \frac{1}{1+\left(\frac{x-1}{x+1}\right)^4}\cdot 2\left(\frac{x-1}{x+1}\right)\frac{(x+1)-(x-1)}{(x+1)^2} = \frac{4(x-1)(x+1)}{(x+1)^4+(x-1)^4} = \frac{2(x-1)(x+1)}{x^4+6x^2+1}$$

$$f''(x) = 2\,\frac{(2x)(x^4+6x^2+1)-(x^2-1)(4x^3+12x)}{(x^4+6x^2+1)^2} = 4\,\frac{(x^5+6x^3+x)-(2x^5+4x^3-6x)}{(x^4+6x^2+1)^2}$$

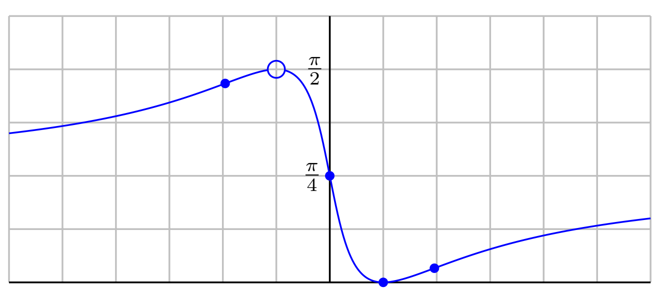
$$= \frac{-4\,x\,(x^4-2x^2-7)}{(x^4+6x^2+1)^2} = \frac{-4\,x\left(x-\sqrt{1+\sqrt{8}}\right)\left(x+\sqrt{1+\sqrt{8}}\right)\left(x^2+(\sqrt{8}-1)\right)}{(x^4+6x^2+1)^2}\,.$$

At the last step, we factored x^4-2x^2-7 as follows: using the Quadratic Formula, we have $x^4-2x^2-7=0$ when $x=\frac{2\pm\sqrt{4+28}}{2}=1\pm\sqrt{8}$, and so $x^4-2x^2-7=\big(x^2-(1+\sqrt{8})\big)\big(x^2-(1-\sqrt{8})\big)$. Note that x^4+6x^2+1 has no real roots, since we would need $x^2=\frac{-6\pm\sqrt{36-4}}{2}=-3\pm\sqrt{8}$, but $-3+\sqrt{8}<0$. We indicate where $f(x)$, $f'(x)$ and $f''(x)$ are positive, negative, zero and undefined in the following table.

x		$-\sqrt{1+\sqrt{8}}$		-1		0		1		$\sqrt{1+\sqrt{8}}$	
$f(x)$	$+$	$+$	$+$	#	$+$	$+$	$+$	0	$+$	$+$	$+$
$f'(x)$	$+$	$+$	$+$	#	$-$	$-$	$-$	0	$+$	$+$	$+$
$f''(x)$	$+$	0	$-$	#	$-$	0	$+$	$+$	$+$	0	$-$

The graph always lies above the x-axis except at $x=1$ where there is an x-intercept. The graph is increasing on $(-\infty,-1)$ then decreasing on $(-1,1)$ then increasing again on $(1,\infty)$. The graph has a local minimum at $x=1$. The behaviour of the graph near $x=-1$ is a bit subtle. Note that as $x\to-1$ we have $\frac{(x-1)^2}{(x+1)^2}\to\infty$ and so $f(x)=\tan^{-1}\frac{(x-1)^2}{(x+1)^2}\to\frac{\pi}{2}$, so there is a hole in the graph at the point $\big(-1,\frac{\pi}{2}\big)$. Also note that as $x\to-1$ we have $f'(x)\to 0$, so the slope of the graph near the hole approaches zero. If we were to fill the hole in the graph by adding the point $\big(-1,\frac{\pi}{2}\big)$, then the new graph would have a local maximum at $\big(-1,\frac{\pi}{2}\big)$. Furthermore, writing $a=\sqrt{1+\sqrt{8}}$, we note that the graph is concave up in $(-\infty,-a)$, then concave down in $(-a,0)$ (except at the hole), then concave up in $(0,a)$, then concave down in (a,∞). We make a table of values and limits and sketch the curve.

x	y	
$\to-\infty$	$\frac{\pi}{4}$	assymptote
$-\sqrt{1+\sqrt{8}}$	ugly	inflection
$\to-1$	$\frac{\pi}{2}$	hole
0	$\frac{\pi}{4}$	inflection
1	0	minimum
$\sqrt{1+\sqrt{8}}$	ugly	inflection
$\to\infty$	$\frac{\pi}{4}$	assymptote



5: (a) Prove that $\ln x \geq -\frac{1}{2}(x-1)(x-3)$ for all $x \geq 1$.

