MATH 137 Calculus 1, Solutions to Assignment 7

1: (a) A population $P = P(t)$ grows exponentially with $P(3) = 60$ and $P(6) = 90$. Find the population when $t = 10$, and find the time at which the population is $P = 200$.

Solution: Since $P(3) = 60$ and $P(6) = 90$, between $t = 3$ and 6 the population increases by a factor of $\frac{3}{2}$, and so the formula for $P(t)$ can be written in the form $P(t) = P(0)\left(\frac{3}{2}\right)^{t/3}$. Put in $t = 3$ to get $60 = P(0) \cdot \frac{3}{3}$, so $P(0) = \frac{2 \cdot 60}{3} = 40$, and so we have

$$P(t) = 40\left(\tfrac{3}{2}\right)^{t/3}$$

$\big($this can also be written as $P(t) = 40\,e^{kt}$ with $k = \frac{1}{3}\ln\frac{3}{2}\big)$. In particular, we have $P(10) = 40\left(\frac{3}{2}\right)^{10/3}$. Also,

$$P(t) = 200 \iff 40\left(\tfrac{3}{2}\right)^{t/3} = 40 \iff \left(\tfrac{3}{2}\right)^{t/3} = 5 \iff \tfrac{t}{3}\ln\tfrac{3}{2} = \ln 5 \iff t = \tfrac{3\ln 5}{\ln(3/2)}\,.$$

(b) Living vegetation contains a certain amount of Carbon-14, written as C^{14}, and when the vegetation dies, the amount of C^{14} begins to decay exponentially with a half-life of 5730 years. If a parchment contains 80% as much C^{14} as it contained when it was first made, then how old is the parchment?

Solution: The amount $C(t)$ of C^{14} in the parchment at time t, in years, is given by

$$C(t) = C_0\,2^{-t/5730}$$

where $C_0 = C(0)$ $\big($this can also be written as $C(t) = C_0\,e^{-kt}$ with $k = \frac{\ln 2}{5730}\big)$. We need to find the value of t such that $C(t) = 0.80\,C_0 = \frac{4}{5}\,C_0$. We have

$$C(t) = \tfrac{4}{5}\,C_0 \iff C_0\,2^{-t/5730} = \tfrac{4}{5}\,C_0 \iff 2^{-t/5730} = \tfrac{4}{5} \iff 2^{t/5730} = \tfrac{5}{4}$$
$$\iff \tfrac{t}{5730}\ln 2 = \ln\tfrac{5}{4} \iff t = \tfrac{5730\ln(5/4)}{\ln 2} \cong 1845$$

(the final approximation was done on a calculator), so the parchment is about 1845 years old.

2: (a) The position of an object moving along the x-axis is given by $x(t) = \dfrac{t^4}{e^t}$. Find the velocity $v(t)$ and the acceleration $a(t)$, and find all of the values of t at which $a(t) = 0$.

Solution: The velocity and acceleration are

$$\begin{aligned} v(t) &= x'(t) = \frac{4t^3\,e^t - t^4\,e^t}{(e^t)^2} = \frac{4t^3 - t^4}{e^t} \\ a(t) &= v'(t) = \frac{(12t^2 - 4t^3)e^t - (4t^3 - t^4)e^t}{(e^t)^2} = \frac{12t^2 - 8t^3 + t^4}{e^t} = \frac{t^2(t-2)(t-6)}{e^t}\,. \end{aligned}$$

We see that $a(t) = 0$ when $t = 0$, 2 and 6.

(b) A rod of length 3 m lies along the x-axis with one end at $x = 0$ and the other at $x = 3$. The mass, in kg, of that part of the rod which lies between $x = 0$ and $x = l$ is given by

$$M(l) = l + \tfrac{1}{2}\,l^2 - \tfrac{1}{12}\,l^3\,.$$

The *average linear density* of the rod (measured in kg/m) is defined to be $\overline{\rho} = \frac{M}{L}$ where M is the total mass of the rod and L is the length of the rod, and the *linear density* of the rod (in kg/m) at the point $x = l$ is defined to be $\rho(l) = M'(l)$. Find the average linear density of the rod, find the linear density of the rod at each point $x = l$, and find the maximum value of the linear density.

