MATH 137 Calculus 1, Solutions to Assignment 10

1: (a) Find the function $f(x)$, defined for $x > 0$, such that $f'(x) = \dfrac{\sqrt{x}+1}{x^2}$ with $f(1) = 0$.

Solution: We have $f'(x) = \dfrac{x^{1/2}+1}{x^2} = x^{-3/2} + x^{-2}$, so

$$f(x) = \int x^{-3/2} + x^{-2}\ dx = -2\,x^{-1/2} - x^{-1} + c\,.$$

Since $f(1) = 0$ we have $-2 - 1 + c = 0$ so $c = 3$, and so $f(x) = 3 - 2\,x^{-1/2} - x^{-1}$.

(b) Find the function $f(x)$ such that $f''(x) = e^x - \sin x$ with $f(0) = 2$ and $f'(0) = 1$.

Solution: Since $f''(x) = e^x - \sin x$ we have

$$f'(x) = \int e^x - \sin x\ dx = e^x + \cos x + c_1\,.$$

Put in $f'(0) = 1$ to get $1 + 1 + c_1 = 1$ so $c_1 = -1$, and so $f'(x) = e^x + \cos x - 1$. Then we have

$$f(x) = \int e^x + \cos x - 1\ dx = e^x + \sin x - x + c_2\,.$$

Put in $f(0) = 2$ to get $1 + c_2 = 2$ so $c_2 = 1$, and so $f(x) = e^x + \sin x - x + 1$.

(c) An object moves along the x-axis with $a(t) = |t - 1| - 1$. Given that $x(t)$ and $v(t)$ are both continuous and $x(0) = 0$ and $v(2) = 0$, find $x(2)$.

Solution: We have

$$a(t) = \begin{cases} -t\,, & \text{if } t \leq 1 \\ t - 2, & \text{if } t \geq 1 \end{cases}$$

and so

$$v(t) = \begin{cases} -\frac{1}{2}t^2 + b\,, & \text{if } t \leq 1 \\ \frac{1}{2}t^2 - 2t + c, & \text{if } t \geq 1 \end{cases}$$

for some constants b, c. Since $v(2) = 0$ we have $2 - 4 + c = 0$ and so $c = 2$. Also, since $v(t)$ is continuous at $t = 1$ we must have $-\frac{1}{2} + b = \frac{1}{2} - 2 + 2$ and so $b = 1$. Thus

$$v(t) = \begin{cases} -\frac{1}{2}t^2 + 1\,, & \text{if } t \leq 1 \\ \frac{1}{2}t^2 - 2t + 2, & \text{if } t \geq 1\,. \end{cases}$$

Find the antiderivative again to get

$$x(t) = \begin{cases} -\frac{1}{6}t^3 + t + d\,, & \text{if } t \leq 1 \\ \frac{1}{6}t^3 - t^2 + 2t + e, & \text{if } t \geq 1 \end{cases}$$

for some d, e. Since $x(0) = 0$ we must have $d = 0$. Also, since $x(t)$ is continuous at $t = 1$ we must have $-\frac{1}{6} + 1 = \frac{1}{6} - 1 + 2 + e$ and so $e = -\frac{1}{3}$. Thus

$$x(t) = \begin{cases} -\frac{1}{6}t^3 + t\,, & \text{if } t \leq 1 \\ \frac{1}{6}t^3 - t^2 + 2t - \frac{1}{3}, & \text{if } t \geq 1\,. \end{cases}$$

Finally, we have $x(2) = \frac{8}{6} - 4 + 4 - \frac{1}{3} = 1$.

