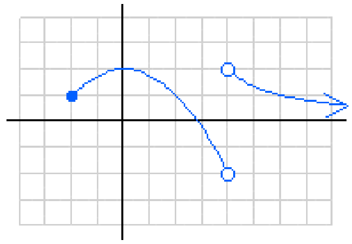
MATH 137 Calculus 1, Solutions to Assignment 1

1: Let f be the function whose graph is shown below.

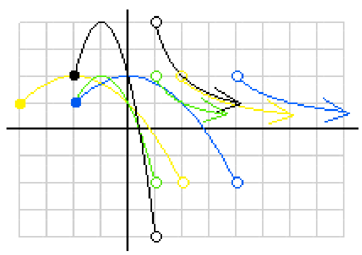


(a) What are the domain and range of f?

Solution: From the graph, the domain is $[-2,4)\cup(4,\infty)$ and the range is $(-2,2]$.

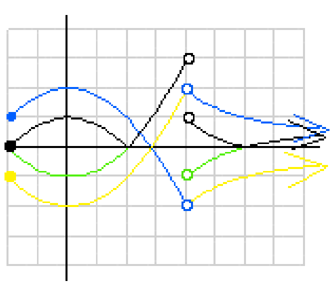
(b) Sketch the graph of $y=2\,f(2x+2)$.

Solution: Translate the graph 2 units to the left to get the graph of $y=f(x+2)$ (shown below in yellow), then scale horizontally by a factor of $\frac{1}{2}$ to get the graph of $y=f(2x+2)$ (shown in green), and then scale vertically by a factor of 2 to get the graph of $y=2\,f(2x+2)$ (shown in black).



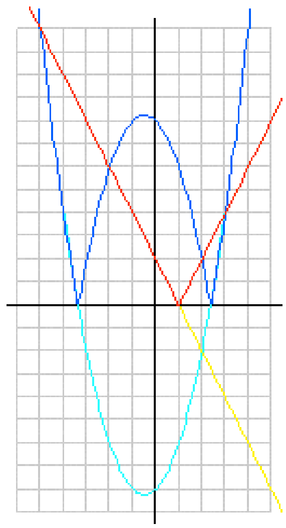
(c) Sketch the graph of $y=\big|1-f(x)\big|$.

Solution: Reflect the graph in the x-axis to get the graph of $y=-f(x)$ (shown below in yellow), then translate vertically by 1 unit to get the graph of $y=1-f(x)$ (shown in green), and then reflect that portion of the green graph which lies below the x-axis in the x-axis to get the graph of $y=|1-f(x)|$ (shown in black).



2: (a) Sketch the graphs of $y = |x^2 + x - 8|$ and $y = |2 - 2x|$ on the same grid.

Solution: The graphs of $y = x^2 + x - 8$, $y = |x^2 + x - 8|$, $y = 2 - 2x$ and $y = |2 - 2x|$ are shown below in cyan, blue, yellow and red respectively.



(b) Use the sketch to determine the solution to the inequality $|x^2 + x - 8| \le |2 - 2x|$.

Solution: The solution set is the set of all values of x for which the blue graph lies below the red graph. From the above sketch, the points of intersection of the blue and red graphs occur when $x = -5$, -2, 2 and 3, and the solution set is $[-5, -2] \cup [2, 3]$. Note that in order to be able to determine the solution set from the sketch, the sketch must clearly show the x-values of all points of intersection.

(c) Solve the inequality $|x^2 + x - 8| \le |2 - 2x|$ algebraically.

Solution: First we note that $2 - 2x \ge 0$ when $x \le 1$ and $2 - 2x \le 0$ when $x \ge 1$, and we note that $x^2 + x - 8 \le 0$ when $\frac{-1-\sqrt{33}}{2} \le x \le \frac{-1+\sqrt{33}}{2}$ and $x^2 + x - 8 \le 0$ when $x \le \frac{-1-\sqrt{33}}{2}$ or $x \ge \frac{-1+\sqrt{33}}{2}$, and so we consider the following four cases: $x \le \frac{-1-\sqrt{33}}{2}$, $\frac{-1-\sqrt{33}}{2} \le x \le 1$, $1 \le x \le \frac{-1+\sqrt{33}}{2}$ and $\frac{-1+\sqrt{33}}{2} \le x$.