Solution: The total mass of the rod is $M(3) = 3 + \frac{9}{2} - \frac{9}{4} = \frac{21}{4}$ and the length is 3, so the average density is

$$\overline{\rho} = \tfrac{M}{L} = \tfrac{21/4}{3} = \tfrac{7}{4}\,.$$

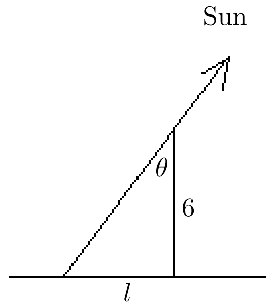
The linear density at $x = l$ is given by

$$\rho(l) = M'(l) = 1 + l - \tfrac{1}{4}\,l^2\,.$$

Note that $\rho(l) = -\frac{1}{4}\,l^2 + l + 1 = -\frac{1}{4}(l-2)^2 + 2$, so the graph of $\rho = \rho(l)$ is a parabola with the highest point at the vertex $(\rho, l) = (2, 2)$, and so the maximum linear density is $\rho(2) = 2$.

3: (a) A 6 m tall flagpole stands out in the sun one afternoon. At 12:00 noon, the sun is directly overhead. At 3:00 in the afternoon, how long is the pole's shadow, in m, and how fast is the tip of the pole's shadow moving along the ground, in m/hr?

Solution: Let t be the time, in hours, elapsed since noon, let $l = l(t)$ be the length of the shadow in meters, and let $\theta = \theta(t)$ be the angle from the sun to the vertical. Note that since the sun revolves around the Earth (or the Earth spins around its axis) one time every 24 hours, we have $\theta'(t) = \frac{2\pi}{24} = \frac{\pi}{12}$ and so $\theta(t) = \frac{\pi}{12}\,t$. At 3:00 we have $t = 3$ and $\theta = \theta(3) = \frac{\pi}{4}$. Note (see the picture below) that at all times, we have $\tan\theta = l/6$, that is $l = 6\tan\theta$. Differentiate both sides with respect to t to get $l' = 6 \cdot \sec^2\theta \cdot \theta'$. To find $l'(3)$, put in $\theta = \theta(3) = \frac{\pi}{4}$ and $\theta' = \theta'(3) = \frac{\pi}{12}$ to get $l' = 6 \cdot \sec^2\frac{\pi}{4} \cdot \frac{\pi}{12} = 6 \cdot 2 \cdot \frac{\pi}{12} = \pi$. Thus, at 3:00, the tip of the shadow is moving at π m/hr.



(b) A video camera is placed 400 meters away from a rocket launch pad, and is used to film a rocket which flies vertically from the pad. When the rocket is 200 meters high, it is moving at 50 meters per second. Find the rate of change of the distance between the camera and the rocket when the rocket is 200 meters high, and find the rate of change of the angle of elevation of the camera when the rocket is 200 meters high.

Solution: Let $y = y(t)$ be the height of the rocket (in meters) at time t (in seconds). Let $l = l(t)$ be the distance (in meters) between the camera and the rocket. By Pythagoras' Theorem, we have $l^2 = 400^2 + y^2$. Note that when $y = 200$ we have $l = \sqrt{400^2 + 200^2} = 100\sqrt{4^2 + 2^2} = 100\sqrt{20} = 200\sqrt{5}$. Differentiate both sides of the equation $l^2 = 400^2 + y^2$ to get $2l\,l' = 2y\,y'$ so $l' = \dfrac{y\,y'}{l}$. Put in $y = 200$, $l = 200\sqrt{5}$ and $y' = 50$ to get $l' = \frac{200\cdot 50}{200\sqrt{5}} = 10\sqrt{5}$. Thus the distance is changing at $10\sqrt{5}$ meters per second.

