MATH 137 Calculus 1, Solutions to Assignment 2

1: Solve each of the following equalities for x.

(a) $(\sqrt{2})^x = 8$

Solution: We have

$$(\sqrt{2})^x = 8 \iff (2^{1/2})^x = 8 \iff 2^{x/2} = 2^3 \iff x/2 = 3 \iff x = 6\,.$$

(b) $e^{2\ln x} = 9$

Solution: We need $x > 0$ (so that $\ln x$ is defined). For $x > 0$ we have

$$e^{2\ln x} = 9 \iff x^2 = 9 \iff x = \pm 3 \iff x = 3 \text{ (since } x > 0).$$

(c) $\ln(x+9) = \ln(x-1) + \ln(x+3)$

Solution: First note that we need $x+9 > 0$, $x-1 > 0$ and $x+3 > 0$, that is $x > 1$. For $x > 1$ we have

$$\begin{aligned}\ln(x+9) = \ln(x-1) + \ln(x+3) &\iff \ln(x+9) = \ln((x-1)(x+3)) \iff x+9 = (x-1)(x+3)\\ &\iff x+9 = x^2+2x-3 \iff x^2+x-12 = 0 \iff (x+4)(x-3) = 0\\ &\iff \big(x = -4 \text{ or } x = 3\big) \iff x = 3 \text{ (since } x > 1).\end{aligned}$$

(d) $e^x - e^{-x} = 2$

Solution: Write $u = e^x$. Note that $u > 0$. For $u > 0$ we have

$$\begin{aligned}e^x - e^{-x} = 2 &\iff u - \frac{1}{u} = 2 \iff u^2 - 1 = 2u \iff u^2 - 2u - 1 = 0 \iff u = \tfrac{2\pm\sqrt{8}}{2} = 1 \pm \sqrt{2}\\ &\iff u = 1+\sqrt{2} \text{ (since } u > 0) \iff e^x = 1+\sqrt{2} \iff x = \ln(1+\sqrt{2})\,.\end{aligned}$$

2: (a) Find the sine, cosine and tangent of the angle $\alpha = \frac{10\pi}{3}$.

Solution: We have

$$\begin{aligned} \sin\alpha &= -\sin(\alpha - 3\pi) = -\sin\left(\tfrac{\pi}{3}\right) = -\tfrac{\sqrt{3}}{2}\,, \\ \cos\alpha &= -\cos(\alpha - 3\pi) = -\cos\left(\tfrac{\pi}{3}\right) = -\tfrac{1}{2}\,, \text{ and} \\ \tan\alpha &= \frac{\sin\alpha}{\cos\alpha} = \tfrac{-\sqrt{3}/2}{-1/2} = \sqrt{3}\,. \end{aligned}$$

(b) Find the angle $\beta \in [\pi, 2\pi]$ with $\tan\beta = -\sqrt{3}$.

Solution: $\beta = \frac{5\pi}{3}$ since $\frac{5\pi}{3} \in [\pi, 2\pi]$ and $\tan\frac{5\pi}{3} = \tan\left(\frac{5\pi}{3} - 2\pi\right) = \tan\left(-\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$.

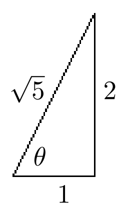
(c) Find the exact value of $\tan\left(\frac{\pi}{12}\right)$.

Solution: We have

$$\begin{aligned} \tan\tfrac{\pi}{12} &= \tan\left(\tfrac{\pi}{4} - \tfrac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)} = \frac{\sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}}{\cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}}{\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}} \\ &= \tfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \tfrac{(\sqrt{6}-\sqrt{2})^2}{6-2} = \tfrac{6-2\sqrt{12}+2}{4} = \tfrac{8-4\sqrt{3}}{4} = 2 - \sqrt{3}\,. \end{aligned}$$

(d) Find the exact value of $\sin\left(\tan^{-1} 2\right)$.

