MATH 137 Calculus 1, Solutions to Assignment 4

1: Evaluate the following limits, if they exist.

(a) $\lim\limits_{x\to-\infty} \dfrac{(2x-1)^3}{(4x+2)(x-1)^2}$

Solution: Divide the numerator and denominator by x^3 to get

$$\lim_{x\to-\infty} \frac{(2x-1)^3}{(4x+2)(x-1)^2} = \lim_{x\to-\infty} \frac{\left(2-\frac{1}{x}\right)^3}{\left(4+\frac{2}{x}\right)\left(1-\frac{1}{x}\right)^2} = \frac{2^3}{(4)(1)^2} = 2\,.$$

(b) $\lim\limits_{x\to\infty} \dfrac{x\left(\sqrt{x}+1\right)}{x^2+3x+2}$

Solution: Divide the numerator and denominator by x^2 to get

$$\lim_{x\to\infty} \frac{x\left(\sqrt{x}+1\right)}{x^2+3x+1} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}+\frac{1}{x}}{1+\frac{3}{x}+\frac{2}{x^2}} = \frac{0}{1} = 0\,.$$

(c) $\lim\limits_{x\to-\infty} \dfrac{3x+2}{\sqrt{4x^2+x+1}}$

Solution: For $x<0$, divide the numerator and the denominator by $-x$ (which is positive), and note that $-x=|x|=\sqrt{x^2}$ (so we divide by x^2 inside the root sign), to get

$$\lim_{x\to-\infty} \frac{3x+2}{\sqrt{4x^2+x+1}} = \lim_{x\to-\infty} \frac{-3-\frac{2}{x}}{\sqrt{4+\frac{1}{x}+\frac{1}{x^2}}} = -\tfrac{3}{2}\,.$$

(d) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+6x}-x\right)$

Solution: Rationalize the numerator then, for $x>0$, divide top and bottom by $x=\sqrt{x^2}$ to get

$$\lim_{x\to\infty} \left(\sqrt{x^2+6x}-x\right) = \lim_{x\to\infty} \frac{(x^2+6x)-(x^2)}{\sqrt{x^2+6x}+x} = \lim_{x\to\infty} \frac{6}{\sqrt{1+\frac{6}{x}}+1} = \tfrac{6}{2} = 3\,.$$

2: Evaluate the following limits if they exist.

(a) $\lim_{x \to \frac{\pi}{2}} e^{\sin x}$

Solution: Since the function $f(x) = e^{\sin x}$ is continuous, $\lim_{x \to \frac{\pi}{2}} e^{\sin x} = e^{\sin(\pi/2)} = e^1 = e$.

(b) $\lim_{x \to \infty} \tan^{-1}(\ln x)$

Solution: Write $u = \ln x$. Since $\lim_{x \to \infty} u = \lim_{x \to \infty} \ln x = \infty$ and $\lim_{u \to \infty} \tan^{-1} u = \frac{\pi}{2}$, we have $\lim_{x \to \infty} \tan^{-1}(\ln x) = \frac{\pi}{2}$.

(c) $\lim_{x \to \infty} \dfrac{\cos x}{x^2 + 1}$

Solution: Note that $-1 \leq \cos x \leq 1$ and so $\frac{-1}{x^2+1} \leq \frac{\cos x}{x^2+1} \leq \frac{1}{x^2+1}$ for all x. Since $\lim_{x \to \infty} \dfrac{-1}{x^2+1} = 0$ and $\lim_{x \to \infty} \dfrac{1}{x^2+1} = 0$, we have $\lim_{x \to \infty} \dfrac{\cos x}{x^2+1} = 0$ by the Squeeze Theorem (which also holds for limits at infinity).

(d) $\lim_{x \to 1^+} \big(2\log(x-1) - \log(x^2-1)\big)$

Solution: For $x > 1$ we have

$$2\log(x-1) - \log(x^2-1) = \log\frac{(x-1)^2}{x^2-1} = \log\frac{(x-1)^2}{(x-1)(x+1)} = \log\frac{x-1}{x+1}\,.$$

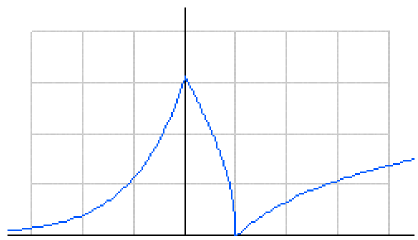
As $x \to 1^+$ we have $(x-1) \to 0^+$ and $(x+1) \to 2$ and so $\frac{x-1}{x+1} \to 0^+$. Thus, writing $u = \frac{x-1}{x+1}$, we have

$$\lim_{x \to 1^+} \big(2\log(x-1) - \log(x^2-1)\big) = \lim_{x \to 1^+} \log\frac{x-1}{x+1} = \lim_{u \to 0^+} \ln u = -\infty\,.$$

3: (a) Sketch the graph of $y = f(x)$ and find all points where f is continuous, where

$$f(x) = \begin{cases} \pi\, e^x & x \le 0 \\ 2\, \cos^{-1} x & 0 < x \le 1 \\ \ln x & 1 < x \end{cases}$$