2: Let $g(x) = \int_0^x f(t)\,dt$ where $f(t)$ is the function whose graph is shown below. Sketch the graph of $y = g(x)$ showing all intercepts, all local maxima and minima, and all points of inflection.
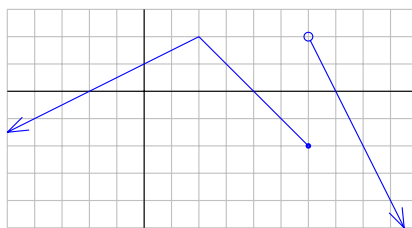
Solution: By the Fundamental Theorem of Calculus, we know that $g'(x) = f(x)$. When $f(x) > 0$ we have $g'(x) > 0$ so $g(x)$ is increasing and when $f(x) < 0$ we have $g'(x) < 0$ so $g(x)$ is decreasing, and each time $f(x)$ changes sign, $g(x)$ has a local maximum or minimum according to the First Derivative Test. Thus $g(x)$ is decreasing in $(-\infty, -2)$, increasing in $(-2, 4)$, decreasing in $(4, 6)$, increasing in $(6, 7)$, and decreasing in $(7, \infty)$, and $g(x)$ has a local minimum at $x = -2$, a local maximum at $x = 4$, a local minimum at $x = 6$, and a local maximum at $x = 7$. Also, when $f'(x) > 0$ we have $g''(x) > 0$ so $g(x)$ is concave up and when $f'(x) < 0$ we have $g''(x) < 0$ so $g(x)$ is concave down, and each time $f'(x)$ changes sign, $g(x)$ has a point of inflection. Thus $g(x)$ is concave up in $(-\infty, 2)$, concave down in $(2, 6)$, and again concave down in $(6, \infty)$, and $g(x)$ has a point of inflection at $x = 2$. We can find the exact value of $g(x)$ at each of the points of interest $x = -2, 0, 2, 4, 6, 7$ by interpreting the integral $\int_0^x f(t)\,dt$ as a signed area. For example, $g(2) = \int_0^2 f(t)\,dt$ is the area under $y = f(x)$ between $x = 0$ and $x = 2$, which is equal to 3, so $g(2) = 3$. As another example, $g(6)$ is the area under $y = f(x)$ with $0 \le x \le 4$ minus the are over $y = f(x)$ with $4 \le x \le 6$, so $g(6) = 5 - 2 = 3$. As a final example, $g(-2) = \int_0^{-2} f(t)\,dt = -\int_{-2}^0 f(t)\,dt$, which is the negative of the area under $y = f(x)$ with $-2 \le x \le 0$, so $g(-2) = -1$. Alternatively, we can find an explicit formula for $g(x)$ as follows. From the graph of $f(x)$ we can see that

$$f(x) = \begin{cases} 1 + \frac{1}{2}x\,, & \text{if } x \le 2 \\ 4 - x\,, & \text{if } 2 \le x \le 6 \\ 14 - 2x\,, & \text{if } 6 < x\,. \end{cases}$$

For $x \le 2$ we have

$$g(x) = \int_0^x f(t)\,dt = \int_0^x 1 + \tfrac{1}{2}t\ dx = \left[t + \tfrac{1}{4}t^2\right]_0^x = x + \tfrac{1}{4}x^2\,,$$

for $2 \le x \le 6$ we have

$$g(x) = \int_0^x f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^x f(t)\,dt = g(2) + \int_2^x 4 - t\ dt$$
$$= \left(2 + \tfrac{1}{4}\cdot 2^2\right) + \left[4t - \tfrac{1}{2}t^2\right]_2^x = 3 + \left(4x - \tfrac{1}{2}x^2\right) - (8 - 2) = -3 + 4x - \tfrac{1}{2}x^2\,,$$
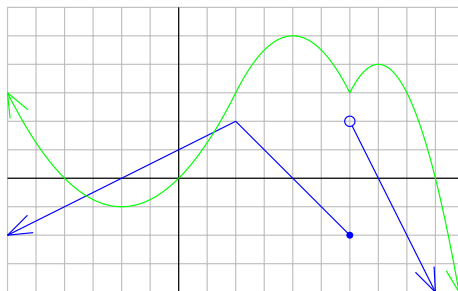
and for $6 \le x$ we have

$$g(x) = \int_0^x f(t)\,dt = \int_0^6 f(t)\,dt + \int_6^x f(t)\,dt = g(6) + \int_6^x 14 - 2x\ dx$$
$$= \left(-3 + 4\cdot 6 - \tfrac{1}{2}\cdot 6^2\right) + \left[14t - t^2\right]_6^x = 3 + \left(14x - x^2\right) - \left(4\cdot 6 - \cdot 6^2\right) = -45 + 14x - x^2\,.$$

Thus we have

$$g(x) = \begin{cases} x + \tfrac{1}{4}x^2\,, & \text{if } x \le 2 \\ -3 + 4x - \tfrac{1}{2}x^2\,, & \text{if } 2 \le x \le 6 \\ 45 + 14x - x^2\,, & \text{if } 6 \le x \end{cases}$$

The graph of $y = g(x)$ is shown in green on the next page.

3: (a) Let $f(x) = \dfrac{8x}{2^{3x}}$. Approximate the integral $\displaystyle\int_0^2 f(x)\,dx$ using the Riemann sum for $f(x)$ which uses the right endpoints of 6 equal-sized subintervals.