Solution: For θ as in the picture, we have $\theta = \tan^{-1} 2$ and $\sin\theta = \frac{2}{\sqrt{5}}$, so $\sin\left(\tan^{-1} 2\right) = \frac{2}{\sqrt{5}}$.



(e) Find the exact value of $\cos^{-1}\left(\sin\left(\frac{12\pi}{5}\right)\right)$.

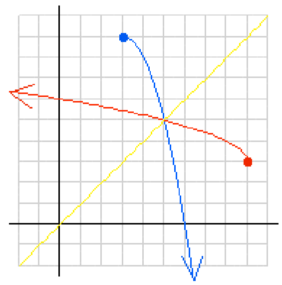
Solution: Note that $\sin\frac{12\pi}{5} = \sin\left(\frac{12\pi}{5} - 2\pi\right) = \sin\frac{2\pi}{5} = \sin\left(\frac{\pi}{2} - \frac{\pi}{10}\right) = \cos\frac{\pi}{10}$. It follows that

$$\cos^{-1}\left(\sin\left(\tfrac{12\pi}{5}\right)\right) = \cos^{-1}\left(\cos\tfrac{\pi}{10}\right) = \tfrac{\pi}{10}\,.$$

3: For each of the following functions $f(x)$, sketch the graphs $y = f(x)$ and $y = f^{-1}(x)$ on the same grid, find a formula for f^{-1}, and find the domain and range of f^{-1}.

(a) $f(x) = x(6-x)$ for $x \geq 3$.

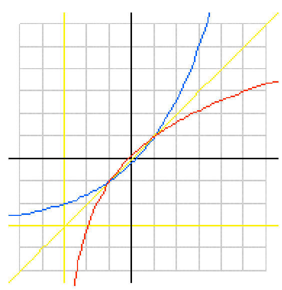
Solution: The graph of f is shown in blue and the graph of f^{-1} is shown in red.



For $x \geq 3$, we have $y = f(x) \iff y = x(6-x) = 6x - x^2 \iff x^2 - 6x + y = 0 \iff x = \frac{6 \pm \sqrt{36-4y}}{2} \iff x = 3 \pm \sqrt{9-y} \iff x = 3 + \sqrt{9-y}$, since $x \geq 3$. Thus we have $x = f^{-1}(y) = 3 + \sqrt{9-y}$. The domain of f^{-1} is $(-\infty, 9]$ and the range of f^{-1} is $[3, \infty)$.

(b) $f(x) = 2^{(x+3)/2} - 3$ for all x.

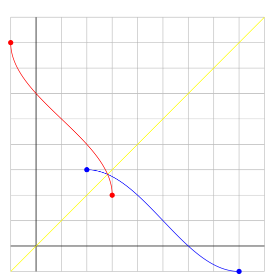
Solution: The graph of f is shown in blue and the graph of f^{-1} is shown in red.



We have $y = f(x) \iff y = 2^{(x+3)/2} - 3 \iff y + 3 = 2^{(x+3)/2} \iff \log_2(y+3) = (x+3)/2 \iff 2\log_2(y+3) = x + 3 \iff x = 2\,\log_2(y+3) - 3$. Thus we have $f^{-1}(y) = 2\,\log_2(y+3) - 3$. The domain of f^{-1} is $(-3, \infty)$ and the range is $(-\infty, \infty)$.

(c) $f(x) = 1 + 2\sin\left(\frac{\pi}{6}(x+1)\right)$ for $2 \leq x \leq 8$.

Solution: The graph of f is shown in blue and the graph of f^{-1} is shown in red.



