MATH 137 Calculus 1, Assignment 6 Due Fri Nov 2

1: (a) Let $f(x)=(x^2-3)\sqrt{x-1}$. Find the tangent line to $y=f(x)$ at the point $(2,1)$.

(b) Let $f(x)=\dfrac{\cos(\sqrt{\pi x})}{\sqrt{\sin x}}$. Find $f'(\frac{\pi}{4})$.

(c) Let $f(x)=\tan^{-1}\sqrt{5x^2-1}$. Find $f'(1)$ and $f''(1)$.

2: (a) Let $f(x)=\dfrac{x+\sqrt{x}}{\sqrt[3]{x}}$. Find the tangent line to $y=f(x)$ at the point where $x=1$.

(b) Let $f(x)=\ln\left(\dfrac{x^2-3}{(x-1)^3}\right)$. Find $f'(2)$ and $f''(2)$.

(c) $f(x)=x^{\,x^2}$. Find $f'(1)$ and $f''(1)$.

3: (a) Find the tangent line to the curve $y^3+x\,y^2+1=x\,(1+xy)$ at the point $(3,2)$.

(b) Find the tangent line to the curve $y+\ln(x^2y-1)=2x$ at the point $(1,2)$.

(c) Let $f(x)=x^3-3x^2+6x-2$ and let $g=f^{-1}$. Find $g'(6)$ and $g''(6)$.

4: (a) Find constants $a>0$ and $r>1$ so that f is differentiable at $x=1$, where

$$f(x)=\begin{cases}\sqrt{r^2-x^2}\,, & \text{if } -r<x\le 1\,,\\ \dfrac{a}{x^2+1}\,, & \text{if } 1<x\,.\end{cases}$$

(b) Find all points x at which f is differentiable, where

$$f(x)=\begin{cases}x\,, & \text{if } x\le 0\,,\\ x\sin(1/x)\,, & \text{if } 0<x\le \frac{1}{\pi}\,,\\ \pi x-1\,, & \text{if } \frac{1}{\pi}<x\,.\end{cases}$$

5: Consider the curve $x^2+y^2=(x^2+y^2-2x)^2$. Use MAPLE to help solve the following problems.

(a) Find the values of y' and y'' for the given curve at the point $(0,1)$.

(b) Find the centre (a,b) and the radius r of the **osculating circle** at the point $(0,1)$, that is the circle $(x-a)^2+(y-b)^2=r^2$ which passes through $(0,1)$ and, at this point, and has the same values of y' and y'' as the given curve.

(c) On the same set of axes, in the rectangle $-2\le x\le 4$, $-3\le y\le 3$, plot the given curve in blue, the tangent line at $(0,1)$ in grey, and the osculating circle at $(0,1)$ in green.

Hints and Comments

1: (c) You should simplify $f'(x)$ before you attempt to find $f''(x)$.

2: (a),(b) You can simplify $f(x)$ before finding $f'(x)$.

(c) By definition, $x^{x^2} = e^{x^2 \ln x}$.

3: (a),(b) Use implicit differentiation.

(c) In this problem, it is implicitly assumed that the given function $f(x)$ is 1:1 so that $g = f^{-1}$ exists. Later (in Section 4.3) we shall prove the (intuitively clear) fact that, for any function f, if $f'(x) > 0$ for all x then $f(x)$ is increasing (and hence 1:1). For the given function $f(x) = x^3 - 3x^2 + 6x - 2$ we have $f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 1) = 3((x-1)^2 + 1) \geq 3$ for all x and so $f(x)$ is increasing.

We can also prove directly (without using calculus) that the given function $f(x)$ is increasing as follows. Let $a < b$. Then

$$\begin{aligned} f(b) - f(a) &= (b^3 - 3b^2 + 6b - 2) - (a^3 - 3a^2 + 6a - 2) \\ &= (b^3 - a^3) - 3(b^2 - a^2) + 6(b - a) \\ &= (b-a)(b^2 + ab + a^2) - 3(b-a)(b+a) + 6(b-a) \\ &= (b-a)(a^2 + ab + b^2 - 3a - 3b + 6) \\ &= (b-a)\Big(\big(a + \tfrac{b-3}{2}\big)^2 - \tfrac{(b^2-6b+9)^2}{4} + (b^2 - 3b + 6)\Big) \\ &= (b-a)\Big(\big(a + \tfrac{b-3}{2}\big)^2 + \tfrac{3}{4}(b^2 - 2b + 5)\Big) \\ &= (b-a)\Big(\big(a + \tfrac{b-3}{2}\big)^2 + \tfrac{3}{4}(b-1)^2 + 3\Big) \geq 3(b-a) > 0\,. \end{aligned}$$

If we accept that the given function $f(x)$ is 1:1, so that $g = f^{-1}$ does exist, we can solve problem 3(c) as follows. Note that

$$y = g(x) \iff x = g(y) \iff x = y^3 - 3y^2 + 6y - 2\,.$$

To find $y = g(6)$, solve $6 = y^3 - 3y^2 + 6y - 2$ (by inspection). To find $y' = g'(6)$ and $y'' = g''(6)$, use implicit differentiation on the equation $x = y^3 - 3y^2 + 6y - 2$.

4: (a),(b) In order for f to be differentiable at $x = a$ it must be continuous at $x = a$. Note that f is continuous at $x = a$ when $\lim\limits_{x \to a^-} f(x) = f(a) = \lim\limits_{x \to a^+} f(x)$, and f is differentiable at $x = a$ when $\lim\limits_{x \to a^-} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a^+} \dfrac{f(x) - f(a)}{x - a}$ (both limits exist and are equal).

5: (a) You can use the MAPLE commands `implicitdiff` and `subs`.

(b) You do not need MAPLE for this part.

(c) You can use the MAPLE commands `with(plots)`, `implicitplot` and `display`. To make an implicit plot more accurate, you can use the plot option `gridrefine`.