When $x \le \frac{-1-\sqrt{33}}{2}$ we have $x^2 + x - 8 \ge 0$ and $2 - 2x \ge 0$ and so

$$\begin{aligned}\left|x^2 + x - 8\right| \le \left|2 - 2x\right| &\iff x^2 + x - 8 \le 2 - 2x \iff x^2 + 3x - 10 \le 0 \iff (x+5)(x-2) \le 0 \\ &\iff -5 \le x \le 2 \iff -5 \le x \le \tfrac{-1-\sqrt{33}}{2}\,,\end{aligned}$$

when $\frac{-1-\sqrt{33}}{2} \le x \le 1$ we have $x^2 + x - 8 \le 0$ and $2 - 2x \ge 0$ and so

$$\begin{aligned}\left|x^2 + x - 8\right| \le \left|2 - 2x\right| &\iff -(x^2 + x - 8) \le 2 - 2x \iff x^2 - x - 6 \ge 0 \iff (x-3)(x+2) \ge 0 \\ &\iff \big(x \le -2 \text{ or } x \ge 3\big) \iff \tfrac{-1-\sqrt{33}}{2} \le x \le -2\,,\end{aligned}$$

when $1 \le x \le \frac{-1+\sqrt{33}}{2}$ we have $x^2 + x - 8 \le 0$ and $2 - 2x \le 0$ and so

$$\begin{aligned}\left|x^2 + x - 8\right| \le \left|2 - 2x\right| &\iff -(x^2 + x - 8) \le -(2 - 2x) \iff x^2 + 3x - 10 \ge 0 \iff (x+5)(x-2) \ge 0 \\ &\iff \big(x \le -5 \text{ or } x \ge 2\big) \iff 2 \le x \le \tfrac{-1+\sqrt{33}}{2}\,,\end{aligned}$$

and when $\frac{-1+\sqrt{33}}{2} \le x$ we have $x^2 + x - 8 \ge 0$ and $2 - 2x \le 0$ and so

$$\begin{aligned}\left|x^2 + x - 8\right| \le \left|2 - 2x\right| &\iff x^2 + x - 8 \le -(2 - 2x) \iff x^2 - x - 6 \le 0 \iff (x-3)(x+2) \le 0 \\ &\iff -2 \le x \le 3 \iff \tfrac{-1+\sqrt{33}}{2} \le x \le 3\,.\end{aligned}$$

Thus the solution set is $\left[-5\,, \frac{-1-\sqrt{33}}{2}\right] \cup \left[\frac{-1-\sqrt{33}}{2}\,, -2\right] \cup \left[2\,, \frac{-1+\sqrt{33}}{2}\right] \cup \left[\frac{-1+\sqrt{33}}{2}\,, 3\right] = [-5, -2] \cup [2, 3]$.

There is an alternate solution which, for this particular inequality, is easier. We have

$$\begin{aligned}\left|x^2+x-8\right| \leq \left|2-2x\right| &\iff (x^2+x-8)^2 \leq (2-2x)^2 \\ &\iff x^4+2x^3-15x^2-16x+64 \leq 4-8x+4x^2 \\ &\iff x^4+2x^3-19x^2-8x+60 \leq 0 \\ &\iff (x+5)(x+2)(x-2)(x-3) \leq 0 \\ &\iff x \in [-5,-2] \cup [2,3]\,.\end{aligned}$$

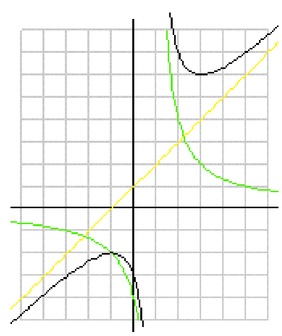
Notice that we can use our result from part (b) to help factor the quartic polynomial.

