Last Name (print): ______________________

First Name (print): ______________________

Signature: ______________________

ID Number: ______________________

Section (circle): 1 2 3 4 5 6 7 8 9

MATH 135, Algebra for Honours Mathematics

Faculty of Mathematics, University of Waterloo
Final Examination, Fall Term 2009
Date: Friday, December 18th
Time: 12:30 – 3:00 pm

Section	Time	Instructor
1	10:30-11:20	C. Hewitt
2	12:30-1:20	E. Teske
3	9:30-10:20	S. Furino
4	10:30-11:20	E. Teske
5	11:30-12:20	S. New
6	1:30-2:20	Y.-R. Liu
7	2:30-3:20	R. Moosa
8	12:30-1:20	J. Koeller
9	8:30-9:20	J. Koeller

Pages: This test contains 9 pages, including this cover sheet and a page at the end for rough work.

Instructions: Write your name, signature and ID number, and circle your section, at the top of this page. Answer all questions, and provide **full explanations**. If you need more space to show your work, then use the back of the previous page.

Aids: Only faculty approved calculators are allowed.

Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/ 10
Total	/70

[6] **1:** (a) Let $a_0 = 1$ and $a_1 = 3$, and for $n \geq 2$ let $a_n = 3a_{n-1} - 2a_{n-2} - 1$. Show that $a_n = 2^n + n$ for all $n \geq 0$.

[4] (b) Find the term containing x^8 in the binomial expansion of $\left(\frac{18}{x} - \frac{x^2}{3}\right)^7$.

[3] **2:** (a) Let $a = -215$ and $b = 17$. Find the integers q and r wth $0 \leq r < b$ such that $a = qb + r$.

[7] (b) List all pairs of integers (x, y) with $|x| \leq 50$ such that $245x + 189y = 84$.

[4] **3:** (a) List all elements $[x] \in \mathbf{Z}_{13}$ such that $[5][x]^2 = [6]$.

[6] (b) Solve the pair of congruences $x \equiv 5 \pmod 9$ and $10x \equiv 6 \pmod{28}$.

[5] **4:** (a) Use the Square and Multiply Algorithm to encrypt the message $m = 4$ using the RSA public key $(n, e) = (253, 29)$.

[5] (b) Determine the private key (n, d) which corresponds to the public key $(n, e) = (253, 29)$.

[2] **5:** (a) Define $\phi(n)$, where n is a positive integer and ϕ is the Euler phi function.

[3] (b) State the Chinese Remainder Theorem.

[5] (c) Let $n = pq$ where p and q are distinct primes, and let $\phi = \phi(n) = (p-1)(q-1)$. Prove that for all integers a we have $a^{\phi+1} \equiv a \pmod{n}$. (This is part of Proposition 7.41).

[5] **6:** (a) Determine the number of positive integers a such that $a \big| 9!$ and $\gcd(a, 3600) = 180$.

[5] (b) Prove that $\gcd\left(5^{98} + 3, 5^{99} + 1\right) = 14$.

[3] **7:** (a) Simplify $z = \dfrac{(1+3i)^2 + (5-i)}{(1+i)}$.

[3] (b) Solve $z = \dfrac{1+8i}{2-z}$ for $z \in \mathbf{C}$.

[4] (c) Solve $z^5 + 16\,\overline{z} = 0$ for $z \in \mathbf{C}$. Draw a picture showing all of the solutions.

This page may be used for rough work. It will not be marked.