MATH 137 Calculus 1, Solutions to Assignment 3

1: Evaluate the following limits, if they exist.

(a) $\lim_{x\to 2}\sqrt{\dfrac{x^2+5x-14}{x^2-4}}$

Solution: We have

$$\lim_{x\to 2}\frac{x^2+5x-14}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x+7)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x+7}{x+2}=\tfrac{9}{4}\,,$$

and so

$$\lim_{x\to 2}\sqrt{\frac{x^2+5x-14}{x^2-4}}=\sqrt{\tfrac{9}{4}}=\tfrac{3}{2}\,.$$

(b) $\lim_{x\to 1}\dfrac{\sqrt{x+3}-2}{x-1}$

Solution: We have

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}&=\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}=\lim_{x\to 1}\frac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)}\\&=\lim_{x\to 1}\frac{(x-1)}{(x-1)(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{1}{\sqrt{x+3}+2}=\tfrac{1}{4}\,.\end{aligned}$$

(c) $\lim_{x\to 4}\dfrac{x-\sqrt{x}-2}{x-4}$

Solution: We have

$$\begin{aligned}\lim_{x\to 4}\frac{x-\sqrt{x}-2}{x-4}&=\lim_{x\to 4}\frac{(x-2)-\sqrt{x}}{x-4}\cdot\frac{(x-2)+\sqrt{x}}{(x-2)+\sqrt{x}}=\lim_{x\to 4}\frac{(x-2)^2-x}{(x-4)(x-2+\sqrt{x})}\\&=\lim_{x\to 4}\frac{x^2-5x+4}{(x-4)(x-2+\sqrt{x})}=\lim_{x\to 4}\frac{(x-4)(x-1)}{(x-4)(x-2+\sqrt{x})}=\lim_{x\to 4}\frac{x-1}{x-2+\sqrt{x}}=\tfrac{3}{4}\end{aligned}$$

2: Evaluate the following limits, if they exist.

(a) $\lim\limits_{x\to 1^-} \dfrac{x^2-3x+2}{x^3+x^2-5x+3}$

Solution: We have

$$\lim_{x\to 1^-} \frac{x^2-3x+2}{x^3+x^2-5x+3} = \lim_{x\to 1^-} \frac{(x-1)(x-2)}{(x-1)^2(x+3)} = \lim_{x\to 1^-} \frac{(x-2)}{(x-1)(x+3)} = \infty$$

since as $x \to 1^-$ we have $x-2 \to -1$, $x+3 \to 4$ and $x-1 \to 0^-$.

(b) $\lim\limits_{x\to 2^-} \dfrac{x-2}{\sqrt{4-x^2}}$

Solution: We have

$$\lim_{x\to 2^-} \frac{x-2}{\sqrt{4-x^2}} = \lim_{x\to 2^-} \frac{-(2-x)}{\sqrt{(2-x)(2+x)}} = \lim_{x\to 2^-} \frac{-\sqrt{2-x}}{\sqrt{2+x}} = -\tfrac{0}{4} = 0\,.$$

(c) $\lim\limits_{x\to 3^-} \dfrac{|6+x-x^2|}{1-|4-x|}$

Solution: Note that for $-2 < x < 3$ we have $x-3<0$, $x+2>0$ and $x-4<0$ so that $|x-3| = (3-x)$, $|x+2| = (x+2)$ and $|x-4| = (4-x)$. Thus

$$\begin{aligned}\lim_{x\to 3^-} \frac{|6+x-x^2|}{1-|4-x|} &= \lim_{x\to 3^-} \frac{|x^2-x-6|}{1-|x-4|} = \lim_{x\to 3^-} \frac{|x-3|\,|x+2|}{1-|x-4|} = \lim_{x\to 3^-} \frac{(3-x)(x+2)}{1-(4-x)} \\ &= \lim_{x\to 3^-} \frac{(3-x)(x+2)}{(x-3)} = \lim_{x\to 3^-} -(x+2) = -5\,.\end{aligned}$$

3: Evaluate the following limits, if they exist.

(a) $\lim\limits_{x\to 0^+} x\sqrt{1+\frac{1}{x}}$

Solution: For $x>0$ we have $x = \sqrt{x^2}$ so

$$\lim_{x\to 0^+} x\sqrt{1+\tfrac{1}{x}} = \lim_{x\to 0^+} \sqrt{x^2\left(1+\tfrac{1}{x}\right)} = \lim_{x\to 0^+} \sqrt{x^2+x} = 0\,.$$

(b) $\lim\limits_{x\to 0} x^2\left(1+\sin\frac{1}{x}\right)$

Solution: Since $-1 \le \sin\frac{1}{x} \le 1$ we have $0 \le 1+\sin\frac{1}{x} \le 2$ and so $0 \le x^2\left(1+\sin\frac{1}{x}\right) \le 2x^2$. Thus, by the Squeeze Theorem, we have $\lim\limits_{x\to 0} x^2\left(1+\sin\frac{1}{x}\right) = 0$.

