MATH 135 Algebra, Solutions to Term Test 1

1: Recall that the symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are alternate notations for the connectives NOT, AND, OR, $\Longrightarrow$, and $\Longleftrightarrow$, respectively.

[4] (a) Determine whether $P \leftrightarrow (Q \rightarrow \neg P)$ is equivalent to $\neg(P \rightarrow Q)$.

Solution: We make a truth table.

P	Q	$\neg P$	$Q \rightarrow \neg P$	$P \leftrightarrow (Q \rightarrow \neg P)$	$P \rightarrow Q$	$\neg(P \rightarrow Q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	T	F
F	F	T	T	F	T	F

The given two statements are equivalent, because their truth table columns are identical.

[3] (b) Express the statement "x is the greatest integer such that $2x \leq y$", taking the universe of discourse to be $\mathbf{Z}$, and using only symbols from the following list:

$$\neg \,,\ \wedge \,,\ \vee \,,\ \rightarrow \,,\ \leftrightarrow \,,\ (\,,\ ) \,,\ \forall \,,\ \exists \,,\ 0 \,,\ 1 \,, + \,,\ \times \,,\ = \,,\ < \,,\ \leq \,,\ x \,,\ y \,,\ z$$

Solution: Here are two ways to express the given statement symbolically.

$$x + x \leq y \wedge \forall z \, (z + z \leq y \rightarrow z \leq x)$$
$$x + x = y \vee x + x = y + 1$$

[3] (c) Determine whether the statement "$\forall x \; x \leq x \times x$" is true when the universe of discourse is $\mathbf{Z}$ and whether it is true when the universe of discourse is $\mathbf{R}$.

Solution: For all $x \in \mathbf{R}$ (and for all $x \in \mathbf{Z}$) we have

$$x \leq x^2 \Longleftrightarrow x^2 - x \geq 0 \Longleftrightarrow x(x-1) \geq 0 \Longleftrightarrow x \leq 0 \text{ or } x \geq 1 \,.$$

Thus the statement is true in $\mathbf{Z}$ (since for every $x \in \mathbf{Z}$ we have $x \leq 0$ or $x \geq 1$) but false in $\mathbf{R}$ (for example, when $x = \frac{1}{2}$ we have $x > x^2$).

[5] **2:** (a) Let $a_0 = 0$ and $a_1 = 1$, and for $n \geq 2$ let $a_n = 5a_{n-1} - 6a_{n-2}$. Show that $a_n = 3^n - 2^n$ for all $n \geq 0$.

Solution: We claim that $a_n = 3^n - 2^n$ for all $n \geq 0$. When $n = 0$ we have $a_n = a_0 = 0$ and $3^n - 2^n = 3^0 - 2^0 = 1 - 1 = 0$, and when $n = 1$ we have $a_n = a_1 = 1$ and $3^n - 2^n = 3^1 - 2^1 = 3 - 2 = 1$, so the claim is true when $n = 0$ and when $n = 1$. Let $k \geq 2$ and suppose the claim is true when $n = k - 1$ and when $n = k - 2$, that is suppose that

$$a_{k-1} = 3^{k-1} - 2^{k-1} \text{ and } a_{k-2} = 3^{k-2} - 2^{k-2}.$$

Then when $n = k$ we have

$$\begin{aligned} a_n = a_k &= 5a_{k-1} - 6a_{k-2} \\ &= 5\big(3^{k-1} - 2^{k-1}\big) - 6\big(3^{k-2} - 2^{k-2}\big) \\ &= 5 \cdot 3^{k-1} - 6 \cdot 3^{k-2} - 5 \cdot 2^{k-1} + 6 \cdot 2^{k-2} \\ &= 15 \cdot 3^{k-2} - 6 \cdot 3^{k-2} - 10 \cdot 2^{k-2} + 6 \cdot 2^{k-2} \\ &= 9 \cdot 3^{k-2} - 4 \cdot 2^{k-2} \\ &= 3^k - 2^k = 3^n - 2^n, \end{aligned}$$

so the claim is true when $n = k$. By Strong Mathematical Induction, the claim is true for all $n \geq 0$.