Solution: The six intervals are of size $\Delta x = \frac{2-0}{6} = \frac{1}{3}$ and the right endpoints are the points $x_i = 0 + i\,\Delta x = \frac{i}{3}$, that is the points $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}$ and 2. We have

$$\sum_{i=1}^n f(x_i)\Delta_i x = \left(f\left(\tfrac{1}{3}\right) + f\left(\tfrac{2}{3}\right) + f(1) + f\left(\tfrac{4}{3}\right) + f\left(\tfrac{5}{3}\right) + f(2)\right)\left(\tfrac{1}{3}\right)$$

$$= \left(\tfrac{8\cdot 1}{3\cdot 2} + \tfrac{8\cdot 2}{3\cdot 4} + \tfrac{8\cdot 3}{3\cdot 8} + \tfrac{8\cdot 4}{3\cdot 16} + \tfrac{8\cdot 5}{3\cdot 32} + \tfrac{8\cdot 6}{3\cdot 64}\right)\left(\tfrac{1}{3}\right)$$

$$= \left(\tfrac{4}{3} + \tfrac{4}{3} + \tfrac{3}{3} + \tfrac{2}{3} + \tfrac{5}{12} + \tfrac{3}{12}\right)\left(\tfrac{1}{3}\right)$$

$$= \left(\tfrac{15}{3}\right)\left(\tfrac{1}{3}\right)$$

$$= \tfrac{5}{3}\,.$$

We remark that it can be shown, using methods that you will learn in Calculus II, that the exact value of the integral is

$$\int_0^2 f(x)\,dx = \tfrac{21 - 2\ln 2}{24(\ln 2)^2}\,.$$

(b) Let $f(x) = \dfrac{1}{x}$. Approximate the integral $\displaystyle\int_{1/5}^{13/5} f(x)\,dx$ using the Riemann sum for $f(x)$ which uses the midpoints of 6 equal-sized subintervals.

Solution: Let $f(x) = \frac{1}{x}$. We divide $\left[\frac{1}{5}, \frac{13}{5}\right]$ into 6 equal intervals using $\Delta x = \frac{b-a}{n} = \frac{\frac{13}{5} - \frac{1}{5}}{6} = \frac{2}{5}$. The endpoints of these intervals are given by $x_i = a + \frac{b-a}{n}\,i = \frac{1}{5} + \frac{2}{5}i$ so that $x_0, x_1, x_2, \cdots x_6 = \frac{1}{5}, \frac{3}{5}, \frac{5}{5}, \cdots, \frac{13}{5}$. The midpoints are $c_i = \dfrac{x_i + x_{i-1}}{2}$ so that $c_1, c_2, c_3, \cdots, c_6 = \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \cdots, \frac{12}{5}$. We have

$$\int_{1/5}^{13/5} f(x)\,dx \cong \sum_{i=1}^6 f(c_i)\Delta x = \left(f(c_1) + f(c_2) + \cdots + f(c_6)\right)\left(\tfrac{2}{5}\right) = \tfrac{2}{5}\left(f\left(\tfrac{2}{5}\right) + f\left(\tfrac{4}{5}\right) + \cdots f\left(\tfrac{12}{5}\right)\right)$$

$$= \tfrac{2}{5}\left(\tfrac{5}{2} + \tfrac{5}{4} + \tfrac{5}{6} + \cdots + \tfrac{5}{12}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} = \tfrac{60+30+20+15+12+10}{60} = \tfrac{147}{60}\,.$$

We remark that, by the FTC, the exact value of this integral is

$$\int_{1/5}^{13/5} \frac{dx}{x} = \Big[\ln x\Big]_{1/5}^{13/5} = \ln \tfrac{13}{5} - \ln \tfrac{1}{5} = \ln 13\,,$$

so the above approximation shows that $\ln 13 \cong \frac{147}{60}$.

4: (a) Evaluate $\int_1^3 x^3 - 3x\ dx$ by finding a limit of Riemann sums.

Solution: Let $f(x) = x^3 - 3x$, $a = 1$, $b = 3$, $\Delta x = \frac{b-a}{n} = \frac{2}{n}$, and $x_i = a + \frac{b-a}{n}i = 1 + \frac{2}{n}i$. Then

$$\int_1^3 x^3 - 3x\ dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^n f\left(1 + \tfrac{2}{n}i\right)\left(\tfrac{2}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^n \left(\left(1 + \tfrac{2}{n}i\right)^3 - 3\left(1 + \tfrac{2}{n}i\right)\right)\left(\tfrac{2}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^n \left(\left(1 + \tfrac{6}{n}i + \tfrac{12}{n^2}i^2 + \tfrac{8}{n^3}i^3\right) - 3\left(1 + \tfrac{2}{n}i\right)\right)\left(\tfrac{2}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^n \left(-2 + \tfrac{12}{n^2}i^2 + \tfrac{8}{n^3}i^3\right)\left(\tfrac{2}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^n \left(-\tfrac{4}{n} + \tfrac{24}{n^3}i^2 + \tfrac{16}{n^4}i^3\right)$$

