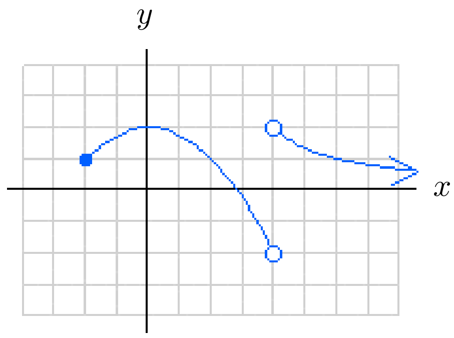
MATH 137 Calculus 1, Assignment 1 Due Fri Sept 21

1: Let f be the function whose graph is shown below.



(a) What are the domain and range of f?

(b) Sketch the graph of $y = 2\,f(2x+2)$.

(c) Sketch the graph of $y = \big|1 - f(x)\big|$.

2: (a) Sketch the graphs of $y = |x^2 + x - 8|$ and $y = |2 - 2x|$ on the same grid.

(b) Use the sketch to determine the solution to the inequality $|x^2 + x - 8| \leq |2 - 2x|$.

(c) Solve the inequality $|x^2 + x - 8| \leq |2 - 2x|$ algebraically.

3: Let $f(x) = \dfrac{x^2+3}{x-1}$. Note that $f(x) = (x+1) + \dfrac{4}{x-1}$.

(a) Sketch the graphs of $y = x+1$, $y = \dfrac{4}{x-1}$ and $y = f(x)$ all on the same grid.

(b) Use the sketch to guess what the range of f is.

(c) Find the range of f algebraically.

4: Let $f(x) = \sqrt{x-3}$ and $g(x) = x^2 - 1$.

(a) Find the domain and the range and sketch the graph of $f \circ g$.

(b) Find the domain and the range and sketch the graph of $g \circ f$.

(c) Find a function h such that $f(h(x)) = x^2$ for all x.

5: Given a polynomial $f(x)$, we can find the tangent line of the graph $y = f(x)$ at the point $\big(a, f(a)\big)$ *without* using calculus as follows. The line through $\big(a, f(a)\big)$ with slope m has equation $y = l_m(x)$ where $l_m(x) = f(a) + m(x-a)$. The tangent line will be the line $y = l_m(x)$ when m is chosen so that $(x-a)^2$ is a factor of $f(x) - l_m(x)$.

(a) Find the equation of the tangent line to $y = x^2$ at $(1,1)$ without using calculus.

(b) Find the equation of the tangent line to $y = x^3 + x$ at $(1,2)$ without using calculus.

(c) Without using calculus, find all points on the graph $y = x^3 - 3x$ at which the tangent line is horizontal.

Hints and Comments

1: (b) Some students might prefer to write $2x + 2$ as $2(x + 1)$.

2: (b) In order to use your sketch to solve the inequality, your sketch must be sufficiently accurate that you can determine the exact coordinates of all points of intersection.

(c) This is quite challenging. Consider 4 cases depending on whether $x^2 + x - 8$ and $2 - 2x$ are positive or negative.

3: (c) To find the range of f, solve $y = f(x)$ for x in terms of y.

4: (a) To sketch the graph of $f \circ g$, first find a formula for $f\big(g(x)\big)$. You could then sketch the graph by making a table of values and plotting points (you might find a calculator useful for this). Alternatively, you can determine the graph of any equation of the form $y = \sqrt{Ax^2 + Bx + C}$ without using a calculator as follows. Square both sides to get $y^2 = Ax^2 + Bx + C$ then complete the square to put this in the form $y^2 = A(x + b)^2 + c$. This is the equation of an ellipse (if $A < 0$ and $c > 0$) or a hyperbola (if $A > 0$ and $c \neq 0$), as described in Appendix C. The graph of the original equation $y = \sqrt{Ax^2 + Bx + C}$ will be the top half of this ellipse or hyperbola.

5: You should try to understand why this method for finding the tangent line works. The situation is illustrated below. The graph of $y = f(x)$ is shown in blue, the graph of the line $y = l(x)$ is shown in red, and the graph of $y = h(x) = f(x) - l(x)$ is shown in green. For this illustration we have chosen $l(x) = \frac{1}{2}x + \frac{5}{2}$ and $h(x) = -\frac{1}{20}(x + 1)^2(x - 1)(x - 3)^3$. Note that the line $y = l(x)$ will be tangent to $y = f(x)$ at the point where $x = a$ if and only if the line $y = 0$ (that is the x-axis) is tangent to $y = h(x)$ at the point $(a, 0)$ and this happens precisely when $(x - a)^k$ is a factor of $h(x)$ for some $k \geq 2$. In the picture below, we see that $y = l(x)$ is tangent to $y = f(x)$ at the two points $(-1, 2)$ and $(3, 4)$ because the x-axis is tangent to $y = h(x)$ at the two x-intercepts $(-1, 0)$ and $(3, 0)$, and this happens because $(x + 1)^2$ and $(x - 3)^3$ are both factors of $h(x)$.