[5] (b) Show that $\displaystyle\sum_{i=0}^{n} (-1)^i\, i^2 = (-1)^n \frac{n\,(n+1)}{2}$ for all $n \geq 0$.

Solution: We claim that $\sum_{i=0}^{n} (-1)^i\, i^2 = (-1)^n \frac{n\,(n+1)}{2}$ for all $n \geq 0$. When $n = 0$ we have

$$\sum_{i=0}^{n} (-1)^i\, i^2 = \sum_{i=0}^{0} (-1)^i\, i^2 = (-1)^0\, 0^2 = 1 \cdot 0 = 0 \text{ , and } (-1)^n \tfrac{n(n+1)}{2} = (-1)^0 \tfrac{0 \cdot 1}{2} = 1 \cdot 0 = 0$$

so the claim is true when $n = 0$. Let $k \geq 0$ and suppose the claim is true when $n = k$, that is suppose $\sum_{i=0}^{k} (-1)^i\, i^2 = (-1)^k \frac{k(k+1)}{2}$. Then when $n = k + 1$ we have

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i\, i^2 = \sum_{i=0}^{k+1} (-1)^i\, i^2 &= \sum_{i=0}^{k} (-1)^i\, i^2 + (-1)^{k+1}(k+1)^2 \\ &= (-1)^k \tfrac{k(k+1)}{2} + (-1)^{k+1}(k+1)^2 \\ &= (-1)^{k+1} \Big((k+1)^2 - \tfrac{k(k+1)}{2}\Big) \\ &= (-1)^{k+1}(k+1)\,\big(k + 1 - \tfrac{k}{2}\big) \\ &= (-1)^{k+1}(k+1)\,\big(\tfrac{2k+2-k}{2}\big) \\ &= (-1)^{k+1} \tfrac{(k+1)(k+2)}{2} = (-1)^n \tfrac{n(n+1)}{2}, \end{aligned}$$

so the claim is true when $n = k + 1$. By Mathematical Induction, the claim is true for all $n \geq 0$.

[5] **3:** (a) Find the term involving x^1 in the expansion of $\left(x^2+\frac{1}{2x}\right)^8$.

Solution: The i^{th} term in the expansion (counting the term $\binom{8}{0}(x^2)^8$ as the 0^{th} term) is

$$\binom{8}{i}\left(x^2\right)^{8-i}\left(\tfrac{1}{2x}\right)^i=\tfrac{1}{2^i}\binom{8}{i}\,x^{2(8-i)-i}=\tfrac{1}{2^i}\binom{8}{i}\,x^{16-3i}\,.$$

The term involving x^1 occurs when $16-3i=1$, that is when $i=5$. The 5^{th} term is

$$\tfrac{1}{2^5}\binom{8}{5}\,x^1=\tfrac{1}{2^5}\cdot\tfrac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2\cdot1}\,x=\tfrac{7}{4}\,x\,.$$

[5] (b) Evaluate the sum $\displaystyle\sum_{i=0}^{n}\binom{2n+1}{i}$ for all $n\geq 0$. (Prove that your answer is correct).

Solution: Notice that the odd-numbered rows in Pascal's Triangle (counting the top 1 as the 0^{th} row) have an even number of terms, and the given sum is the sum of the first half of the terms in row $2n+1$. By symmetry, this is the same as one half of the sum of all the terms in row $2n+1$. By the Binomial Theorem, the sum of all these terms is equal to $(1+1)^{2n+1}=2^{2n+1}$. Thus the given sum is equal to 2^{2n}.