Let $\theta = \theta(t)$ be the angle of elevation of the camera (in radians) at time t (in seconds). Then we have $\tan\theta = \dfrac{y}{400}$. Differentiate both sides to get $\sec^2\theta\ \theta' = \dfrac{y'}{400}$ so $\theta' = \dfrac{\cos\theta\ y'}{400}$. Note that when $y = 200$ we have $\tan\theta = \frac{1}{2}$ and so, by considering the triangle with vertices at $(0,0)$, $(2,0)$ and $(2,1)$, we see that $\cos\theta = \frac{2}{\sqrt{5}}$. So when $y = 200$ we have $\theta' = \dfrac{\cos^2\theta\ y'}{400} = \dfrac{\frac{4}{5}\cdot 50}{400} = \frac{1}{10}$. Thus the angle of elevation of the camera is changing at $\frac{1}{10}$ of a radian per second.

4: (a) Approximate $\sqrt[5]{e}$ using the linearization of $f(x) = e^x$ at $x = 0$.

Solution: For $f(x) = e^x$ we have $f'(x) = e^x$ so $f(0) = 1$ and $f'(0) = 1$, and so the linearization of f at $x = 0$ is $L(x) = f(0) + f'(0)(x - 0) = 1 + x$. For $x \cong 0$ we have $f(x) \cong L(x) = 1 + x$. Put in $x = \frac{1}{5}$ to get

$$\sqrt[5]{e} = f\left(\tfrac{1}{5}\right) \cong L\left(\tfrac{1}{5}\right) = 1 + \tfrac{1}{5} = \tfrac{6}{5}\,.$$

(b) Approximate $\sqrt[5]{30}$ using the linearization of $f(x) = \sqrt[5]{x}$ at $x = 32$.

Solution: For $f(x) = \sqrt[5]{x} = x^{1/5}$ we have $f'(x) = \frac{1}{5}\,x^{-4/5}$ so $f(32) = 2$ and $f'(32) = \frac{1}{5} \cdot \frac{1}{16} = \frac{1}{80}$, and so the linearization of f at $x = 32$ is $L(x) = 2 + \frac{1}{80}(x - 32)$. Thus

$$\sqrt[5]{30} = f(30) \cong L(30) = 2 + \tfrac{1}{80}(-2) = 2 - \tfrac{1}{40} = \tfrac{79}{40}\,.$$

(c) Approximate $\sin\left(27^\circ\right) = \sin\left(\frac{3\pi}{20}\right)$ using the linearization of $f(x) = \sin x$ at $x = \frac{\pi}{6}$.

Solution: For $g(x) = \sin x$ we have $g' = \cos x$ so $g\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$ and $g'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, and so the linearization of g at $x = \frac{\pi}{6}$ is $L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$. Thus

$$\sin(27^\circ) = f\left(\tfrac{3\pi}{20}\right) \cong L\left(\tfrac{3\pi}{20}\right) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(\tfrac{3\pi}{20} - \tfrac{\pi}{6}\right) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(-\tfrac{\pi}{60}\right) = \tfrac{1}{2} - \tfrac{\sqrt{3}\,\pi}{120}\,.$$

(d) Use the linearization of $f(x) = \ln x$ at $x = 1$ and the linearization of $g(x) = \ln\frac{1}{x}$ at $x = 1$ to obtain two approximations for the value of $\ln\frac{6}{5}$. Explain why the exact value of $\ln\frac{6}{5}$ must lie between these two approximations.

