MATH 137 Calculus 1, Solutions to Assignment 6

1: (a) Let $f(x) = (x^2-3)\sqrt{x-1}$. Find the tangent line to $y = f(x)$ at the point $(2,1)$.

Solution: We have $f'(x) = 2x \cdot \sqrt{x-1} + (x^2-3)\cdot \dfrac{1}{2\sqrt{x-1}}$ and so $f'(2) = 4 + \frac{1}{2} = \frac{9}{2}$. The equation of the tangent line is $y = 1 + \frac{9}{2}(x-2)$, or equivalently $9x - 2y = 16$.

(b) Let $f(x) = \dfrac{\cos(\sqrt{\pi x})}{\sqrt{\sin x}}$. Find $f'(\frac{\pi}{4})$.

Solution: We have $f'(x) = \dfrac{-\sin(\sqrt{\pi x})\cdot \frac{1}{2\sqrt{\pi x}}\cdot \pi \cdot \sqrt{\sin x} - \cos(\sqrt{\pi x})\cdot \frac{1}{2\sqrt{\sin x}}\cdot \cos x}{\sin x}$. Note that when $x = \frac{\pi}{4}$, we have $\sqrt{\pi x} = \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$, and so $f'(\frac{\pi}{4}) = \dfrac{-1\cdot \frac{1}{\pi}\cdot \pi \cdot \frac{1}{\sqrt[4]{2}} - 0}{\frac{1}{\sqrt{2}}} = -\sqrt[4]{2}$

(c) Let $f(x) = \tan^{-1}\sqrt{5x^2-1}$. Find $f'(1)$ and $f''(1)$.

Solution: For $x > 0$ we have

$$f'(x) = \frac{1}{1+(5x^2-1)}\cdot\frac{1}{2\sqrt{5x^2-1}}\cdot 10\,x = \frac{10\,x}{5x^2\cdot 2\sqrt{5x^2-1}} = \frac{1}{x\sqrt{5x^2-1}} = (5x^4-x^2)^{-1/2}$$
$$f''(x) = -\tfrac{1}{2}(5x^4-x^2)^{-3/2}\cdot(20x^3-2x)$$

and so $f'(1) = 4^{-1/2} = \frac{1}{2}$ and $f''(1) = -\frac{1}{2}\,4^{-3/2}\cdot 18 = -\frac{18}{16} = -\frac{9}{8}$.

2: (a) Let $f(x) = \dfrac{x+\sqrt{x}}{\sqrt[3]{x}}$. Find the tangent line to $y = f(x)$ at the point where $x = 1$.

Solution: Note that $f(x) = x^{2/3} + x^{1/6}$ so $f'(x) = \frac{2}{3}\,x^{-1/3} + \frac{1}{6}\,x^{-5/6}$. We have $f(1) = 2$ and $f'(1) = \frac{2}{3} + \frac{1}{6} = \frac{5}{6}$ and so the equation of the tangent line is $y = 2 + \frac{5}{6}(x-1)$, or equivalently $5x - 6y + 7 = 0$.

(b) Let $f(x) = \ln\left(\dfrac{x^2-3}{(x-1)^3}\right)$. Find $f'(2)$ and $f''(2)$.

Solution: Note that $f(x) = \ln(x^2-3) - 3\ln(x-1)$ and so we have

$$f'(x) = \frac{2x}{x^2-3} - \frac{3}{x-1}$$
$$f''(x) = \frac{2(x^2-3)-4x^2}{(x^2-3)^2} + \frac{3}{(x-1)^2}$$

and so $f'(2) = 4 - 3 = 1$ and $f''(2) = (2-16) + 3 = -11$.

(c) $f(x) = x^{x^2}$. Find $f'(1)$ and $f''(1)$.

Solution: Since $f(x) = e^{x^2\ln x}$, we have

$$f'(x) = e^{x^2\ln x}\cdot\left(2x\ln x + x^2\cdot\tfrac{1}{x}\right) = x^{x^2}(2x\ln x + x)$$
$$f''(x) = x^{x^2}(2x\ln x + x)^2 + x^{x^2}(2\ln x + 2x\cdot\tfrac{1}{x} + 1) = x^{x^2}\left((2x\ln x + x)^2 + (2\ln x + 3)\right)$$

and so $f'(1) = 1\cdot(0+1) = 1$ and $f''(1) = 1\cdot((0+1)^2 + 3) = 4$.

3: (a) Find the tangent line to the curve $y^3 + x\,y^2 + 1 = x\,(1 + xy)$ at the point $(3, 2)$.

