MATH 137 Calculus 1, Solutions to Assignment 5

1: (a) Let $f(x)=\sqrt{5-x^2}$. Using the definition of the derivative as a limit, find $f'(2)$ and then find the equation of the tangent line to the curve $y=f(x)$ at the point where $x=2$.

Solution: We have

$$\begin{aligned} f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{\sqrt{5-x^2}-1}{x-2} = \lim_{x\to 2}\frac{\sqrt{5-x^2}-1}{x-2}\cdot\frac{\sqrt{5-x^2}+1}{\sqrt{5-x^2}+1} \\ &= \lim_{x\to 2}\frac{(5-x^2)-1}{(x-2)(\sqrt{5-x^2}+1)} = \lim_{x\to 2}\frac{-(x-2)(x+2)}{(x-2)(\sqrt{5-x^2}+1)} = \lim_{x\to 2}\frac{-(x+2)}{\sqrt{5-x^2}+1} = -\frac{4}{2} = -2\,. \end{aligned}$$

Alternatively, we have

$$\begin{aligned} f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\sqrt{5-(2+h)^2}-1}{h} = \lim_{h\to 0}\frac{\sqrt{5-(2+h)^2}-1}{h}\cdot\frac{\sqrt{5-(2+h)^2}+1}{\sqrt{5-(2+h)^2}+1} \\ &= \lim_{h\to 0}\frac{\big(5-(2+h)^2\big)-1}{h\sqrt{5-(2+h)^2}+1} = \lim_{h\to 0}\frac{5-(4+4h+h^2)-1}{h\sqrt{5-(4+4h+h^2)}+1} = \lim_{h\to 0}\frac{-4h-h^2}{h\sqrt{1-4h-h^2}+1} \\ &= \lim_{h\to 0}\frac{-4-h}{\sqrt{1-4h-h^2}+1} = -\frac{4}{2} = -2\,. \end{aligned}$$

Also $f(2)=1$ and so the equation of the tangent line is $y-1=-2(x-2)$, or equivalently $y=5-2x$. (Incidentally, this curve is the top half of a circle).

(b) Let $f(x)=x^3+x-1$. Find $f'(1)$ using the definition of the derivative. and then find the equation of the tangent line to $y=f(x)$ at the point where $x=1$.

Solution: We have

$$\begin{aligned} f'(1) &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{(x^3+x-1)-1}{x-1} = \lim_{x\to 1}\frac{x^3+x-2}{x-1} \\ &= \lim_{x\to 1}\frac{(x-1)(x^2+x+2)}{x-1} = \lim_{x\to 1}(x^2+x+2) = 4\,. \end{aligned}$$

Alternatively,

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\big((1+h)^3+(1+h)-1\big)-1}{h} = \lim_{h\to 0}\frac{1+3h+3h^2+h^3+1+h-1}{h} \\ &= \lim_{h\to 0}\frac{4h+3h^2+h^3}{h} = \lim_{h\to 0}(4+3h+h^2) = 4\,. \end{aligned}$$

Since $f(1)=1$ and $f'(1)=4$, the equation of the tangent line is $y-1=4(x-1)$, or $y=4x-3$.

2: (a) Let $f(x)=\dfrac{1}{x}$. Find the derivative $f'(x)$ using the definition of the derivative.

Solution: We have

$$f'(x)=\lim_{u\to x}\frac{f(u)-f(x)}{u-x}=\lim_{u\to x}\frac{\frac{1}{u}-\frac{1}{x}}{u-x}=\lim_{u\to x}\frac{\frac{x-u}{ux}}{u-x}=\lim_{u\to x}\frac{-(u-x)}{u\,x\,(u-x)}=\lim_{u\to x}\frac{-1}{u\,x}=\frac{-1}{x^2}\,.$$

Alternatively,

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\frac{x-(x+h)}{(x+h)(x)}}{h}=\lim_{h\to 0}\frac{-h}{(x+h)(x)(h)}\\&=\lim_{h\to 0}\frac{-1}{(x+h)(x)}=\frac{-1}{x^2}\,.\end{aligned}$$

(b) Let $f(x)=x^{1/3}$. Use the definition of the derivative to show that $f'(0)$ does not exists and to show that $f'(x)=\frac{1}{3}x^{-2/3}$ for $x\neq 0$.

