MATH 135 Algebra, Solutions to Assignment 7

1: (a) Find the smallest non-negative integer x such that $x \equiv 41 \pmod 9$.

Solution: The smallest such x is the remainder when 41 is divided by 9. We have $41 = 9 \cdot 4 + 5$, so $x = 5$.

(b) Find the integer x which has the smallest absolute value such that $x \equiv 568 \pmod{41}$.

Solution: We have $568 = 41 \cdot 13 + 35$ and so $568 \equiv 35 \pmod{41}$. Thus we have

$$x = 568 \pmod{41} \iff x \equiv 35 \pmod{41} \iff x \in \{\cdots, -47, -6, 35, 76, \cdots\}.$$

Thus the integer x with the smallest absolute value such that $x \equiv 568 \pmod{41}$ is $x = -6$.

(c) What day of the week will it be 1000 days after a Monday?

Solution: We have $1000 = 7 \cdot 142 + 6$ so $1000 \equiv 6 \pmod 7$. Thus 1000 days after a Monday, it will be the same day of the week as it is 6 days after a Monday, that is, it will be Sunday.

(d) What time of day will it be 1000 hours after 5:00 pm?

Solution: We have $1000 = 24 \cdot 41 + 16$ so $1000 \equiv 16 \pmod{24}$. Thus 1000 hours after 5:00 pm it will be the same time of day as it is 16 hours after 5:00 pm, that is, it will be 9:00 am.

(e) Exactly what time of day will it be 1 million seconds after 5:00 pm?

Solution: We have $1,000,000 = 60 \cdot 16,666 + 40$ so 1 million seconds is equal to 16,666 minutes and 40 seconds. Also, we have $16,666 = 60 \cdot 277 + 46$, so 16,666 minutes is equal to 277 hours and 46 minutes. Finally, we have $277 = 24 \cdot 11 + 13$ so 277 hours is equal to 11 days and 13 hours. Thus 1 million seconds is equal to 11 days, 13 hours, 46 minutes and 40 seconds. It follows that 1 million seconds after 5:00 pm, it will be the same time of day as it is 13 hours, 46 minutes and 40 seconds after 5:00 pm, that is it will be exactly 40 seconds past 6:46 am.

2: (a) Find all positive integers m such that $126 \equiv 35 \pmod{m}$.

Solution: We have

$$126 \equiv 35 \pmod{m} \iff m\,\big|\,(126-35) \iff m\,\big|\,91 \iff m\,\big|\,7\cdot 13 \iff m = 1, 7, 13 \text{ or } 91\,.$$

(b) Find the remainder when the integer $\sum_{k=1}^{100} k!$ is divided by 13.

Solution: We have $1! = 1 \equiv 1 \pmod{13}$, $2! = 2 \equiv 2 \pmod{13}$, $3! = 6 \equiv 6 \pmod{13}$, $4! = 24 \equiv 11 \pmod{13}$, $5! = 5\cdot 4! \equiv 5\cdot 11 \equiv 55 \equiv 3 \pmod{13}$, $6! = 6\cdot 5! \equiv 6\cdot 3 \equiv 18 \equiv 5 \pmod{13}$, $7! = 7\cdot 6! \equiv 7\cdot 5 \equiv 35 \equiv 9 \pmod{13}$, $8! = 8\cdot 7! \equiv 8\cdot 9 \equiv 7 \pmod{13}$, $9! = 9\cdot 8! \equiv 9\cdot 7 \equiv 11 \pmod{13}$, $10! = 10\cdot 9! \equiv 10\cdot 11 \equiv 6 \pmod{13}$, $11! = 11\cdot 10! \equiv 11\cdot 6 \equiv 1 \pmod{13}$, $12! = 12\cdot 11! \equiv 12\cdot 1 \equiv 12 \pmod{13}$, and for all $k \geq 13$ we have $13\,\big|\,k!$ so $k! \equiv 0 \pmod{13}$. Thus

$$\begin{aligned}\sum_{k=1}^{100} k! &= 1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10! + 11! + 12! \\ &= 1 + 2 + 6 + 11 + 3 + 5 + 9 + 7 + 11 + 6 + 1 + 12 \\ &\equiv 9 \pmod{13}\,.\end{aligned}$$

(c) Find the remainder when the integer $\dfrac{40!}{2^{20}\cdot 20!}$ is divided by 8.