Solution: Differentiate both sides of the formula $y^3 + x\,y^2 + 1 = x + x^2y$, remembering that y is a function of x, to get $3y^2\,y' + y^2 + 2xy\,y' = 1 + 2xy + x^2\,y'$. Put in $(x, y) = (3, 2)$ to get $12y' + 4 + 12y' = 1 + 12 + 9y'$, that is $15\,y' = 9$ so $y' = \frac{3}{5}$. The equation of the tangent line is $y = 2 + \frac{3}{5}(x - 3)$ or $3x - 5y + 1 = 0$.

(b) Find the tangent line to the curve $y + \ln(x^2y - 1) = 2x$ at the point $(1, 2)$.

Solution: Differentiate both sides to get $y' + \dfrac{2xy + x^2\,y'}{x^2y - 1} = 2$. Put in $(x, y) = (1, 2)$ to get $y' + \dfrac{4 + y'}{2 - 1} = 2$, that is $y' + 4 + y' = 2$, so $2y' = -2$ and hence $y' = -1$. The equation of the tangent line is $y - 2 = (-1)(x - 1)$, or equivalently $y = -x + 3$.

(c) Let $f(x) = x^3 - 3x^2 + 6x - 2$ and let $g = f^{-1}$. Find $g'(6)$ and $g''(6)$.

Solution: As mentioned in the hints, $y = g(x) \iff x = f(y) \iff x = y^3 - 3y^2 + 6y - 2$ and so the curve $y = g(x)$ is given implicitly by the equation

$$x = y^3 - 3y^2 + 6y - 2\,.$$

Note that $f(2) = 6$ so we have $g(6) = 2$, so we are interested in the point $(x, y) = (6, 2)$ on this curve. We differentiate both sides of the above equation (remembering that y is a function of x), then we solve for y' in terms of x and y, and then we differentiate again to get

$$\begin{aligned} 1 &= 3y^2\,y' - 6y\,y' + 6y' = 3y'(y^2 - 2y + 2)\,, \\ y' &= \frac{1}{3(y^2 - 2y + 2)}\,, \\ y'' &= \frac{-(2y\,y' - 2y')}{3(y^2 - 2y + 2)^2} = \frac{-2(y - 1)\,y'}{3(y^2 - 2y + 2)^2}\,. \end{aligned}$$

Put $x = 6$ and $y = 2$ into the formula for y' to get $g'(6) = y' = \frac{1}{3\cdot 2} = \frac{1}{6}$, and then put $x = 6$, $y = 2$ and $y' = \frac{1}{6}$ into the formula for y'' to get $g''(6) = \dfrac{-2 \cdot 1 \cdot \frac{1}{6}}{3 \cdot 2^2} = -\frac{1}{36}\,.$

4: (a) Find constants $a>0$ and $r>1$ so that f is differentiable at $x=1$, where

$$f(x)=\begin{cases}\sqrt{r^2-x^2}\ , \text{ if } -r<x\le 1\,,\\ \dfrac{a}{x^2+1}\ , \text{ if } 1<x\,.\end{cases}$$

Solution: Let $g(x)=\sqrt{r^2-x^2}$ and $h(x)=\dfrac{a}{x^2+1}$. Note that $g'(x)=\dfrac{-x}{\sqrt{r^2-x^2}}$ and $h'(x)=\dfrac{-2ax}{(x^2+1)^2}$. In order for $f(x)$ to be differentiable at $x=1$, it must be continuous at $x=1$, so we must have

$$\lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)\,.$$

Since $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} g(x)=g(1)$, $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} h(x)=h(1)$, and $f(1)=g(1)$, we need $g(1)=h(1)$, that is

$$\sqrt{r^2-1}=\frac{a}{2}\qquad (1)\,.$$

In order for f to be differentiable at $x=1$ we need

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}\,.$$

In the case that $f(1)=g(1)=h(1)$ we have $\lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^-}\dfrac{g(x)-g(1)}{x-1}=g'(1)$ and we have $\lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^+}\dfrac{h(x)-h(1)}{x-1}=h'(1)$, and so f is differentiable at $x=1$ when we have $g'(1)=h'(1)$, that is when

$$\frac{-1}{\sqrt{r^2-1}}=\frac{-2a}{4}\qquad (2)\,.$$

Solve equations (1) and (2) to get $a=2$ and $r=\sqrt{2}$.