Solution: We have

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^{1/3}-0}{x-0}=\lim_{x\to 0}\frac{1}{x^{2/3}}=\infty\,,$$

or alternatively

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^{1/3}-0}{h}=\lim_{h\to 0}\frac{1}{h^{2/3}}=\infty\,,$$

and so $f'(0)$ does not exist (as a real number). For $x\neq 0$ we use the formula $(a-b)(a^2+ab+b^2)=a^3-b^3$ with $a=u^{1/3}$ and $b=x^{1/3}$ to get

$$\begin{aligned}f'(x)&=\lim_{u\to x}\frac{f(u)-f(x)}{u-x}=\lim_{u\to x}\frac{u^{1/3}-x^{1/3}}{u-x}=\lim_{u\to x}\frac{u^{1/3}-x^{1/3}}{(u^{1/3}-x^{1/3})(u^{2/3}+u^{1/3}x^{1/3}+x^{2/3})}\\&=\lim_{u\to x}\frac{1}{u^{2/3}+u^{1/3}x^{1/3}+x^{2/3}}=\frac{1}{3\,x^{2/3}}\,.\end{aligned}$$

Alternatively, we use the formula $(a-b)(a^2+ab+b^2)=a^3-b^3$ with $a=(x+h)^{1/3}$ and $b=x^{1/3}$ to get

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^{1/3}-x^{1/3}}{h}\\&=\lim_{h\to 0}\frac{(x+h)^{1/3}-x^{1/3}}{h}\cdot\frac{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}\\&=\lim_{h\to 0}\frac{(x+h)-(x)}{h\big((x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}\big)}=\lim_{h\to 0}\frac{h}{h\,\big((x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}\big)}\\&=\lim_{h\to 0}\frac{1}{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}=\frac{1}{3\,x^{2/3}}\,.\end{aligned}$$

3: (a) Let $f(x) = \dfrac{x^2-5}{x-2}$. Find the equation of the tangent line to $y = f(x)$ at $(3,4)$.

Solution: We have

$$f'(x) = \frac{(x^2-5)'(x-2)-(x^2-5)(x-2)'}{(x-2)^2} = \frac{(2x)(x-2)-(x^2-5)(1)}{(x-2)^2} = \frac{x^2-4x+5}{(x-2)^2}\,.$$

When $x = 3$ we have $f(3) = \dfrac{9-5}{3-2} = 4$ and $f'(3) = \dfrac{9-12+5}{(3-2)^2} = 2$ and so the equation of the tangent line is $y - 4 = 2(x-3)$, or $y = 2x-2$.

(b) Let $f(x) = \dfrac{\sqrt{x}}{e^x}$. Find all the values of x where the tangent line to $y = f(x)$ is horizontal.

Solution: We have

$$f'(x) = \frac{(\sqrt{x})'\,e^x - \sqrt{x}\,(e^x)'}{(e^x)^2} = \frac{\frac{1}{2\sqrt{x}}\,e^x - \sqrt{x}\,e^x}{(e^x)^2} = \frac{\frac{1}{2\sqrt{x}} - \sqrt{x}}{e^x} = \frac{1-2x}{2\,e^x\sqrt{x}}\,.$$

The tangent line is horizontal when $f'(x) = 0$, that is when $x = \frac{1}{2}$.

(c) Find the equations of the two lines which are tangent to the curve $y = \dfrac{x^2}{x-1}$ and which pass through the point $(2,0)$.

Solution: Let $f(x) = \dfrac{x^2}{x-1}$. Then

$$f'(x) = \frac{2x(x-1)-x^2}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}\,.$$

Thus $f(a) = \dfrac{a^2}{a-1}$ and $f'(a) = \dfrac{a(a-2)}{(a-1)^2}$, and so the equation of the tangent line to $y = f(x)$ at $x = a$ is

$$y - \frac{a^2}{a-1} = \frac{a(a-2)}{(a-1)^2}\,(x-a)\,.$$

This line will pass through the point $(2,0)$ provided that $0 - \frac{a^2}{a-1} = \frac{a(a-2)}{(a-1)^2}(2-a)$. Multiply both sides by $(a-1)^2$ to get $-a^2(a-1) = a(a-2)(2-a)$, that is $-a^3+a^2 = -a^3+4a^2-4a$ or $3a^2-4a = 0$. Thus $a = 0$ or $a = \frac{4}{3}$. We have $f(0) = 0$ and $f'(0) = 0$, and so when $a = 0$, the equation of the tangent line is $y = 0$. We have $f(\frac{4}{3}) = \frac{16}{3}$ and $f'(\frac{4}{3}) = -8$ so when $a = \frac{4}{3}$ the equation of the tangent line is $y - \frac{16}{3} = -8(x-\frac{4}{3})$ or equivalently, $y = 16-8x$. (Incidentally, this curve is a hyperbola with asymptotes $x = 1$ and $y = x+1$).