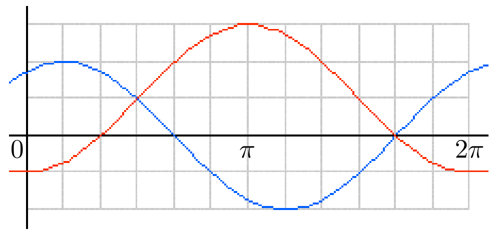
For $2 \leq x \leq 8$ we have $\frac{\pi}{2} \leq \frac{\pi}{6}(x+1) \leq \frac{3\pi}{2}$ hence $-\frac{\pi}{2} \leq \pi - \frac{\pi}{6}(x+1) \leq \frac{\pi}{2}$ and so

$$\begin{aligned} y = f(x) &\iff y = 1 + 2\sin\left(\tfrac{\pi}{6}(x+1)\right) \iff \tfrac{1}{2}(y-1) = \sin\left(\tfrac{\pi}{6}(x+1)\right) = \sin\left(\pi - \tfrac{\pi}{6}(x+1)\right) \\ &\iff \sin^{-1}\left(\tfrac{1}{2}(y-1)\right) = \pi - \tfrac{\pi}{6}(x+1) = \tfrac{\pi}{6}(5-x) \iff x = 5 - \tfrac{6}{\pi}\sin^{-1}\left(\tfrac{1}{2}(y-1)\right). \end{aligned}$$

Thus we have $f^{-1}(y) = 5 - \frac{6}{\pi}\sin^{-1}\left(\frac{1}{2}(y-1)\right)$. The domain of f^{-1} is $[-1, 3]$ and the range is $[2, 8]$.

4: (a) Sketch the graphs of $y = 2\,\sin(x + \frac{\pi}{3})$ and $y = 1 - 2\,\cos x$ on the same grid.

Solution: The graph of $y = 2\,\sin(x + \frac{\pi}{3})$ is shown in blue; it can be obtained from the graph of $y = \sin x$ by translating $\frac{\pi}{3}$ units to the left (to get the graph of $y = \sin\left(x + \frac{\pi}{3}\right)$), then scaling vertically by a factor of 2. The graph of $y = 1 - 2\,\cos x$ is shown in red; it can be obtained from the graph of $y = \cos x$ by reflecting in the x-axis (to get the graph of $y = -\cos x$), scaling vertically by a factor of 2 (to get $y = -2\,\cos x$), then translating 1 unit upwards.



(b) Use the sketch to solve the equality $\sin\left(x + \frac{\pi}{3}\right) + \cos x = \frac{1}{2}$, for $0 \le x \le 2\pi$.

Solution: Note that we have $\sin\left(x + \frac{\pi}{3}\right) + \cos x = \frac{1}{2}$ when $2\,\sin\left(x + \frac{\pi}{3}\right) = 1 - 2\,\cos x$, and this happens when the above two graphs intersect, that is when $x = \frac{\pi}{2}$ and $x = \frac{5\pi}{3}$.

(c) Solve the equality $\sin\left(x + \frac{\pi}{3}\right) + \cos x = \frac{1}{2}$ algebraically.

Solution: For $0 \le x \le 2\pi$, we have

$$\begin{aligned}
\sin\left(x + \tfrac{\pi}{3}\right) + \cos x = \tfrac{1}{2} \iff\ & \sin x \cos \tfrac{\pi}{3} + \cos x \sin \tfrac{\pi}{3} + \cos x = \tfrac{1}{2} \\
\iff\ & \tfrac{1}{2}\,\sin x + \tfrac{\sqrt{3}}{2}\,\cos x + \cos x = \tfrac{1}{2} \\
\iff\ & \sin x + (\sqrt{3} + 2)\cos x = 1 \\
\iff\ & (\sqrt{3} + 2)\cos x = 1 - \sin x \\
\implies\ & (7 + 4\sqrt{3})\cos^2 x = (1 - \sin x)^2 \\
\iff\ & (7 + 4\sqrt{3})(1 - \sin^2 x) = 1 - 2\sin x + \sin^2 x \\
\iff\ & (8 + 4\sqrt{3})\sin^2 x - 2\sin x - (6 + 4\sqrt{3}) = 0 \\
\iff\ & 2(2 + \sqrt{3})\sin^2 x - \sin x - (3 + 2\sqrt{3}) = 0 \\
\iff\ & 2(2 + \sqrt{3})(2 - \sqrt{3})\sin^2 x - (2 - \sqrt{3})\sin x - (3 + 2\sqrt{3})(2 - \sqrt{3}) = 0 \\
\iff\ & 2\sin^2 x - (2 - \sqrt{3})\sin x - \sqrt{3} = 0 \\
\iff\ & \sin x = \tfrac{(2-\sqrt{3})\pm\sqrt{(7-4\sqrt{3})+8\sqrt{3}}}{4} = \tfrac{(2-\sqrt{3})\pm\sqrt{7+4\sqrt{3}}}{4} = \tfrac{(2-\sqrt{3})\pm(2+\sqrt{3})}{4} \\
\iff\ & \sin x = 1 \text{ or } -\tfrac{\sqrt{3}}{2} \\
\iff\ & x = \tfrac{\pi}{2},\ \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3}
\end{aligned}$$