We repeat the above proof, formally. Write $S_n=\displaystyle\sum_{i=0}^{n}\binom{2n+1}{i}=\binom{2n+1}{0}+\binom{2n+1}{1}+\binom{2n+1}{2}+\cdots+\binom{2n+1}{n}$. Recall that $\binom{m}{k}=\binom{m}{m-k}$, so we also have $S_n=\binom{2n+1}{2n+1}+\binom{2n+1}{2n}+\binom{2n+1}{2n-1}+\cdots+\binom{2n+1}{n+1}$. Adding these two expressions for S_n then using the Binomial Theorem gives

$$2S_n=\binom{2n+1}{0}+\binom{2n+1}{1}+\cdots+\binom{2n+1}{n}+\binom{2n+1}{n+1}+\cdots+\binom{2n+1}{2n}+\binom{2n+1}{2n+1}=(1+1)^{2n+1}=2^{2n+1}\,.$$

Thus $S_n=2^{2n}$. (It is also possible to prove this result using induction).

[2] **4:** (a) Define the statement "a divides b", for integers a and b.

Solution: The statement "a divides b" means that $b = ka$ for some integer k.

[3] (b) State the Division Algorithm.

Solution: The Division Algorithm states that for all integers a and b with $b > 0$, there exist unique integers q and r with $a = qb + r$ and $0 \le r < b$.

[5] (c) Prove Proposition 2.21 from the text, which states that for all integers a, b, q and r, if $a = qb + r$ then $\gcd(a, b) = \gcd(b, r)$.

Solution: Let $a, b, q, r \in \mathbf{Z}$. Suppose that $a = qb + r$. Let $d = \gcd(a, b)$ and let $e = \gcd(b, r)$. We must show that $d = e$. Note that since $a = qb + r$ we have $a = b = 0 \iff b = r = 0$, and in this case $d = 0 = e$. Suppose that a and b are not both zero (hence b and r are not both zero). We have

$$d = \gcd(a, b) \implies \big(d \big| a \text{ and } d \big| b\big) \implies d \big| (ax + by) \text{ for all } x, y \implies d \big| (a - qb) \implies d \big| r\,.$$

Since d is a common divisor of b and r (and e is the greatest common divisor of b and r) we must have $d \le e$. On the other hand,

$$e = \gcd(b, r) \implies \big(e \big| b \text{ and } e \big| r\big) \implies e \big| (bx + ry) \text{ for all } x, y \implies e \big| (qb + r) \implies e \big| a\,.$$

Since e is a common divisor of a and b (and d is the greatest common divisor of a and b) we must have $e \le d$.

[5] **5:** (a) Let $a = 231$ and $b = 182$. Find integers s and t such that $as + bt = d$, where $d = \gcd(a, b)$.

Solution: The Euclidean Algorithm gives

$$231 = 1 \cdot 182 + 49 \;,\quad 182 = 3 \cdot 49 + 35 \;,\quad 49 = 1 \cdot 35 + 14 \;,\quad 35 = 2 \cdot 14 + 7 \;,\quad 14 = 2 \cdot 7 + 0$$

so we have $d = 7$. Back-Substitution then gives rise to the sequence

$$1 \;,\quad -2 \;,\quad 3 \;,\quad -11 \;,\quad 14 \,,$$

so we can take $s = -11$ and $t = 14$.

Alternatively, the Extended Euclidean Algorithm gives rise to the table

0	1	231
1	0	182
−1	1	49
4	−3	35
−5	4	14
14	−11	7

so we can take $s = -11$ and $t = 14$.

[5] (b) Prove that for all integers a, b and c, if $a\big|c$ and $b\big|c$ and $\gcd(a, b) = d$ then $ab\big|cd$.

Solution: Let $a, b, c \in \mathbf{Z}$. Suppose that $a\big|c$ and $b\big|c$ and $\gcd(a, b) = d$. Since $a\big|c$ and $b\big|c$ we can choose integers k and l so that $c = ak$ and $c = bl$. Since $d = \gcd(a, b)$ we can use the Euclidean Algorithm with Back-Substitution to find integers s and t such that $as + bt = d$. Then we have

$$as + bt = d \Longrightarrow cas + cbt = cd \Longrightarrow blas + akbt = cd \Longrightarrow ab(ls + kt) = cd \Longrightarrow ab\big|cd\,.$$