MATH 137 Calculus 1, Assignment 8 Due Fri Nov 16

1: Evaluate each of the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\ln(1-x^2)}$.

(b) $\lim\limits_{x\to 1^-} \dfrac{\sqrt{1-x}}{\cos^{-1} x}$.

(c) $\lim\limits_{x\to \frac{1}{2}^-} (2x)^{\tan(\pi x)}$.

2: (a) Let $f(x) = \dfrac{x+1}{x^2+3}$. Find all the local maximum and minimum values of f for $x \in \mathbf{R}$, and find the absolute maximum and minimum values of f for $x \in [0,5]$.

(b) Let $f(x) = \dfrac{(2x-1)}{e^{x^2}}$. Find all the local maximum and minimum values of f for $x \in \mathbf{R}$, and find the absolute maximum and minimum values of f for $x \in [-1,2]$.

3: (a) Let $f(x) = 2 - \dfrac{3}{x} + \dfrac{1}{x^3}$. Sketch the graph $y = f(x)$, showing all x-intercepts, all asymptotes, all local maxima and minima, and all points of inflection.

(b) Let $f(x) = \dfrac{x}{\sqrt{x^4+1}}$. Sketch the graph $y = f(x)$, showing all x-intercepts, all asymptotes, all local maxima and minima, and all points of inflection.

4: (a) Let $f(x) = 2\sin x + \sin^2 x$ for $0 \le x \le 2\pi$. Sketch the graph $y = f(x)$ showing all x-intercepts, all local maxima and minima, and all points of inflection.

(b) Let $f(x) = \tan^{-1}\left(\dfrac{(x-1)^2}{(x+1)^2}\right)$. Sketch the graph of $y = f(x)$ showing all intercepts, all asymptotes, all local maxima and minima, and find the x-value of each point of inflection.

5: (a) Use the Mean Value Theorem to prove that $\ln x \ge -\frac{1}{2}(x-1)(x-3)$ for all $x \ge 1$, without using the Concavity Test.

(b) Prove that $\sqrt{x}^{\sqrt{x+1}} > \sqrt{x+1}^{\sqrt{x}}$ for all $x > e^2$.

(c) Let $f(x)$ be differentiable for all $x \in \mathbf{R}$ with $f(0) = 3$. Suppose $f'(x) \le 1$ for all $x > 0$. Prove that there is a number $a > 0$ such that $f(a) = 2a$.

Hints and Comments

1: (a),(b) Use l'Hôpital's Rule.

(c) Note that $(2x)^{\tan(\pi x)} = e^{\tan(\pi x)\ln(2x)}$. Use l'Hôpital's Rule to find the limit of the exponent.

2: This should be fairly straightforward.

3,4: To sketch the graphs in problems 3 and 4, you should follow the following (fairly lengthy) procedure. Calculate $f'(x)$ and $f''(x)$. Express each of the functions $f(x)$, $f'(x)$ and $f''(x)$ in factored form (as a single fraction with the numerator factored and the denominator factored). From the factored form, you should be able to determine where each of the functions $f(x)$, $f'(x)$ and $f''(x)$ is positive, negative, zero and undefined. This gives a great deal of information about the graph of $f(x)$ $\big($the graph lies above that x-axis when $f(x) > 0$ and below the x-axis when $f(x) < 0$, the graph is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$, the graph is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$, the x-intercepts occur at the points where $f(x) = 0$, the local maximum and minimum values occur at points where $f'(x)$ changes sign and the points of inflection occur at points where $f''(x)$ changes sign$\big)$. To help sketch the graph, you should make a table of values (including the values at important points) and you should find any important limits (the limits of $f(x)$ as x approaches any boundary point of the domain).

5: (a) Let $f(x) = \ln x + \frac{1}{2}(x-1)(x-3)$. Show that $f'(x) \geq 0$ for all $x \geq 1$. Use the Mean Value Theorem (as in the Increasing/Decreasing Test) to show that when $f'(x) \geq 0$ for all $x \geq 1$ we have $f(x) \geq f(1)$ for all $x \geq 1$.

(b) First show that $\sqrt{x}^{\sqrt{x+1}} > \sqrt{x+1}^{\sqrt{x}} \iff \dfrac{\ln x}{\sqrt{x}} > \dfrac{\ln(x+1)}{\sqrt{x+1}}$. Then let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ and show that $f(x)$ is decreasing for $x > e^2$.

(c) First show that $f(3) \leq 6$ by applying the Mean Value Theorem to $f(x)$ on $[0,3]$. Then apply the Intermediate Value Theorem to the function $g(x) = f(x) - 2x$ on a suitable interval to find the required point a.