These are the only possible solutions, but they need not all be solutions since at one point we squared both sides (note the line that begins with $\implies$ instead of $\iff$). We check each of these three values of x and find that the only solutions are $x = \frac{\pi}{2}$ and $x = \frac{5\pi}{3}$.

5: Let $f(x) = x^3 + 3x$ for all $x \in \mathbf{R}$, let $g(t) = t - \frac{1}{t}$ for $t > 0$, and let $h(t) = f\big(g(t)\big)$ for $t > 0$.

(a) Show that for every $x \in \mathbf{R}$ there exists a unique $t > 0$ such that $x = g(t)$.

Solution: Let $x \in \mathbf{R}$. For $t > 0$ we have

$$x = g(t) \iff x = t - \tfrac{1}{t} \iff tx = t^2 - 1 \iff t^2 - xt - 1 = 0 \iff t = \frac{x \pm \sqrt{x^2+4}}{2}\,.$$

Note that $\sqrt{x^2+4} > \sqrt{x^2} = |x|$ so that $x + \sqrt{x^2+4} > 0$ and $x - \sqrt{x^2+4} < 0$. Thus the unique number $t > 0$ with $x = g(t)$ is given by $t = g^{-1}(x) = \dfrac{x + \sqrt{x^2+4}}{2}$.

(b) Expand and simplify $h(t)$ then find a formula for h^{-1}.

Solution: We have

$$h(t) = f\big(g(t)\big) = f\big(t - \tfrac{1}{t}\big) = \big(t - \tfrac{1}{t}\big)^3 + 3\big(t - \tfrac{1}{t}\big) = \big(t^3 - 3t + \tfrac{3}{t} - \tfrac{1}{t^3}\big) + \big(3t - \tfrac{3}{t}\big) = t^3 - \tfrac{1}{t^3}\,.$$

For $t > 0$ we have

$$y = h(t) \iff y = t^3 - \tfrac{1}{t^3} \iff t^6 - y\,t^3 - 1 = 0 \iff t^3 = \frac{y \pm \sqrt{y^2+4}}{2} \iff t = \sqrt[3]{\frac{y + \sqrt{y^2+4}}{2}}$$

where we must use the + sign since $t > 0$. Thus $h^{-1}(y) = \sqrt[3]{\dfrac{y + \sqrt{y^2+4}}{2}}$.

(c) Use the results of parts (a) and (b) to find a formula for f^{-1}.

Solution: For $x = g(t)$ with $t > 0$ we have

$$\begin{aligned} y = f(x) &\iff y = f\big(g(t)\big) = h(t) \iff t = h^{-1}(y) = \sqrt[3]{\frac{y + \sqrt{y^2+4}}{2}} \\ &\iff x = g(t) = t - \tfrac{1}{t} = \sqrt[3]{\frac{y + \sqrt{y^2+4}}{2}} - \sqrt[3]{\frac{2}{y + \sqrt{y^2+4}}}\,. \end{aligned}$$

Thus we have $f^{-1}(y) = \sqrt[3]{\dfrac{y + \sqrt{y^2+4}}{2}} - \sqrt[3]{\dfrac{2}{y + \sqrt{y^2+4}}} = \sqrt[3]{\dfrac{y + \sqrt{y^2+4}}{2}} + \sqrt[3]{\dfrac{y - \sqrt{y^2+4}}{2}}$.