We remark that if the given inequality was modified slightly, then this second method of solution might give rise to a quartic polynomial that is very difficult to factor. In that case, the first solution method would be easier.

3: Let $f(x) = \dfrac{x^2+3}{x-1}$. Note that $f(x) = (x+1) + \dfrac{4}{x-1}$.

(a) Sketch the graphs of $y = x+1$, $y = \dfrac{4}{x-1}$ and $y = f(x)$ all on the same grid.

Solution: The graph of $y = x+1$ is shown in yellow, the graph of $y = \frac{4}{x-1}$ is shown in green, and the graph of $y = f(x)$, which is obtained by adding the y-values of the yellow and green graphs, is shown in black. $\big(y = f(x)$ is a hyperbola with asymptotes $x = 1$ and $y = x+1\big)$.



(b) Use the sketch to guess what the range of f is.

Solution: From the sketch, it appears that the range is $(-\infty,-2] \cup [6,\infty)$.

(c) Find the range of f algebraically.

Solution: we can find the range of f using algebra by solving the equation $y = f(x)$ for x, using the quadratic formula, as follows:

$$\begin{aligned}y &= \frac{x^2+3}{x-1} \\ xy - y &= x^2+3 \\ x^2 - xy + (y+3) &= 0 \\ x &= \frac{y \pm \sqrt{y^2-4(y+3)}}{2}\end{aligned}$$

A solution for x exists if and only if $y^2-4(y+3) \geq 0$, and we have $y^2-4(y+3) = y^2-4y-12 = (y-6)(y+2)$, and so a solution for x exists when $y \leq -2$ or $y \geq 6$. Thus the range is indeed $(-\infty,-2] \cup [6,\infty)$.

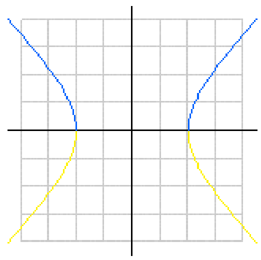
4: Let $f(x)=\sqrt{x-3}$ and $g(x)=x^2-1$.

(a) Find the domain and the range and sketch the graph of $f\circ g$.

Solution: The domain of f is $[3,\infty)$; the domain of g is all of $\mathbf{R}$; the domain of $f\circ g$ is the set of $x\in\mathbf{R}$ such that $g(x)\in[3,\infty)$, that is $x^2-1\geq 3$, or $x^2\geq 4$, and so the domain of $f\circ g$ is $(-\infty,-2]\cup[2,\infty)$. We have

$$(f\circ g)(x)=f(g(x))=f(x^2-1)=\sqrt{(x^2-1)-3}=\sqrt{x^2-4}\,.$$

When $y=\sqrt{x^2-4}$ we have $y^2=x^2-4$ or $x^2-y^2=4$, which is the equation of the hyperbola with vertices at $(\pm 2,0)$ and asymptotes along $y=\pm x$. The graph of $y=(f\circ g)(x)=\sqrt{x^2-4}$ is the top half of this hyperbola; it is shown below, in blue (the bottom half of the hyperbola is in yellow). From the sketch, we can see that the range of $f\circ g$ is $[0,\infty)$.

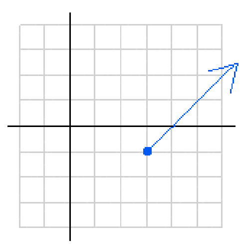


(b) Find the domain and the range and sketch the graph of $g\circ f$.

