MATH 135 Algebra, Assignment 6 — Due: Wed Oct 28, 8:30 am

1: Find the prime factorization of each of the following integers.

(a) $30!$

(b) $\binom{30}{10}$

(c) $2^{36} - 1$

2: (a) Let $a = 8400$. Find the number of positive factors of a.

(b) Find the number of positive integers whose prime factors are 2, 3 and 5 and which have exactly 100 positive divisors.

(c) Let $a = \prod_{k=1}^{6} k^k$. Find the number of factors (positive or negative) of a which are either perfect squares or perfect cubes (or both).

3: In parts (a) and (b), find the prime factorization of $\gcd(a, b)$ and of $\operatorname{lcm}(a, b)$.

(a) $a = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $b = 2^2 \cdot 5^3 \cdot 7 \cdot 11$

(b) $a = 25!$ and $b = (5500)^3(1001)^2$.

(c) Find the number of pairs of positive integers (a, b) with $a \leq b$ such that $\gcd(a, b) = 60$ and $\operatorname{lcm}(a, b) = 4200$.

4: (a) Show that for all positive integers a and b we have $a \big| b$ if and only if $a^2 \big| b^2$.

(b) Show that for all positive integers a, b and c, if $c \big| ab$ then $c \big| \gcd(a, c) \gcd(b, c)$.

(c) Show that for all positive integers a and b we have $\gcd(a, b) = \gcd\big(a + b, \operatorname{lcm}(a, b)\big)$.

5: A **Hilbert number** is a positive integer of the form $n = 1 + 4k$ for some integer $k \geq 0$. A **Hilbert prime** is a Hilbert number $n > 1$ whose only Hilbert number factors are 1 and n.

(a) List the first 20 Hilbert primes.

(b) Show that every Hilbert number greater than 1 is either a Hilbert prime or a product of Hilbert primes.

(c) Show that the factorization of a Hilbert number into Hilbert primes is not always unique.