Solution: Let $f(x) = \ln x + \frac{1}{2}(x-1)(x-3)$ and note that

$$f'(x) = \frac{1}{x} + \tfrac{1}{2}\big((x-3)+(x-1)\big) = \frac{1}{x} + (x-2) = \frac{x^2-2x+1}{x} = \frac{(x-1)^2}{x}\,.$$

We must show that $f(x) \geq 0$ for all $x \geq 1$. When $x = 1$ we have $f(x) = f(1) = 0$. Suppose that $x = a > 1$. Since $f(x)$ is differentiable in $(1,a)$ and continuous on $[1,a]$, by the Mean Value Theorem, we can choose a number $c \in (1,a)$ such that $f'(c) = \frac{f(a)-f(1)}{a-1} = \frac{f(a)}{a-1}$. Then we have $f(a) = (a-1)f'(c) = (a-1)\,\frac{(c-1)^2}{c} > 0$.

(b) Prove that ${\sqrt{x}}^{\sqrt{x+1}} > {\sqrt{x+1}}^{\sqrt{x}}$ for all $x > e^2$.

Solution: Note that for $x > 0$ we have

$$\begin{aligned} {\sqrt{x}}^{\sqrt{x+1}} > {\sqrt{x+1}}^{\sqrt{x}} &\iff \ln\left({\sqrt{x}}^{\sqrt{x+1}}\right) > \ln\left({\sqrt{x+1}}^{\sqrt{x}}\right) \\ &\iff \sqrt{x+1}\cdot\tfrac{1}{2}\ln x > \sqrt{x}\cdot\tfrac{1}{2}\ln(x+1) \\ &\iff \frac{\ln x}{\sqrt{x}} > \frac{\ln(x+1)}{\sqrt{x+1}}\,. \end{aligned}$$

Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ for $x > 0$. Then

$$f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \ln x\cdot\frac{1}{2\sqrt{x}}}{x} = \frac{2-\ln x}{2x^{3/2}}\,.$$

We see that $f'(x) < 0$ when $\ln x > 2$, that is when $x > e^2$, and so $f(x)$ is decreasing for $x > e^2$. In particular, when $x > e^2$ we have $f(x) > f(x+1)$, that is $\frac{\ln x}{\sqrt{x}} > \frac{\ln(x+1)}{\sqrt{x+1}}$, as required.

(c) Let $f(x)$ be differentiable for all $x \in \mathbf{R}$ with $f(0) = 3$. Suppose $f'(x) \leq 1$ for all $x > 0$. Prove that there is a number $a > 0$ such that $f(a) = 2a$.

Solution: We claim that $f(3) \leq 6$. Since f is differentiable in $(0,3)$ and continuous on $[0,3]$, by the Mean Value Theorem we can choose a point $c \in (0,3)$ such that $f'(c) = \frac{f(3)-f(0)}{3-0} = \frac{f(3)-3}{3}$. Since $f'(c) \leq 1$ we have $f(3) = 3f'(c) + 3 \leq 6$, as claimed. If $f(3) = 6$ then we can take $a = 3$ to get $f(a) = 2a$. Suppose that $f(3) < 6$. Let $g(x) = f(x) - 2x$. Then $g(x)$ is continuous on $[0,3]$ and $g(0) = 3 > 0$ and $g(3) = f(3) - 6 < 0$, and so by the Intermediate Value Theorem we can choose a number $a \in (0,3)$ such that $g(a) = 0$. Then we have $0 = g(a) = f(a) - 2a$ and so $f(a) = 2a$, as required.