Solution: Since each of the functions $\pi\, e^x$, $2\, \cos^{-1} x$ and $\ln x$ is continuous in its domain, the function $f(x)$ is continuous everywhere except possibly at $x = 0$ and $x = 1$. Let us determine whether f is continuous at $x = 0$. We have $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \pi\, e^x = \pi\, e^0 = \pi$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2\, \cos^{-1} x = 2\, \cos^{-1}(0) = 2\, \frac{\pi}{2} = \pi$, and so $\lim\limits_{x\to 0} f(x) = \pi = f(0)$. Thus f is continuous at $x = 0$. Now, let us determine whether f is continuous at $x = 1$. We have $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2\, \cos^{-1} x = 2\, \cos^{-1}(1) = 0$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \ln x = \ln 1 = 0$, and so $\lim\limits_{x\to 1} f(x) = 0 = f(1)$. Thus f is also continuous at $x = 1$, so f is continuous everywhere. The graph of $y = f(x)$ is shown below. The y-intercept is at $(0, \pi)$.



(b) Find the values of a and b such that $f(x)$ is continuous for all x, where

$$f(x) = \begin{cases} \dfrac{x^2 + ax + b}{x - 1} & x < 1 \\ ax + b & x \ge 1 \end{cases}$$

Solution: Note that f is continuous for $x < 1$ (since $\frac{x^2+ax+b}{x-1}$ is continuous for $x < 1$) and f is continuous for $x > 1$ (since $ax + b$ is continuous for $x > 1$) so we only need to ensure that f is continuous at $x = 1$. This happens when $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$ both exist and are both equal to $f(1)$, which is equal to $a + b$. Note that $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (ax + b) = a + b$, so it suffices to ensure that $\lim\limits_{x\to 1^-} f(x)$ exists and is equal to $a + b$.

Consider $\lim\limits_{x\to 1^-} f(x)$, that is $\lim\limits_{x\to 1^-} \dfrac{x^2 + ax + b}{x - 1}$. Note that as $x \to 1^-$ we have $(x^2 + ax + b) \to 1 + a + b$ and $(x - 1) \to 0^-$. It follows that if $1 + a + b < 0$ then $\lim\limits_{x\to 1^-} \dfrac{x^2 + ax + b}{x - 1} = \infty$ while if $1 + a + b > 0$ then $\lim\limits_{x\to 1^-} \dfrac{x^2 + ax + b}{x - 1} = -\infty$. Thus in order for $\lim\limits_{x\to 1^-} f(x)$ to exist and be finite, we must have $1 + a + b = 0$. In this case, $x^2 + ax + b$ factors as $x^2 + ax + b = (x - 1)(x + a + 1)$, and so we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2 + ax + b}{x - 1} = \lim_{x\to 1^-} \frac{(x - 1)(x + a + 1)}{x - 1} = \lim_{x\to 1^-} (x + a + 1) = a + 2\,.$$

Thus

$$\begin{aligned} f \text{ is continuous for all } x &\iff \Big(\lim_{x\to 1^-} f(x) \text{ exists and } \lim_{x\to 1^-} f(x) = a + b \Big) \\ &\iff \big(a + b + 1 = 0 \text{ and } a + 2 = a + b \big) \\ &\iff b = 2 \text{ and } a = -3\,. \end{aligned}$$

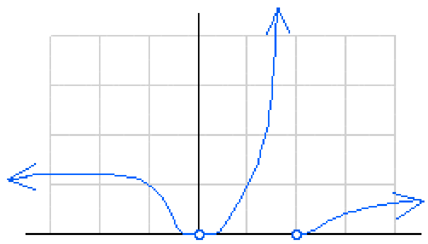
4: Let $f(x) = \dfrac{x^2-1}{2x^2-x^3}$ and let $g(x) = e^{f(x)}$.

(a) Find $\lim\limits_{x\to-\infty} g(x)$, $\lim\limits_{x\to-1} g(x)$, $\lim\limits_{x\to 0} g(x)$, $\lim\limits_{x\to 1} g(x)$, $\lim\limits_{x\to 2^-} g(x)$, $\lim\limits_{x\to 2^+} g(x)$ and $\lim\limits_{x\to\infty} g(x)$.

Solution: Verify that $\lim\limits_{x\to-\infty} f(x) = 0$, $\lim\limits_{x\to-1} f(x) = f(-1) = 0$, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = -\infty$, $\lim\limits_{x\to 1} f(x) = f(1) = 0$, $\lim\limits_{x\to 2^-} f(x) = +\infty$, $\lim\limits_{x\to 2^+} f(x) = -\infty$ and $\lim\limits_{x\to\infty} f(x) = 0$. So we have $\lim\limits_{x\to-\infty} g(x) = e^0 = 1$, $\lim\limits_{x\to-1} g(x) = g(-1) = e^0 = 1$, $\lim\limits_{x\to 0} g(x) = \lim\limits_{u\to-\infty} e^u = 0$, $\lim\limits_{x\to 1} g(x) = g(1) = e^0 = 1$, $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{u\to\infty} e^u = \infty$, $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{u\to-\infty} e^u = 0$, and $\lim\limits_{x\to\infty} g(x) = e^0 = 1$.