Solution: Using the formula that we found in Assignment 6, Problem 1(a), we have

$$\begin{aligned}40! &= 2^{20+10+5+2+1}\cdot 3^{13+4+1}\cdot 5^{8+1}\cdot 7^5\cdot 11^3\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37 \\ 20! &= 2^{10+5+2+1}\cdot 3^{6+2}\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\end{aligned}$$

and so

$$\begin{aligned}\frac{40!}{2^{20}20!} &= 3^{10}\cdot 5^5\cdot 7^3\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37 \\ &\equiv 3^{10}\cdot 5^5\cdot (-1)^3\cdot 3^2\cdot 5^2\cdot 1\cdot 3\cdot (-1)\cdot 5\cdot (-1)\cdot 5 \pmod{8} \\ &\equiv -3^{13}\cdot 5^9 \pmod{8}\,.\end{aligned}$$

Note that $3^2 = 9 \equiv 1 \pmod{8}$, so $3^{13} \equiv 3\cdot (3^2)^6 \equiv 3 \pmod{8}$, and similarly $5^2 = 25 \equiv 1 \pmod{8}$ so $5^9 = 5\cdot (5^2)^4 \equiv 5 \pmod{8}$. Thus

$$\frac{40!}{2^{20}20!} \equiv -3\cdot 5 \equiv -15 \equiv 1 \pmod{8}\,.$$

3: Most recent books are identified by their *International Standard Book Number*, or ISBN, which is a 10-digit number, separated into four blocks. The ISBN for our course text book is 0-13-184868-2. Here the first block of digits, 0, represents the language of the book (English), the second block, 13, represents the publisher (Pearson Prentice Hall), the third block, 184868, is the number assigned to the book by the publisher, and the last block, 2, is the *check digit*. If $a_1, a_2, \cdots, a_{10}$ are the 10-digits of the ISBN then for $1 \leq i \leq 9$ we have $a_i \in \{0, 1, 2, \cdots, 9\}$ while $a_{10} \in \{0, 1, 2, \cdots, 9, X\}$. The check digit a_{10} is used to determine whether an error has been made when an ISBN is copied. It is chosen so that

$$\sum_{i=1}^{10} i\, a_i \equiv 0 \pmod{11}.$$

(a) Determine whether the number 2-14-013862-5 is a valid ISBN.

Solution: For this ISBN we have

$$\begin{aligned}\sum_{i=1}^{10} i\, a_i &= 1\cdot 2 + 2\cdot 1 + 3\cdot 4 + 4\cdot 0 + 5\cdot 1 + 6\cdot 3 + 7\cdot 8 + 8\cdot 6 + 9\cdot 2 + 10\cdot 5 \\ &\equiv 2 + 2 + 1 + 0 + 5 + 7 + 1 + 4 + 7 + 6 \pmod{11} \\ &\equiv 2 \pmod{11}\end{aligned}$$

so this is not a valid ISBN.

(b) Determine the value of the digit a such that the number 1-29-14a238-X is a valid ISBN.

Solution: For thiss ISBN we have

$$\begin{aligned}\sum_{i=1}^{10} i\, a_i &= 1\cdot 1 + 2\cdot 2 + 3\cdot 9 + 4\cdot 1 + 5\cdot 4 + 6a + 7\cdot 2 + 8\cdot 3 + 9\cdot 8 + 10\cdot 10 \\ &\equiv 1 + 4 + 5 + 4 + 9 + 6a + 3 + 2 + 6 + 1 \pmod{11} \\ &\equiv 2 + 6a \pmod{11}.\end{aligned}$$

To be a valid ISBN we need $6a \equiv -2 \pmod{11}$, that is $6a = \cdots, -13, -2, 9, 20, 31, 42, \cdots$, so we can take $6a = 42$, that is $a = 7$.

(c) When an ISBN was copied, two adjacent digits were interchanged resulting in the number 0-07-286593-4. Determine the original ISBN.

Solution: For this ISBN we have

$$\begin{aligned}\sum_{i=1}^{10} i\, a_i &= 1\cdot 0 + 2\cdot 0 + 3\cdot 7 + 4\cdot 2 + 5\cdot 8 + 6\cdot 6 + 7\cdot 5 + 8\cdot 9 + 9\cdot 3 + 10\cdot 4 \\ &\equiv 0 + 0 + 10 + 8 + 7 + 3 + 2 + 6 + 5 + 7 \pmod{11} \\ &\equiv 4 \pmod{11}.\end{aligned}$$

When we interchange a_k with a_{k+1}, the sum changes by the amount

$$\big(ka_{k+1} + (k+1)a_k\big) - \big(ka_k + (k+1)a_{k+1}\big) = a_k - a_{k+1}$$

so we need to find k so that $a_k - a_{k+1} \equiv -4 \pmod{11}$, that is $a_{k+1} = a_k + 4$ or $a_{k+1} = a_k - 7$. The only such index k is $k = 7$, so we interchange a_7 and a_8 to get the original ISBN 0-07-286953-4.

4: (a) Use mathematical induction to show that $5^n \equiv 1+4n \pmod{16}$ for all integers $n \geq 0$. We claim that $5^n \equiv 1+4n \pmod{16}$ for all $n \geq 0$. When $n=0$ we have $5^n = 5^0 = 1$ and $1+4n = 1+4\cdot 0 = 1$ so the claim is true when $n=0$. Let $k \geq 0$ and suppose the claim is true when $n=k$, that is suppose that $5^k \equiv 1+4k \pmod{16}$. Then when $n=k+1$ we have

$$5^n = 5^{k+1} = 5\cdot 5^k \equiv 5(1+4k) \equiv 5+20k \equiv 5+4k \equiv 1+4+4k \equiv 1+4(k+1) \equiv 1+4n \pmod{16}$$

so the claim is true when $n=k+1$. Thus by Mathematical Induction, the claim is true for all $n \geq 0$.