(c) $\lim\limits_{x\to 0} (x^2+1)\cos\frac{1}{x}$

Solution: We claim that the limit does not exist. When $x = \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \cdots$ we have $\cos\frac{1}{x} = 0$ and hence $(x^2+1)\cos\frac{1}{x} = 0$. It follows that if the limit existed then it would have to be 0. On the other hand, when $x = \frac{1}{2\pi}, \frac{1}{4\pi}, \frac{1}{6\pi}, \frac{1}{8\pi}, \cdots$ we have $\cos\frac{1}{x} = 1$ and hence $(x^2+1)\cos\frac{1}{x} = x^2+1 \ge 1$. It follows that if the limit existed then it would have to be ≥ 1.

4: (a) Use the definition of the limit to show that $\lim\limits_{x\to -2}(3x+2) = -4$.

Solution: Note that $\Big|(3x+2)-(-4)\Big| = |3x+6| = 3\,|x+2|$. Given $\epsilon > 0$ we choose $\delta = \frac{\epsilon}{3}$ and then

$$0 < |x+2| < \delta \Longrightarrow |x+2| < \tfrac{\epsilon}{3} \Longrightarrow \Big|(3x+2)-(-4)\Big| = 3\,|x+2| < 3\cdot\tfrac{\epsilon}{3} = \epsilon\,.$$

(b) Use the definition of the limit to show that $\lim\limits_{x\to -1}\dfrac{x+1}{x^2-1} = -\frac{1}{2}$.

Solution: Note that for $x \neq \pm 1$ we have

$$\left|\frac{x+1}{x^2-1}+\frac{1}{2}\right| = \left|\frac{x+1}{(x-1)(x+1)}+\frac{1}{2}\right| = \left|\frac{1}{x-1}+\frac{1}{2}\right| = \left|\frac{2+(x-1)}{2(x-1)}\right| = \frac{|x+1|}{2|x-1|}\,.$$

Also note that

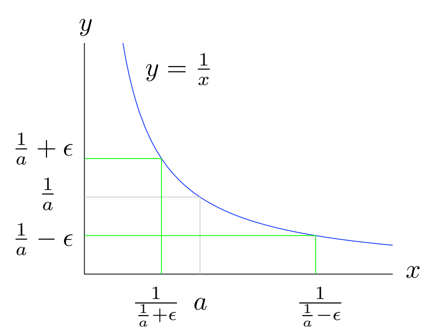
$$|x+1| < 1 \Longrightarrow -2 < x < 0 \Longrightarrow -3 < x-1 < -1 \Longrightarrow 1 < |x-1| < 3 \Longrightarrow \tfrac{1}{3} < \frac{1}{|x-1|} < 1\,.$$

Given $\epsilon > 0$ we choose $\delta = \min\big(1, 2\epsilon\big)$ and then we have

$$\begin{aligned} 0 < |x+1| < \delta &\Longrightarrow \big(x \neq \pm 1 \text{ and } |x+1| < 1 \text{ and } x+1| < 2\epsilon\big) \\ &\Longrightarrow \left(x \neq \pm 1 \text{ and } \frac{1}{|x-1|} < 1 \text{ and } |x+1| < 2\epsilon\right) \\ &\Longrightarrow \left|\frac{x+1}{x^2-1}+\frac{1}{2}\right| = \frac{|x+1|}{2|x-1|} < \frac{2\epsilon}{2\cdot 1} = \epsilon\,. \end{aligned}$$

(c) Let $a > 0$ and let $0 < \epsilon < \frac{1}{a}$. Find the *largest* value of $\delta > 0$ with the property that for all x with $0 < |x-a| < \delta$ we have $\left|\frac{1}{x}-\frac{1}{a}\right| < \epsilon$.