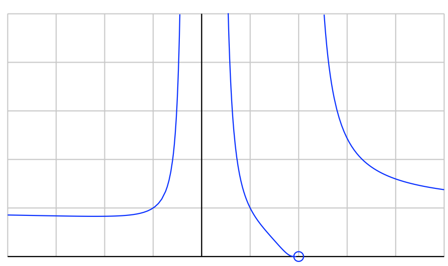
(b) Sketch the graph of $y = g(x)$.

Solution: The above limits give us almost all the information we need to sketch the graph of $y = g(x)$. There is a horizontal asymptote along $y = 1$ and a vertical asymptote along $x = 2$.



(c) Sketch the graph of $y = 1/g(x)$.

Solution: We can obtain the graph of $y = 1/g(x)$ from the graph of $y = g(x)$ $\big($for each point (x_0, y_0) on the graph of $y = g(x)$, the point $(x_0, 1/y_0)$ lies on the graph of $y = 1/g(x)\big)$. We can also calculate various limits, for example $\lim\limits_{x\to-\infty} \dfrac{1}{g(x)} = 1$, $\lim\limits_{x\to 0} \dfrac{1}{g(x)} = \infty$, $\lim\limits_{x\to 2^-} \dfrac{1}{g(x)} = 0$, $\lim\limits_{x\to 2^+} \dfrac{1}{g(x)} = \infty$ and $\lim\limits_{x\to\infty} \dfrac{1}{g(x)} = 1$.



5: (a) Show that there exist (at least) 3 distinct values of x such that $8x^3 = 6x + 1$.

Solution: Let $f(x) = 8x^3 - 6x - 1$. Notice that $f(x)$ is continuous and we have $f(x) = 0 \iff 8x^3 = 6x + 1$. By the Intermediate Value Theorem, since $f(-1) = -3 < 0$ and $f\big(-\frac{1}{2}\big) = 1 > 0$, there is a number $x_1 \in \big(-1, -\frac{1}{2}\big)$ such that $f(x_1) = 0$. Similarly, since $f\big(-\frac{1}{2}\big) = 1 > 0$ and $f(0) = -1 < 0$, there is a number $x_2 \in \big(-\frac{1}{2}, 0\big)$ with $f(x_2) = 0$, and since $f(0) = -1 < 0$ and $f(1) = 1 > 0$, there is a number $x_3 \in (0, 1)$ with $f(x_3) = 0$. $\big($In fact, the exact values of x_1, x_2 and x_3 are $x_1 = -\cos(40°)$, $x_2 = -\sin(10°)$ and $x_3 = \cos(20°)\big)$.

(b) Let $f(x)$ be continuous on $[0, 2]$ with $f(0) = f(2)$. Show that $f(x) = f(x + 1)$ for some $x \in [0, 1]$.

Solution: Let $g(x) = f(x + 1) - f(x)$. Note that g is continuous and $g(1) = f(2) - f(1) = f(0) - f(1) = -\big(f(1) - f(0)\big) = -g(0)$. By the Intermediate Value Theorem, there is a number $x \in [0, 1]$ with $g(x) = 0$ $\big($indeed if $g(0) \neq 0$ then one of the numbers $g(0)$ and $g(1)$ is positive and the other is negative so there is a number $x \in (0, 1)$ with $g(x) = 0\big)$. Then we have $0 = g(x) = f(x + 1) - f(x)$ and so $f(x) = f(x + 1)$.

(c) Let $f(x)$ be $1:1$ and continuous on the interval $[a, b]$ with $f(a) < f(b)$. Show that the range of f is the interval $\big[f(a), f(b)\big]$.

Solution: First we show that $\big[f(a), f(b)\big] \subseteq \text{Range}(f)$. If $y = f(a)$ then clearly $y \in \text{Range}(f)$, if $y = f(b)$ then clearly $y \in \text{Range}(f)$, and if $y \in \big(f(a), f(b)\big)$ then, by the Intermediate Value Theorem, there is a number $x \in (a, b)$ such that $y = f(x)$ and so again we have $y \in \text{Range}(f)$. This proves that $\big[f(a), f(b)\big] \subseteq \text{Range}(f)$.

Next we show that $\text{Range}(f) \subseteq \big[f(a), f(b)\big]$. Let $y \in \text{Range}(f)$, say $y = f(x)$ where $x \in [a, b]$. We must show that $y \in \big[f(a), f(b)\big]$. Suppose, for a contradiction, that $y \notin \big[f(a), f(b)\big]$. Then either $y < f(a)$ or $y > f(b)$. In the case that $y < f(a)$ we would have $y = f(x) < f(a) < f(b)$ but then, by the Intermediate Value Theorem, there would be a number $c \in (x, b)$ with $f(c) = f(a)$, but this would contradict the fact that f is $1:1$. Similarly, in the case that $y > f(b)$ we would have $f(a) < f(b) < f(x) = y$ but then, by the Intermediate Value Theorem, there would be a number $c \in (a, x)$ with $f(c) = f(b)$, but this would contradict the fact that f is $1:1$.