(b) Show that if $n \equiv 3 \pmod 4$ then n is not the sum of two squares.

Solution: Suppose that n is the sum of two squares, say $n = x^2+y^2$. We make a table of squares modulo 4.

x	0	1	2	3
x^2	0	1	0	1

From the table we see that $x^2 \equiv 0$ or $1 \pmod 4$. Similarly we have $y^2 \equiv 0$ or $1 \pmod 4$. It follows that $n = x^2+y^2 \equiv 0+0,\ 0+1,\ 1+0$ or $1+1 \pmod 4$, that is $n \equiv 0, 1$ or $2 \pmod 4$. Thus $n \not\equiv 3 \pmod 4$.

(c) Show that there is no perfect square whose last three digits are 341.

Solution: Let x be any integer. Note that if x^2 ended with the digits 341, then we would have $x^2 = 1000k+341$ for some integer k, so $x^2 = 1000k+341 \equiv 341 \equiv 5 \pmod 8$. We make a table of squares modulo 8.

x	0	1	2	3	4	5	6	7
x^2	0	1	4	1	0	1	4	1

From the table we see that $x^2 \equiv 0, 1$ or $4 \pmod 8$. Thus x^2 cannot end with the digits 341.

5: (a) Find all possible pairs of digits (a,b) such that $99\big|38a91b$.

Solution: Note that $99\big|38a91b$ implies that $9\big|38a91b$ and $11\big|38191b$. We have

$$9\big|38a91b \Longrightarrow 9\big|(3+8+a+9+1+b) \Longrightarrow a+b\equiv 6 \pmod 9 \Longrightarrow a+b=6 \text{ or } 15\,,$$

and

$$11\big|38a91b \Longrightarrow 3-8+a-9+1-b \Longrightarrow a-b\equiv 2 \pmod{11} \Longrightarrow a-b=2 \text{ or } \;-9\,.$$

The only pair (a,b) with $a-b=-9$ is the pair $(a,b)=(0,9)$, but for this pair we have $a+b=9$, so it does not satisfy the condition that $a+b=6$ or 15. The only pairs (a,b) with $a-b=2$ are the pairs $(a,b)=(2,0),(3,1),(4,2),\cdots,(9,7)$. Of these 8 pairs, only the pair $(a,b)=(4,2)$ satisfies the condition $a+b=6$ or 15. Thus $(a,b)=(4,2)$ is the only such pair.

(b) Show that it is not possible to rearrange the digits of the number 51328167 to form a perfect square or a perfect cube or any higher perfect power.

Solution: If we rearrange the digits of 51328167 in any way, to form a number a, then we have $3\big|a$ since $5+1+3+2+8+1+6+7=33\equiv 0 \pmod 3$, but $9\not\big|a$ since $33\not\equiv 0 \pmod 9$. Thus the exponent of 3 in the prime factorization of a is equal to 1, so a cannot be a square or a cube or any higher perfect power.

(c) Let $n=a_0+a_1\cdot 1000+a_2\cdot 1000^2+\cdots+a_l\cdot 1000^l$ where $a_l\neq 0$ and for each i we have $0\leq a_i<1000$. Show that for $d=7$, 11 and 13 we have

$$d\big|n \Longleftrightarrow d\big|\big(a_0-a_1+a_2-a_3+\cdots+(-1)^l a_l\big)\,.$$

Solution: Let $n=a_0+a_1\cdot 1000+a_2\cdot 1000^2+\cdots+a_l\cdot 1000^l$ where $a_l\neq 0$ and for each i we have $0\leq a_i<1000$. Notice that $1001=7\cdot 11\cdot 13$, so for $d=7$, 11 or 13, we have $1000=-1 \pmod d$, and so

$$\begin{aligned} n &= a_0+a_1\cdot 1000+a_2\cdot 1000^2+\cdots+a_l\cdot 1000^l \\ &\equiv a_0+a_1(-1)+a_2(-1)^2+\cdots+a_l(-1)^l \pmod d \\ &\equiv a_0-a_1+a_2-a_3+\cdots+(-1)^l a_l \pmod d \end{aligned}$$

and

$$\begin{aligned} d\big|n &\Longleftrightarrow n\equiv 0 \pmod d \\ &\Longleftrightarrow a_0-a_1+a_2-a_3+\cdots+(-1)^l a_l\equiv 0 \pmod d \\ &\Longleftrightarrow d\big|\big(a_0-a_1+a_2-a_3+\cdots+(-1)^l a_l\big)\,. \end{aligned}$$