Solution: The domain of f is $[3,\infty)$; the domain of g is all of $\mathbf{R}$; the domain of $g\circ f$ is the set of all $x\in[3,\infty)$ such that $f(x)\in\mathbf{R}$, so the domain of $g\circ f$ is $[3,\infty)$. We have

$$(g\circ f)(x)=g(f(x))=g(\sqrt{x-3})=\left(\sqrt{x-3}\right)^2-1=(x-3)-1=x-4\,,$$

and the sketch is shown below. From the sketch, the range of $g\circ f$ is $[-1,\infty)$.



(c) Find a function h such that $f(h(x))=x^2$ for all x.

Solution: We have $f(h(x))=\sqrt{h(x)-3}$, and so we need

$$\begin{aligned}\sqrt{h(x)-3}&=x^2\\ h(x)-3&=x^4\\ h(x)&=x^4+3\end{aligned}$$

We take $h(x)=x^4+3$, then $f(h(x))=f(x^4+3)=\sqrt{(x^4+3)-3}=\sqrt{x^4}=x^2$ for all x.

5: Given a polynomial $f(x)$, we can find the tangent line of the graph $y = f(x)$ at the point $\big(a, f(a)\big)$ *without* using calculus as follows. The line through the point $\big(a, f(a)\big)$ with slope m has equation $y = l_m(x)$ where $l_m(x) = f(a) + m(x-a)$. The tangent line will be the line $y = l_m(x)$ when m is chosen so that $(x-a)^2$ is a factor of $f(x) - l_m(x)$.

(a) Find the equation of the tangent line to $y = x^2$ at $(1,1)$ without using calculus.

Solution: We have $l_m(x) = 1 + m(x-1)$. Let

$$g_m(x) = f(x) - l_m(x) = x^2 - 1 - m(x-1) = x^2 - mx + (m-1) = (x-1)\big(x + (1-m)\big)\,.$$

In order for $(x-1)^2$ to be a factor of $g_m(x)$, it must be that $x + (1-m) = x - 1$, and so we must have $1 - m = -1$, that is $m = 2$. Thus the tangent line has equation $y = l_2(x) = 1 + 2(x-1)$, that is $y = 2x - 1$.

(b) Find the equation of the tangent line to $y = x^3 + x$ at $(1,2)$ without using calculus.

Solution: We have $l_m(x) = 2 + m(x-1)$. Let

$$g_m(x) = f(x) - l_m(x) = x^3 + x - 2 - m(x-1) = x^3 + (1-m)x + (m-2) = (x-1)\big(x^2 + x + (2-m)\big)\,.$$

In order for $(x-1)^2$ to be a factor of $g_m(x)$, it must be that $(x-1)$ is a factor of $h_m(x) = x^2 + x + (2-m)$, so we must have $h_m(1) = 0$, that is $4 - m = 0$, and so $m = 4$. Thus the equation of the tangent line is $y = l_4(x) = 2 + 4(x-1)$, that is $y = 4x - 2$.

(c) Without using calculus, find all points on the graph $y = x^3 - 3x$ at which the tangent line is horizontal.

Solution: Let $f(x) = x^3 - 3x$, let $l_{a,m}(x) = f(a) + m(x-a) = a^3 - 3a + m(x-a)$, and let

$$\begin{aligned} g_{a,m}(x) &= f(x) - l_{a,m}(x) = x^3 - 3x - a^3 + 3a - m(x-a) \\ &= x^3 - (3+m)x + (ma - a^3 + 3a) = (x-a)h_{a,m}(x) \end{aligned}$$

where $h_{a,m}(x) = x^2 + ax + (a^2 - 3 - m)$. In order for $(x-a)^2$ to be a factor of $g_{a,m}(x)$, it must be that $(x-a)$ is a factor of $h_{a,m}(x)$, so we must have $h_{a,m}(a) = 0$, that is $3a^2 - 3 - m = 0$. Thus the slope of the tangent line at $\big(a, f(a)\big)$ is $m = 3a^2 - 3$. The tangent line is horizontal when $0 = m = 3a^2 - 3$, that is when $a = \pm 1$. Thus the two points at which the tangent line is horizontal are $(1,-2)$ and $(-1,2)$.