Solution: For $f(x) = \ln x$, we have $f'(x) = \frac{1}{x}$ so $f(1) = 0$ and $f'(1) = 1$, and so the linearization of $f(x)$ at $x = 1$ is $L(x) = x - 1$. For x near 1 we have $f(x) \cong L(x)$, so in particular

$$\ln\tfrac{6}{5} = f\left(\tfrac{6}{5}\right) \cong L\left(\tfrac{6}{5}\right) = \tfrac{6}{5} - 1 = \tfrac{1}{5}\,.$$

For $g(x) = \ln\frac{1}{x} = -\ln x$ we have $g'(x) = -\frac{1}{x}$ so $g(1) = 0$ and $g'(1) = -1$, and so the linearization of $g(x)$ at $x = 1$ is $L(x) = -(x - 1)$. For x near 1 we have $f(x) \cong L(x)$, and so in particular we have

$$\ln\tfrac{6}{5} = \ln\tfrac{1}{5/6} = g\left(\tfrac{5}{6}\right) \cong L\left(\tfrac{5}{6}\right) = -\left(\tfrac{5}{6} - 1\right) = \tfrac{1}{6}\,.$$

Since the graph of $y = f(x)$ is concave down, its tangent line at $x = 1$ (that is the graph of the linearization) lies above the graph of $f(x)$, and so our approximation $\frac{1}{5}$ lies above the actual value $\ln\frac{6}{5}$. Similarly, since the graph of $y = g(x)$ is concave down, its tangent line lies below the graph, so our approximation $\frac{1}{6}$ lies below the actual value $\ln\frac{6}{5}$. This shows that in fact we have $\frac{1}{6} < \ln\frac{6}{5} < \frac{1}{5}$.

5: (a) Let $f(x)$ be a function whose derivatives $f^{(k)}(0)$ all exist. For each positive integer n, we define the n^{th} *Taylor polynomial* of $f(x)$ at $x=0$ to be the polynomial $T_n(x)$ of degree at most n with ${T_n}^{(k)}(0)=f^{(k)}(0)$ for all $k=0,1,2,\cdots,n$. Show that

$$T_n(x)=f(0)+f'(0)\,x+\frac{f''(0)}{2!}\,x^2+\frac{f'''(0)}{3!}\,x^3+\cdots+\frac{f^{(n)}(0)}{n!}\,x^n\,.$$

Solution: Let $T_n(x)=a_0+a_1x+a_2x^2+\cdots a_nx^n$. We need to show that $a_k=\dfrac{f^{(k)}(0)}{k!}$ for all $k=0,1,2,\cdots,n$ (in the case $k=0$ we use the convention that $0!=1$). We have

$$\begin{aligned}
{T_n}^{(0)}(x)&=T_n(x)=a_0+a_1\,x^1+a_2\,x^2+a_3\,x^3+\cdots+a_n\,x^n\\
{T_n}^{(1)}(x)&={T_n}'(x)=1\,a_1+2\,a_2\,x^1+3\,a_3x^2+4\,a_4\,x^3+\cdots+n\cdot a_n\,x^{n-1}\\
{T_n}^{(2)}(x)&={T_n}''(x)=2\cdot1\,a_2+3\cdot2\,a_3\,x^1+4\cdot3\,a_4\,x^2+5\cdot4\,a_5\,x^3+\cdots+n(n-1)a_n\,x^{n-2}\\
{T_n}^{(3)}(x)&=3\cdot2\cdot1\,a_3+4\cdot3\cdot2\,a_4\,x^1+5\cdot4\cdot3\,a_5\,x^2+6\cdot5\cdot4\,a_6\,x^3+\cdots+n(n-1)(n-2)\,a_n\,x^{n-3}
\end{aligned}$$

and so on, and so

$$\begin{aligned}
{T_n}^{(0)}(0)&=T_n(0)=a_0\\
{T_n}^{(1)}(0)&={T_n}'(0)=1\,a_1\\
{T_n}^{(2)}(0)&={T_n}''(0)=2\cdot1\,a_2\\
{T_n}^{(3)}(0)&={T_n}'''(0)=3\cdot2\cdot1\,a_3
\end{aligned}$$

and so on. We see that in general, for $k\geq0$ we have

$${T_n}^{(k)}(0)=k!\,a_k\,.$$

For each $k\geq0$, we have ${T_n}^{(k)}(0)=f^{(k)}(0)$ when $k!\,a_k=f^{(k)}(0)$, that is when $a_k=\dfrac{f^{(k)}(0)}{k!}$.

