MATH 137 Calculus 1, Solutions to Assignment 11

1: Evaluate the following definite integrals.

(a) $\displaystyle\int_{-2}^{4} x^3 - x^2 - 3x + 1 \; dx$

Solution: We have

$$\int_{-2}^{4} x^3 - x^2 - 3x + 1 \; dx = \Big[\tfrac{1}{4}\,x^4 - \tfrac{1}{3}\,x^3 - \tfrac{3}{2}\,x^2 + x\Big]_{-2}^{4} = \big(64 - \tfrac{64}{3} - 24 + 4\big) - \big(4 + \tfrac{8}{3} - 6 - 2\big) = 24\,.$$

(b) $\displaystyle\int_{\pi/6}^{\pi/2} \cos x - \sqrt{3}\,\sin x \; dx$

Solution: We have

$$\int_{\pi/6}^{\pi/2} \cos x - \sqrt{3}\,\sin x \; dx = \Big[\sin x + \sqrt{3}\,\cos x\Big]_{\pi/6}^{\pi/2} = \big(1 + 0\big) - \big(\tfrac{1}{2} + \sqrt{3}\cdot\tfrac{\sqrt{3}}{2}\big) = -1\,.$$

(c) $\displaystyle\int_{1}^{4} \frac{(x-1)(x-2)}{\sqrt{x}} \; dx$

Solution: We have

$$\begin{aligned}\int_{1}^{4} \frac{(x-1)(x-2)}{\sqrt{x}} \; dx &= \int_{1}^{4} \frac{x^2 - 3x + 2}{\sqrt{x}} \; dx = \int_{1}^{4} x^{3/2} - 3\,x^{1/2} + 2\,x^{-1/2} \; dx \\ &= \Big[\tfrac{2}{5}\,x^{5/2} - 2\,x^{3/2} + 4\,x^{1/2}\Big]_{1}^{4} = \big(\tfrac{64}{5} - 16 + 8\big) - \big(\tfrac{2}{5} - 2 + 4\big) = \tfrac{12}{5}\,.\end{aligned}$$

2: Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\ln 3} \frac{e^x\,dx}{1+e^x}$

Solution: Let $u = 1 + e^x$ so $du = e^x\,dx$. Then

$$\int_{x=0}^{\ln 3} \frac{e^x\,dx}{1+e^x} = \int_{u=2}^{4} \frac{du}{u} = \Big[\ln u\Big]_2^4 = \ln 4 - \ln 2 = \ln 2\,.$$

(b) $\displaystyle\int_0^{3} \frac{x^2\,dx}{(x+1)^{3/2}}$

Solution: Let $u = x + 1$ so that $x = u - 1$ and $dx = du$. Then

$$\int_{x=0}^{3} \frac{x^2\,dx}{(x+1)^{3/2}} = \int_{u=1}^{4} \frac{(u-1)^2\,du}{u^{3/2}} = \int_1^4 \frac{u^2-2u+1}{u^{3/2}} = \int_1^4 u^{1/2} - 2\,u^{-1/2} + u^{-3/2}\,du$$
$$= \Big[\tfrac{2}{3}\,u^{3/2} - 4\,u^{1/2} - 2\,u^{-1/2}\Big]_1^4 = \big(\tfrac{16}{3} - 8 - 1\big) - \big(\tfrac{2}{3} - 4 - 2\big) = \tfrac{5}{3}\,.$$

(c) $\displaystyle\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\,dx$

Solution: Let $u = \cos x$ so that $du - \sin x\;dx$. Then

$$\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\,dx = \int_{x=0}^{\pi/3} \frac{(1-\cos^2 x)\sin x\;dx}{\cos^2 x} = \int_{u=1}^{1/2} \frac{-(1-u^2)\,du}{u^2} = \int_1^{1/2} 1 - \frac{1}{u^2}\;du$$
$$= \left[u + \frac{1}{u}\right]_1^{1/2} = \big(\tfrac{1}{2} + 2\big) - \big(1+1\big) = \tfrac{1}{2}\,.$$

3: Evaluate the following definite integrals.

