MATH 137 Calculus 1, Assignment 7

Due Fri Nov 9

1: (a) A population $P = P(t)$ grows exponentially with $P(3) = 60$ and $P(6) = 90$. Find the population when $t = 10$, and find the time at which the population is $P = 200$.

(b) Living vegetation contains a certain amount of Carbon-14, written as C^{14}, and when the vegetation dies, the amount of C^{14} begins to decay exponentially with a half-life of 5730 years. If a parchment contains 80% as much C^{14} as it contained when it was first made, then how old is the parchment?

2: (a) The position of an object moving along the x-axis is given by $x(t) = \dfrac{t^4}{e^t}$. Find the velocity $v(t)$ and the acceleration $a(t)$, and find all of the values of t at which $a(t) = 0$.

(b) A rod of length 3 m lies along the x-axis with one end at $x = 0$ and the other at $x = 3$. The mass, in kg, of that part of the rod which lies between $x = 0$ and $x = l$ is given by

$$M(l) = l + \tfrac{1}{2}\,l^2 - \tfrac{1}{12}\,l^3\,.$$

The *average linear density* of the rod (measured in kg/m) is defined to be $\overline{\rho} = \frac{M}{L}$ where M is the total mass of the rod and L is the length of the rod, and the *linear density* of the rod (in kg/m) at the point $x = l$ is defined to be $\rho(l) = M'(l)$. Find the average linear density of the rod, find the linear density of the rod at each point $x = l$, and find the maximum value of the linear density.

3: (a) A 6 m tall flagpole stands out in the sun one afternoon. At 12:00 noon, the sun is directly overhead. At 3:00 in the afternoon, how long is the pole's shadow, in m, and how fast is the tip of the pole's shadow moving along the ground, in m/hr?

(b) A video camera is placed 400 meters away from a rocket launch pad, and is used to film a rocket which flies vertically from the pad. When the rocket is 200 meters high, it is moving at 50 meters per second. Find the rate of change of the distance between the camera and the rocket when the rocket is 200 meters high, and find the rate of change of the angle of elevation of the camera when the rocket is 200 meters high.

4: (a) Approximate $\sqrt[5]{e}$ using the linearization of $f(x) = e^x$ at $x = 0$.

(b) Approximate $\sqrt[5]{30}$ using the linearization of $f(x) = \sqrt[5]{x}$ at $x = 32$.

(c) Approximate $\sin\left(27^\circ\right) = \sin\left(\frac{3\pi}{20}\right)$ using the linearization of $f(x) = \sin x$ at $x = \frac{\pi}{6}$.

(d) Use the linearization of $f(x) = \ln x$ at $x = 1$ and the linearization of $g(x) = \ln\frac{1}{x}$ at $x = 1$ to obtain two approximations for the value of $\ln\frac{6}{5}$. Explain why the exact value of $\ln\frac{6}{5}$ must lie between these two approximations.

5: (a) Let $f(x)$ be a function whose derivatives $f^{(k)}(0)$ all exist. For each positive integer n, we define the n^{th} *Taylor polynomial* of $f(x)$ at $x = 0$ to be the polynomial $T_n(x)$ of degree at most n with ${T_n}^{(k)}(0) = f^{(k)}(0)$ for all $k = 0, 1, 2, \cdots, n$. Show that

$$T_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n\,.$$

(b) Approximate the value of e by using the 5^{th} Taylor polynomial of $f(x) = e^x$ at $x = 0$.

(c) Approximate the value of $\ln 2$ by using the 6^{th} Taylor polynomial of $f(x) = -\ln(1 - x)$ at $x = 0$.

Hints and Comments

1: (b) Find a formula for the amount $C(t)$ of Carbon-14 (measured in any units) at time t (measured in years). Your formula for $C(t)$ will involve the initial amount $C_0 = C(0)$, but you will find that you do not need to determine the value of C_0 to solve this problem.

2: (b) You will find that $\rho(l)$ is a quadratic function of l, so the graph of $\rho = \rho(l)$ is a parabola, and you can find the maximum value of ρ by finding the vertex of the parabola. (You probably also remember, from high school calculus, how to find the maximum value of ρ by finding the derivative $\rho'(l)$, and this method is studied in Chapter 4).

3: (a) Let t denote the time (in hours elapsed since noon) and let $\theta(t)$ be the angle (in radians) from the position of the sun at time t to the vertical. Use the fact that the sun revolves around the Earth (or the Earth spins around its axis) once every 24 hours to determine the rate of change $\theta'(t)$ (in radians per hour).

4: (d) Note that the linear approximation will be too high when the graph of the linearization (the tangent line) lies above the graph of the function.

5: In problem 4, you approximated some values using linear approximations. In this problem, you will make some more accurate approximations using higher order approximations. Notice, from the formula for $T_n(x)$ in part (a), that $T_1(x) = f(0) + f'(0)\,x$, so the 1^{st} Taylor polynomial of $f(x)$ at $x = 0$ is equal to the Linearization of $f(x)$ at $x = 0$. Taylor polynomials wil be studied in more detail in MATH 138.

(a) Let $T_n(x) = a_0 + a_1\,x + a_2\,x^2 + \cdots + a_n\,x^n$. Calculate the first few derivatives ${T_n}'(x)$, ${T_n}''(x)$, ${T_n}'''(x)$ and so on, until you can see the pattern. Then evaluate the first few derivatives at $x = 0$, that is find $T_n(0)$, ${T_n}'(0)$, ${T_n}''(0)$ and so on, until you can see the pattern and discern a general formula for the k^{th} derivative ${T_n}^{(k)}(0)$ in terms of the coefficient a_k (strictly speaking, to be rigorous you should prove that your formula is correct using Mathematical Induction, but you are not required to write up such a proof for this assignment).

(b) Use the formula for $T_n(x)$ from part (a) to calculate $T_5(x)$ for $f(x) = e^x$ $\big($to do this you will need to find the values $f(0), f'(0), f''(0), \cdots, f^{(5)}(0)\big)$. For $x \cong 0$ we can make the approximation $f(x) \cong T_5(x)$. Put in $x = 1$ to get $e = f(1) \cong T_5(1)$.