MATH 137 Calculus 1, Assignment 2 Due Fri Sept 28

1: Solve each of the following equalities for x.

(a) $(\sqrt{2})^x = 8$

(b) $e^{2\ln x} = 9$

(c) $\ln(x+9) = \ln(x-1) + \ln(x+3)$

(d) $e^x - e^{-x} = 2$

2: (a) Find the sine, cosine and tangent of the angle $\alpha = \frac{10\pi}{3}$.

(b) Find the angle $\beta \in [\pi, 2\pi]$ with $\tan\beta = -\sqrt{3}$.

(c) Find the exact value of $\tan\left(\frac{\pi}{12}\right)$.

(d) Find the exact value of $\sin\left(\tan^{-1} 2\right)$.

(e) Find the exact value of $\cos^{-1}\left(\sin\left(\frac{12\pi}{5}\right)\right)$.

3: For each of the following functions $f(x)$, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same grid, find a formula for f^{-1}, and find the domain and range of f^{-1}.

(a) $f(x) = x(6-x)$ for $x \geq 3$.

(b) $f(x) = 2^{(x+3)/2} - 3$ for all x.

(c) $f(x) = 1 + 2\sin\left(\frac{\pi}{6}(x+1)\right)$ for $2 \leq x \leq 8$.

4: (a) Sketch the graphs of $y = 2\,\sin(x + \frac{\pi}{3})$ and $y = 1 - 2\,\cos x$ on the same grid.

(b) Use the sketch to solve the equality $\sin\left(x + \frac{\pi}{3}\right) + \cos x = \frac{1}{2}$, for $0 \leq x \leq 2\pi$.

(c) Solve the equality $\sin\left(x + \frac{\pi}{3}\right) + \cos x = \frac{1}{2}$ algebraically.

5: Let $f(x) = x^3 + 3x$ for all $x \in \mathbf{R}$, let $g(t) = t - \frac{1}{t}$ for $t > 0$, and let $h(t) = f\big(g(t)\big)$ for $t > 0$.

(a) Show that for every $x \in \mathbf{R}$ there exists a unique $t > 0$ such that $x = g(t)$.

(b) Expand and simplify $h(t)$ then find a formula for h^{-1}.

(c) Use the results of parts (a) and (b) to find a formula for f^{-1}.

Hints and Comments

1: (d) First solve for e^x. You might find it easier to let $u = e^x$ then solve for u.

2: (e) First find an angle $\theta \in [0, \pi]$ with $\cos\theta = \sin\frac{12\pi}{5}$.

3: For each function $f(x)$, you can sketch the graph of $y = f(x)$ either by shifting and scaling a known graph, or simply by making a table of values and plotting points.

(c) It is a bit tricky to find a formula for f^{-1}. Remember that $a = \sin b$ is not equivalent to $b = \sin^{-1} a$, unless it is known that $b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4: (a) You can make the sketch either by shifting and scaling known graphs or simply by making tables of values and plotting points.

(b) In order to solve the equality, your sketch must be sufficiently accurate that you can find the exact coordinates of the points of intersection.

(c) This is difficult. Hopefully some of you will enjoy a good challenge.

5: The method that you are using here in Problem 5 to find a formula for f^{-1} can be used to find the inverse of any cubic polynomial or to solve any cubic equation. Sometimes the method requires the use of complex numbers. Historically, complex numbers first arose in the study of cubic equations. An equation of the form $Ax^3 + Bx^2 + Cx + D = 0$ with $A \neq 0$ can be solved as follows. First, divide by A to obtain an equation of the form $x^3 + bx^2 + cx + d = 0$. Next, make the substitution $u = x + \frac{b}{3}$ and rewrite the equation in the form $u^3 + pu + q = 0$. Then make the substitution $u = t - \frac{p}{3t}$ to convert the equation to the form $t^3 - \frac{p^3}{27\,t^3} + q = 0$. Finally, multiply by t^3 to obtain $t^6 + qt^3 - \frac{p^3}{27}$ and solve for t^3 using the Quadratic Formula.

In the given problem, you asked to find a formula for f^{-1} where $f(x) = x^3 + 3x$. To do this, you need to solve the cubic equation $y = f(x)$, that is $x^3 + 3x - y = 0$. Using the notation of the previous paragraph, we have $b = 0$ so the substitution $u = x + \frac{b}{3}$ is unnecessary (indeed it gives $u = x$), and the suggested substitution $u = t - \frac{p}{3t}$ becomes $x = t - \frac{1}{t} = g(t)$.