(a) $\displaystyle\int_0^2 \frac{x^3\,dx}{\sqrt{2x^2+1}}$

Solution: Let $u = 2x^2+1$ so $du = 4x\,dx$ and $x^2 = \frac{1}{2}(u-1)$. Then

$$\int_0^2 \frac{x^3}{\sqrt{2x^2+1}}\,dx = \int_{x=0}^2 \frac{x^2\cdot x\,dx}{\sqrt{2x^2+1}} = \int_{u=1}^9 \frac{\frac{1}{2}(u-1)\cdot\frac{1}{4}\,du}{\sqrt{u}} = \frac{1}{8}\int_1^9 u^{1/2} - u^{-1/2}\,du$$
$$= \frac{1}{8}\Big[\frac{2}{3}u^{3/2} - 2\,u^{1/2}\Big]_1^9 = \frac{1}{8}\Big[(18-6) - (\frac{2}{3}-2)\Big] = \frac{1}{8}\cdot\frac{40}{3} = \frac{5}{3}\,.$$

(b) $\displaystyle\int_1^3 \frac{\sqrt{x}\,dx}{x+1}$

Solution: Let $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\int_{x=1}^3 \frac{\sqrt{x}\,dx}{x+1} = \int_{u=1}^{\sqrt{3}} \frac{u\cdot 2u\,du}{u^2+1} = \int_1^{\sqrt{3}} 2 - \frac{2}{u^2+1}\,du = \Big[2u - 2\tan^{-1}u\Big]_1^{\sqrt{3}}$$
$$= \big(2\sqrt{3} - 2\,\tfrac{\pi}{3}\big) - \big(2 - 2\,\tfrac{\pi}{4}\big) = 2\sqrt{3} - 2 - \tfrac{\pi}{6}\,.$$

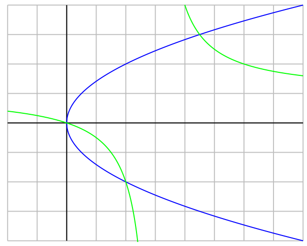
(c) $\displaystyle\int_0^{\pi/4} \tan^3 x\;dx$

Solution: Let $u = \tan x$ so $du = \sec^2 x\,dx$ and let $v = \cos x$ so $dv = -\sin x\,dx$. Then

$$\int_0^{\pi/4} \tan^3 x\;dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan x\;dx = \int_{x=0}^{\pi/4} \tan x\cdot\sec^2 x\,dx - \int_{x=0}^{\pi/4} \frac{\sin x\,dx}{\cos x}$$
$$= \int_{u=0}^1 u\,du - \int_{v=1}^{1/\sqrt{2}} \frac{-dv}{v} = \Big[\tfrac{1}{2}\,u^2\Big]_{u=0}^1 + \Big[\ln v\Big]_{v=1}^{1/\sqrt{2}} = \tfrac{1}{2} + \ln\tfrac{1}{\sqrt{2}} = \tfrac{1}{2} - \tfrac{\ln 2}{2}\,.$$

4: (a) Find the area of the region which is bounded by the curves $y^2 = 2x$ and $y = \dfrac{x}{x-3}$.

Solution: First sketch the two curves. The parabola $y^2 = 2x$ is shown in blue, and the hyperbola $y = \dfrac{x}{x-3}$ is in green.

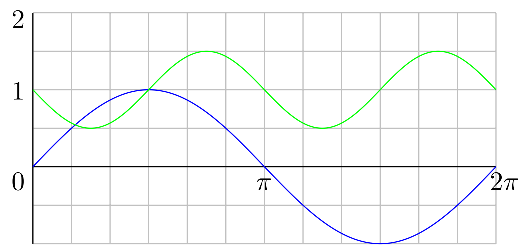


We see that the region in question lies below the hyperbola and above the bottom half of the parabola, which is given by $y = -\sqrt{2x}$, with $0 \le x \le 2$. Thus the area is

$$A = \int_0^2 \left(\frac{x}{x-3}\right) - \big(-\sqrt{2x}\big)\, dx = \int_0^2 \frac{x-3+3}{x-3} + \sqrt{2x}\, dx = \int_0^2 1 + \frac{3}{x-3} + (2x)^{1/2}\, dx$$
$$= \Big[x + 3\ln|x-3| + \tfrac{1}{3}(2x)^{3/2}\Big]_0^2 = \big(2 + 0 + \tfrac{8}{3}\big) - \big(3\ln 3\big) = \tfrac{14}{3} - 3\ln 3\,.$$

(b) Find the area of the region bounded by $y = \sin x$ and $y = 1 - \frac{1}{\sqrt{3}}\sin 2x$ with $0 \le x \le 2\pi$.