$$= \lim_{n\to\infty}\left(-\tfrac{4}{n}\sum_{i=1}^n 1 + \tfrac{24}{n^3}\sum_{i=1}^n i^2 + \tfrac{16}{n^4}\sum_{i=1}^n i^3\right)$$

$$= \lim_{n\to\infty}\left(-\tfrac{4}{n}n + \tfrac{24}{n^3}\tfrac{n\left(n+\frac{1}{2}\right)\left(n+1\right)}{3} + \tfrac{16}{n^4}\tfrac{n^2(n+1)^2}{4}\right)$$

$$= -4 + \tfrac{24}{3} + \tfrac{16}{4}$$

$$= 8$$

(b) Evaluate $\int_0^1 e^x\ dx$ by finding a limit of Riemann sums.

Solution: Let $f(x) = e^x$ and use $a = 0$, $b = 1$, $\Delta x = \frac{b-a}{n} = \frac{1}{n}$ and $x_i = a + \frac{b-a}{n}i = \frac{i}{n}$. Then

$$\int_0^1 e^x\ dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n e^{i/n}\cdot\tfrac{1}{n} = \lim_{n\to\infty}\tfrac{1}{n}\cdot\sum_{i=1}^n (e^{1/n})^i = \lim_{n\to\infty}\tfrac{1}{n}\cdot\frac{(e^{1/n})^{n+1} - e^{1/n}}{e^{1/n} - 1}$$

where, at the last step, we used the formula $\sum_{i=1}^n r^i = \frac{r^{n+1}-r}{r-1}$ for the sum of a geometric series. We have

$$\lim_{n\to\infty}(e^{1/n})^{n+1} = \lim_{n\to\infty}e^{(n+1)/n} = e^1,\ \text{and}\ \lim_{n\to\infty}e^{1/n} = e^0 = 1$$

and by replacing $\frac{1}{n}$ by x and then using l'Hôpital's Rule, we have

$$\lim_{n\to\infty}n\left(e^{1/n} - 1\right) = \lim_{n\to\infty}\frac{e^{1/n} - 1}{1/n} = \lim_{x\to 0^+}\frac{e^x - 1}{x} = \lim_{x\to 0^+}\frac{e^x}{1} = e^0 = 1.$$

and so

$$\int_0^1 e^x\ dx = \lim_{n\to\infty}\frac{(e^{1/n})^{n+1} - e^{1/n}}{n\left(e^{1/n} - 1\right)} = \frac{e - 1}{1} = e - 1.$$

5: (a) Find $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}$.

Solution: Let $f(x) = \frac{1}{1+x}$ and let X_n be the partition of $[0,1]$ into n equal-sized subintervals so $x_{n,i} = \frac{i}{n}$ and $\Delta_{n,i}x = \frac{1}{n}$. Then by the FTC we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}f(x_{n,i})\Delta_{n,i}x = \int_0^1\frac{dx}{1+x} = \Big[\ln(1+x)\Big]_0^1 = \ln 2\,.$$

(b) Find $g'(1)$ where $g(x) = \displaystyle\int_{3x-3}^{x^2+1}\sqrt{1+t^3}\,dt$.

Solution: Let $u(x) = x^2+1$ and let $v(x) = 3x-3$. Also, let $f(t) = \sqrt{1+t^3}$ and let $F(u) = \displaystyle\int_0^u\sqrt{1+t^3}\,dt$ so that $F'(u) = f(u)$, by the FTC. Then

$$g(x) = \int_{3x-3}^{x^2+1}\sqrt{1+t^3}\,dt = \int_0^{x^2+1}\sqrt{1+t^3}\,dt - \int_0^{3x-3}\sqrt{1+t^3}\,dt = F\big(u(x)\big) - F\big(v(x)\big)$$

and so $g'(x) = F'\big(u(x)\big)u'(x) - F'\big(v(x)\big)v'(x) = f\big(u(x)\big)(2x) - f\big(v(x)\big)(3) = 2x\,f(x^2+1) - 3\,f(3x-3)$. Put in $x = 1$ to get $g'(1) = 2f(2) - 3f(0) = 2\sqrt{1+8} - 3\sqrt{1+0} = 6 - 3 = 3$.