Solution: Careful consideration of the following graph



leads us to guess that the largest such value δ is $\delta = a - \frac{1}{\frac{1}{a}+\epsilon}$. To verify this algebraically, we first show that $a - \frac{1}{\frac{1}{a}+\epsilon} < \frac{1}{\frac{1}{a}-\epsilon} - a$ (as can be seen from the graph). Note that $a - \frac{1}{\frac{1}{a}+\epsilon} = a - \frac{a}{1+a\epsilon} = \frac{a+a^2\epsilon-a}{1+a\epsilon} = \frac{a^2\epsilon}{1+a\epsilon}$ and similarly $\frac{1}{\frac{1}{a}-\epsilon} - a = \frac{a^2\epsilon}{1-a\epsilon}$. Since a and ϵ are positive we have

$$-a\epsilon < a\epsilon \Longrightarrow 1-a\epsilon < 1+a\epsilon \Longrightarrow \tfrac{1}{1+a\epsilon} < \tfrac{1}{1-a\epsilon} \Longrightarrow \tfrac{a^2\epsilon}{1+a\epsilon} < \tfrac{a^2\epsilon}{1-a\epsilon} \Longrightarrow a - \tfrac{1}{\frac{1}{a}+\epsilon} < \tfrac{1}{\frac{1}{a}-\epsilon} - a\,,$$

as claimed. It follows that when $\delta = a - \frac{1}{\frac{1}{a}+\epsilon}$ we have

$$\begin{aligned} |x-a| < \delta \Longrightarrow a-\delta < x < a+\delta \Longrightarrow a - \left(a - \tfrac{1}{\frac{1}{a}+\epsilon}\right) < x < a + \left(\tfrac{1}{\frac{1}{a}-\epsilon} - a\right) \\ \Longrightarrow \tfrac{1}{\frac{1}{a}+\epsilon} < x < \tfrac{1}{\frac{1}{a}-\epsilon} \Longrightarrow \tfrac{1}{a} - \epsilon < \tfrac{1}{x} < \tfrac{1}{a} + \epsilon \Longrightarrow \left|\tfrac{1}{x} - \tfrac{1}{a}\right| < \epsilon\,. \end{aligned}$$

On the other hand, when $\delta > a - \frac{1}{\frac{1}{a}+\epsilon}$ we can choose x with $a - \delta < x < \frac{1}{\frac{1}{a}+\epsilon} = \frac{a}{1+a\epsilon}$ but then we have $a-\delta < x < a$ so that $0 < |x-a| < \delta$, and we also have $\frac{1}{x} > \frac{1}{a} + \epsilon$ so that $\left|\frac{1}{x} - \frac{1}{a}\right| > \epsilon$.

5: (a) Use the definition of the limit to show that for all $a > 0$ we have $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.

Solution: Note that for $x > 0$ we have

$$\left|\sqrt{x} - \sqrt{a}\right| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}}\,.$$

Given $\epsilon > 0$ we can choose $\delta = \min(a, \sqrt{a}\,\epsilon)$ and then

$$\begin{aligned} 0 < |x-a| < \delta &\implies \Big(|x-a| < a \text{ and } |x-a| < \sqrt{a}\,\epsilon\Big) \\ &\implies \Big(x > 0 \text{ and } |x-a| < \sqrt{a}\,\epsilon\Big) \\ &\implies \left|\sqrt{x} - \sqrt{a}\right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\sqrt{a}\,\epsilon}{\sqrt{a}} = \epsilon\,. \end{aligned}$$

(b) Use the definition of the limit to show that if the limit exists then it is unique, that is if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$ then $L = M$.

Solution: Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$. Suppose for a contradiction that $L \neq M$. Note that

$$|L - M| = \Big|\big(L - f(x)\big) + \big(f(x) - M\big)\Big| \leq \big|L - f(x)\big| + \big|f(x) - M\big|$$

by the triangle inequality. Let $\epsilon = \frac{1}{2}|L-M|$ and note that since $L \neq M$ we have $\epsilon > 0$. Choose $\delta_1 > 0$ so that $0 < |x-a| < \delta_1 \implies \big|f(x) - L\big| < \epsilon$ and choose $\delta_2 > 0$ so that $0 < |x-a| < \delta_2 \implies \big|f(x) - M\big| < \epsilon$. Let $\delta = \min(\delta_1, \delta_2)$. Choose x with $0 < |x-a| < \delta$. Then

$$\begin{aligned} 0 < |x-a| < \delta &\implies \Big(0 < |x-a| < \delta_1 \text{ and } 0 < |x-a| < \delta_2\Big) \\ &\implies \Big(\big|f(x) - L\big| < \epsilon \text{ and } \big|f(x) - M\big| < \epsilon\Big) \\ &\implies |L-M| \leq \big|L - f(x)\big| + \big|f(x) - M\big| < 2\epsilon = |L-M|\,. \end{aligned}$$

This gives the desired contradiction (since we cannot have $|L-M| < |L-M|$).