Solution: First we sketch the two curves. The curve $y = \sin x$ is shown in blue and the curve $y = 1 - \frac{1}{\sqrt{3}}\sin 2x$ is shown in green.



The region lies below the curve $y = \sin x$ and above the curve $y = 1 - \frac{1}{\sqrt{3}}\sin 2x$ with $\frac{\pi}{6} \le x \le \frac{\pi}{2}$, so the area is given by

$$A = \int_{\pi/6}^{\pi/2} \sin x - \big(1 - \tfrac{1}{\sqrt{3}}\sin 2x\big)\, dx = \int_{\pi/6}^{\pi/2} \sin x + \tfrac{\sqrt{3}}{3}\sin 2x - 1\, dx = \Big[-\cos x - \tfrac{\sqrt{3}}{6}\cos 2x - x\Big]_{\pi/6}^{\pi/2}$$
$$= \big(0 - \tfrac{\sqrt{3}}{6}(-1) - \tfrac{\pi}{2}\big) - \big(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{6}\cdot\tfrac{1}{2} - \tfrac{\pi}{6}\big) = \sqrt{3}\big(\tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{12}\big) - \pi\big(\tfrac{1}{2} - \tfrac{1}{6}\big) = \tfrac{3\sqrt{3}}{4} - \tfrac{\pi}{3}\,.$$

5: (a) Find $\int \tan^{-1} x \, dx$

Solution: We use trial and error. Let $f(x) = x\tan^{-1} x$. Then $f'(x) = \tan^{-1} x + \frac{x}{1+x^2}$. Let $g(x) = \ln(1+x^2)$. Then $g'(x) = \frac{2x}{1+x^2}$. Thus if we let $F(x) = f(x) - \frac{1}{2}\, g(x)$ then we have $F'(x) = f'(x) - \frac{1}{2}\, g'(x) = \tan^{-1} x$. Thus

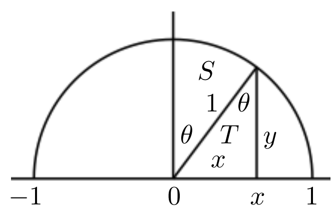
$$\int \tan^{-1} x \, dx = F(x) + c = x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + c\,.$$

(b) Find $\int \sqrt{1-x^2}\, dx$

Solution: We use geometry. Let $f(t) = \sqrt{1-t^2}$ and let

$$F(x) = \int_0^x \sqrt{1-t^2}\, dt\,.$$

By the FTC we know that $F'(x) = f(x) = \sqrt{1-x^2}$. For $0 \le x \le 1$, $F(x)$ is equal to the area of the region R which lies under the semicircle $y = \sqrt{1-t^2}$ with $0 \le t \le x$. Let θ, x and y be as shown below, note that $\sin\theta = x$ and $y = \sqrt{1-x^2}$, and divide the region R into two regions S and T as shown below.



The area of region S is equal to $\frac{\theta}{2\pi}$ times the area of the entire circle of radius 1, that is $|S| = \frac{1}{2}\,\theta$. The area of the region T is $|T| = \frac{1}{2}\, xy$. Thus, for $0 \le x \le 1$, we have

$$F(x) = \int_0^x \sqrt{1-t^2}\, dt = |S| + |T| = \tfrac{1}{2}\theta + \tfrac{1}{2}xy = \tfrac{1}{2}\sin^{-1} x + \tfrac{1}{2}\, x\sqrt{1-x^2}\,.$$

For $-1 < x < 1$ we have

$$\begin{aligned} \frac{d}{dx}\Big(\tfrac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right)\Big) &= \tfrac{1}{2}\left(\frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x\cdot\frac{-x}{\sqrt{1-x^2}}\right) = \tfrac{1}{2}\left(\frac{1+(1-x^2)-x^2}{\sqrt{1-x^2}}\right) \\ &= \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}\,, \end{aligned}$$

and so

$$\int \sqrt{1-x^2}\, dx = \tfrac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) + c\,.$$