(b) Approximate the value of e by using the 5^{th} Taylor polynomial of $f(x)=e^x$ at $x=0$.

Solution: For $f(x)=e^x$ we have $f'(x)=e^x$, $f''(x)=e^x$ and so on, and so $f(0)=e^0=1$, $f'(0)=e^0=1$, $f''(0)=e^0=1$, and in general $f^{(k)}(0)=e^0=1$ for all $k=0,1,2,\cdots,n$. Thus the 5^{th} Taylor polynomial is

$$\begin{aligned}
T_5(x)&=f(0)+f'(0)\,x+\frac{f''(0)}{2!}\,x^2+\frac{f'''(x)}{3!}\,x^3+\frac{f^{(4)}(0)}{4!}\,x^4+\frac{f^{(5)}(0)}{5!}\,x^5\\
&=1+x+\tfrac{1}{2!}\,x^2+\tfrac{1}{3!}\,x^3+\tfrac{1}{4!}\,x^4+\tfrac{1}{5!}\,x^5\,.
\end{aligned}$$

For $x\cong0$ we have $f(x)\cong T_5(x)$. Put in $x=1$ to get

$$e=f(1)\cong T_5(1)=1+1+\tfrac{1}{2!}+\tfrac{1}{3!}+\tfrac{1}{4!}+\tfrac{1}{5!}=1+1+\tfrac12+\tfrac16+\tfrac1{24}+\tfrac1{120}=\tfrac{120+120+60+20+5+1}{120}=\tfrac{163}{60}\,.$$

(c) Approximate the value of $\ln 2$ by using the 6^{th} Taylor polynomial of $f(x)=-\ln(1-x)$ at $x=0$.

Solution: For $f(x)=-\ln(1-x)$, we have $f'(x)=\frac{1}{1-x}$, $f''(x)=\frac{1}{(1-x)^2}$, $f'''(x)=\frac{2}{(1-x)^3}$, $f^{(4)}(x)=\frac{3\cdot2}{(1-x)^4}$, and in general, for $k\geq1$ we have $f^{(k)}(x)=\dfrac{(1-x)!}{(1-k)^k}$. It follows that $f(0)=0$, $f'(0)=1$, $f''(0)=1$, $f'''(0)=2$ and in general, for $k\geq1$ we have $f^{(k)}(0)=(k-1)!$. Thus the 6^{th} Taylor polynomial for $f(x)$ at $x=0$ is

$$\begin{aligned}
T_6(x)&=f(0)+f'(0)\,x+\frac{f''(0)}{2!}\,x^2+\frac{f'''(x)}{3!}\,x^3+\frac{f^{(4)}(0)}{4!}\,x^4+\frac{f^{(5)}(0)}{5!}\,x^5+\frac{f^{(6)}(0)}{6!}\,x^6\\
&=0+\tfrac{0!}{1!}\,x+\tfrac{1!}{2!}\,x^2+\tfrac{2!}{3!}\,x^3+\tfrac{3!}{4!}\,x^4+\tfrac{4!}{5!}\,x^5+\tfrac{5!}{6!}\,x^6\\
&=x+\tfrac12\,x^2+\tfrac13\,x^3+\tfrac14\,x^4+\tfrac15\,x^5+\tfrac16\,x^6\,.
\end{aligned}$$

For $x\cong0$ we have $f(x)\cong T_6(x)$. Put in $x=\frac12$ to get

$$\ln2=-\ln\tfrac12=f\left(\tfrac12\right)\cong T_6\left(\tfrac12\right)=\tfrac12+\tfrac{1}{2\cdot2^2}+\tfrac{1}{3\cdot2^3}+\tfrac{1}{4\cdot2^4}+\tfrac{1}{5\cdot2^5}+\tfrac{1}{6\cdot2^6}=\tfrac{960+240+80+30+12+5}{1920}=\tfrac{1327}{1920}\,.$$