(b) Find all points x at which f is differentiable, where

$$f(x)=\begin{cases} x\ , \text{ if } x\le 0\,,\\ x\sin(1/x)\ , \text{ if } 0<x\le \frac{1}{\pi}\,,\\ \pi x-1\ , \text{ if } \frac{1}{\pi}<x\,.\end{cases}$$

Solution: The only points at which f might not be differentiable are $x=0$ and $\frac{1}{\pi}$. Using the definition of the derivative, we see that f cannot be differentiable at $x=0$ because

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x\,\sin(1/x)}{x}=\lim_{x\to 0^+}\sin(1/x)\,,$$

which does not exist. On the other hand, we can see that f is differentiable at $x=\frac{1}{\pi}$ as follows. if we write $g(x)=x\,\sin(1/x)$ and $h(x)=\pi x-1$ then $g(\frac{1}{\pi})=0$ and $h(\frac{1}{\pi})=0$ so the two curves $y=g(x)$ and $y=h(x)$ both pass through the point $(\frac{1}{\pi},0)$. Further more we have $g'(x)=\sin(1/x)-\frac{1}{x}\cos(1/x)$ so $g'(1/\pi)=\sin\pi-\pi\cos\pi=\pi$ and we have $h'(x)=\pi$ so $h'(1/\pi)=\pi$ so the two curves both have the same slope at $x=\frac{1}{\pi}$. Since $f\left(\frac{1}{\pi}\right)=g\left(\frac{1}{\pi}\right)=h\left(\frac{1}{\pi}\right)$, we have

$$\lim_{x\to\frac{1}{\pi}^-}\frac{f(x)-f\left(\frac{1}{\pi}\right)}{x-\frac{1}{\pi}}=\lim_{x\to\frac{1}{\pi}^-}\frac{g(x)-g\left(\frac{1}{\pi}\right)}{x-\frac{1}{\pi}}=g'\left(\tfrac{1}{\pi}\right)=\pi$$

$$\lim_{x\to\frac{1}{\pi}^+}\frac{f(x)-f\left(\frac{1}{\pi}\right)}{x-\frac{1}{\pi}}=\lim_{x\to\frac{1}{\pi}^+}\frac{h(x)-h\left(\frac{1}{\pi}\right)}{tx\frac{1}{\pi}}=h'\left(\tfrac{1}{\pi}\right)=\pi\,.$$

Since the two one-sided limits are equal, f is differentiable at $x=\frac{1}{\pi}$ with $f'\left(\frac{1}{\pi}\right)=\pi$.

5: Consider the curve $x^2+y^2=(x^2+y^2-2x)^2$. Use MAPLE to help solve the following problems.

(a) Find the values of y' and y'' for the given curve at the point $(0,1)$.

Solution: It is not difficult to calculate the value of y' by hand, but it is quite tedious to calculate y''. We can find these values using the MAPLE commands

```
subs({x=0,y=1},implicitdiff(x^2+y^2=(x^2+y^2-2*x)^2,y,x));
subs({x=0,y=1},implicitdiff(x^2+y^2=(x^2+y^2-2*x)^2,y,x,x));
```

You should find that $y'=2$ and $y''=-9$.

(b) Find the centre (a,b) and the radius r of the **osculating circle** at the point $(0,1)$, that is the circle $(x-a)^2+(y-b)^2=r^2$ which passes through $(0,1)$ and, at this point, and has the same values of y' and y'' as the given curve.

Solution: Differentiate both sides of the equation $(x-a)^2+(y-b)^2=r^2$, solve for y', and then differentiate again to get

$$\begin{aligned} &2(x-a)+2(y-b)\,y'=0\\ &y'=\frac{-(x-a)}{y-b}\\ &y''=\frac{-(y-b)+(x-a)y'}{(y-b)^2}\,. \end{aligned}$$

In order for the point $(0,1)$ to lie on the circle we must have $a^2+(1-b)^2=r^2$ (1). The value of y' at the point $(0,1)$ is $y'=\frac{a}{1-b}$, so we have $y'=2$ when $a=2(1-b)$ (2). When $y'=2$, the value of y'' at the point $(0,1)$ is $y''=\frac{-(1-b)-2a}{(1-b)^2}$, so we have $y''=-9$ when $(1-b+2a)=9(1-b)^2$ (3). It is not difficult to solve equations (1), (2) and (3) by hand. Alternatively, we can use the MAPLE command

```
solve({a^2+(1-b)^2=r^2,a=2*(1-b),(1-b+2*a)=9*(1-b)^2},{a,b,r});
```

to get $a=\frac{10}{9}$, $b=\frac{4}{9}$ and $r=\frac{5\sqrt{5}}{9}$.

(c) On the same set of axes, in the rectangle $-2\le x\le 4$, $-3\le y\le 3$, plot the given curve in blue, the tangent line at $(0,1)$ in grey, and the osculating circle at $(0,1)$ in green.

Solution: Using the MAPLE commands

```
with(plots):
p[1]:=implicitplot(x^2+y^2=(x^2+y^2-2*x)^2,x=-2..4,y=-3..3,gridrefine=3,color=blue):
p[2]:=plot(1+2*x,x=-2..4,y=-3..3,color=grey):
p[3]:=implicitplot((x-(10/9))^2+(y-(4/9))^2=125/81,x=-2..4,y=-3..3,color=green):
display({p[1],p[2],p[3]});
```

we obtain the following graph.