4: (a) Suppose that $f\left(\frac{1}{4}\right)=8$ and that $f'(x)=\dfrac{1+x\,f(x)}{\sqrt{x}}$. Find $f''\left(\frac{1}{4}\right)$.

Solution: Since $f'(x)=\dfrac{1+x\,f(x)}{\sqrt{x}}=x^{-1/2}+x^{1/2}\,f(x)$, the Product Rule gives

$$\begin{aligned} f''(x) &= -\tfrac{1}{2}\,x^{-3/2}+\tfrac{1}{2}\,x^{-1/2}\,f(x)+x^{1/2}\,f'(x) \\ &= -\tfrac{1}{2}\,x^{-3/2}+\tfrac{1}{2}\,x^{-1/2}\,f(x)+x^{1/2}\big(x^{-1/2}+x^{1/2}f(x)\big) \\ &= -\tfrac{1}{2}\,x^{-3/2}+\tfrac{1}{2}\,x^{-1/2}\,f(x)+1+x\,f(x) \end{aligned}$$

and so

$$f''\left(\tfrac{1}{4}\right)=-\tfrac{1}{2}\left(\tfrac{1}{4}\right)^{-3/2}+\tfrac{1}{2}\left(\tfrac{1}{4}\right)^{-1/2}f\left(\tfrac{1}{4}\right)+1+\tfrac{1}{4}\cdot f\left(\tfrac{1}{4}\right)=-\tfrac{1}{2}\cdot 8+\tfrac{1}{2}\cdot 2\cdot 8+1+\tfrac{1}{4}\cdot 8=7\,.$$

(b) Suppose that $f(1)=4$, $f'(1)=2$ and $f''(1)=6$, and let $g(x)=\dfrac{x\,f(x)}{1+x}$. Find $g''(1)$.

Solution: We have

$$\begin{aligned} g'(x) &= \frac{\big(f(x)+x\,f'(x)\big)(1+x)-x\,f(x)}{(1+x)^2} \\ &= \frac{f(x)(1+x)-x\,f(x)}{(1+x)^2}+\frac{x\,f'(x)(1+x)}{(1+x)^2} \\ &= \frac{f(x)}{(1+x)^2}+\frac{x\,f'(x)}{1+x} \end{aligned}$$

and so

$$g''(x)=\frac{f'(x)(1+x)^2-2\,f(x)(1+x)}{(1+x)^4}+\frac{\big(f'(x)+x\,f''(x)\big)(1+x)-x\,f'(x)}{(1+x)^2}\,.$$

Thus

$$g''(1)=\frac{f'(1)\cdot 4-2\cdot f(1)\cdot 2}{16}+\frac{\big(f'(1)+f''(1)\big)\cdot 2-f'(1)}{4}=\frac{2\cdot 4-2\cdot 4\cdot 2}{16}+\frac{(2+6)\cdot 2-2}{4}=-\tfrac{1}{2}+\tfrac{7}{2}=3\,.$$

(c) Let $f(x)=x^2e^x$. Find the n^{th} derivative $f^{(n)}(x)$ in terms of n and x.

Solution: We have

$$\begin{aligned} f'(x) &= 2x\,e^x+x^2e^x=(x^2+2x)e^x \\ f''(x) &= (2x+2)e^x+(x^2+2x)e^x=(x^2+4x+2)e^x \\ f'''(x) &= (2x+4)e^x+(x^2+4x+2)e^x=(x^2+6x+6)e^x \\ f''''(x) &= (2x+6)e^x+(x^2+6x+6)e^x=(x^2+8x+12)e^x\,. \end{aligned}$$

It appears that the n^{th} derivative is given by the formula

$$f^{(n)}(x)=\big(x^2+2n\,x+n(n-1)\big)e^x$$

for all $n\geq 1$. We prove this by induction. Fix $k\geq 1$ and suppose that the above formula holds when $n=k$, that is suppose $f^{(k)}(x)=\big(x^2+2k\,x+k(k-1)\big)e^x$. Then when $n=k+1$ we have

$$\begin{aligned} f^{(n)}(x)=f^{(k+1)}(x)=\frac{d}{dx}\,f^{(k)}(x) &= \frac{d}{dx}\Big((x^2+2k\,x+k^2-k)e^x\Big) \\ &= (x^2+2k)e^x+(x^2+2k\,x+k^2-k)e^x \\ &= (x^2+(2k+2)x+k^2+k)e^x \\ &= \big(x^2+2(k+1)x+(k+1)k\big)e^x \\ &= \big(x^2+2n\,x+n(n-1)\big)e^x\,. \end{aligned}$$

Thus by induction, the formula holds for all $n\geq 1$.