MATH 135 Algebra, Solutions to Assignment 5

1: Solve each of the following linear diophantine equations.

(a) $42x + 30y = 24$

Solution: The Euclidean Algorithm gives

$$42 = 30 \cdot 1 + 12\ .\ \ 30 = 12 \cdot 2 + 6\ ,\ \ 12 = 6 \cdot 2 + 0$$

so we have $\gcd(42, 30) = 6$. Back-Substitution then gives

$$1\ ,\ \ -2\ ,\ \ 3$$

so we have $42(-2)+30(3) = 6$. Note that $\frac{24}{6} = 4$, and multiplying both sides by 4 gives $42(-8)+30(12) = 24$, and so one solution is $(x, y) = (-8, 12)$. Note that $\frac{42}{6} = 7$ and $\frac{30}{6} = 5$, so by the Linear Diophantine Equation Theorem, the general solution is

$$(x, y) = (-8, 12) + k(-5, 7)\ ,\ \ k \in \mathbf{Z}\,.$$

(b) $231x + 792y = 513$

Solution: The Euclidean Algorithm gives

$$792 = 3 \cdot 231 + 99\ ,\ \ 231 = 2 \cdot 99 + 33\ ,\ \ 99 = 3 \cdot 33 + 0$$

and so $\gcd(231, 792) = 33$. Back-Substitution gives

$$1\ ,\ \ -2\ ,\ \ 7$$

and so $231(7) + 792(-2) = 33$. Note that $513 = 33 \cdot 15 + 18$, so 33 does not divide 513, and hence there is no solution (by Proposition 2.11(ii) or by the Linear Diophantine Equation Theorem).

(c) $385x - 1183y = 294$

Solution: The Euclidean Algorithm gives

$$1183 = 3 \cdot 385 + 28\ ,\ \ 385 = 13 \cdot 28 + 21\ ,\ \ 28 = 1 \cdot 21 + 7\ ,\ \ 21 = 3 \cdot 7 + 0$$

and so $\gcd(385, 1183) = 7$. Back Substitution gives

$$1\ ,\ \ -1\ ,\ \ 14\ ,\ \ -43$$

and so we have $385(-43) + 1183(14) = 7$. Note that $\frac{294}{7} = 42$, and multiplying both sides by 42 gives $385(-1806) - 1183(-588) = 294$. Thus one solution is $(x, y) = (-1806, -588)$. Note that $\frac{385}{7} = 55$ and $\frac{1183}{7} = 169$, and so by the Linear Diophantine Equation Theorem, the general solution is

$$(x, y) = (-1806, -588) + k(169, 55)\ ,\ \ k \in \mathbf{Z}$$

2: (a) Find all non-negative solutions to the diophantine equation $483x + 336y = 9513$.

Solution: The Euclidean Algorithm gives

$$483 = 1 \cdot 336 + 147\ ,\quad 336 = 2 \cdot 147 + 42\ ,\quad 147 = 3 \cdot 42 + 21\ ,\quad 42 = 2 \cdot 21 + 0$$

so $\gcd(483, 336) = 21$. Back-Substitution gives

$$1\ ,\quad -3\ ,\quad 7\ ,\quad -10$$

so we have $483(7) + 336(-10) = 21$. Note that $\frac{9513}{21} = 453$ and multiplying both sides of the equation by 453 gives $483(3171) + 336(-4530) = 9513$. Thus one solution is $(x, y) = (3171, -4530)$. Note that $\frac{483}{21} = 23$ and $\frac{336}{21} = 16$, so by the Linear Diophantine Equation Theorem, the general solution is

$$(x, y) = (3171, -4530) + k(-16, 23)\ ,\ k \in \mathbf{Z}\,.$$

Note that

$$x \geq 0 \Longrightarrow 3171 - 16k \geq 0 \Longrightarrow 16k \leq 3171 \Longrightarrow k \leq \left\lfloor \tfrac{3171}{16} \right\rfloor = 198$$
$$y \geq 0 \Longrightarrow -4530 + 23k \geq 0 \Longrightarrow 23k \geq 4530 \Longrightarrow k \geq \left\lceil \tfrac{4530}{23} \right\rceil = 197$$

so we obtain non-negative solutions when $k = 197$ and 198. The solutions are

$$(x, y) = (19, 1)\,,\ (3, 24)\,.$$

(b) Find all pairs of integers (x, y) with $x \geq 1000$, $y \leq 1000$ such that $726x - 1578y = 324$.

Solution: The Euclidean Algorithm gives

$$1578 = 2 \cdot 726 + 126\ ,\quad 726 = 5 \cdot 126 + 96\ ,\quad 126 = 1 \cdot 96 + 30\ ,\quad 96 = 3 \cdot 30 + 6\ ,\quad 30 = 5 \cdot 6 + 0$$

so we have $\gcd(726, 1578) = 6$. Back-Substitution gives

$$1\ ,\quad -3\ ,\quad 4\ ,\quad -23\ ,\quad 50$$

so we have $726(50) - 1578(23) = 6$. Note that $\frac{324}{6} = 54$ and multiplying both sides of the equation by 54 gives $726(2700) - 1578(1242) = 324$, and so one solution is $(x, y) = (2700, 1242)$. Note that $\frac{726}{6} = 121$ and $\frac{1578}{6} = 263$, and so by the Linear Diophantine Equation Theorem, the general solution is

$$(x, y) = (2700, 1242) + k(263, 121)\ ,\ k \in \mathbf{Z}\,.$$

Note that

$$x \geq 1000 \Longrightarrow 2700 + 263k \geq 1000 \Longrightarrow 263k \geq -1700 \Longrightarrow k \geq \left\lceil -\tfrac{1700}{263} \right\rceil = -6$$
$$y \leq 1000 \Longrightarrow 1242 + 121k \leq 1000 \Longrightarrow 121k \leq -242 \Longrightarrow k \leq \left\lfloor -\tfrac{242}{121} \right\rfloor = -2\,,$$

so we obtain solutions with $x \geq 1000$ and $y \leq 1000$ when $k = -6, -5, -4, -3, -2$. The solutions are

$$(x, y) = (1122, 516), (1385, 637), (1648, 758), (1911, 879), (2174, 1000)\,.$$

3: (a) What combinations of 18- and 33-cent stamps can be used to mail a package which requires postage of 6 dollars.

Solution: Let x be the number of 18-cent stamps and let y be the number of 33-cent stamps. Then the stamps are worth 6 dollars when

$$18x + 33y = 600\,.$$

We look for non-negative integer solutions to this equation. The Euclidean Algorithm gives

$$33 = 1\cdot 18 + 15\ ,\quad 18 = 1\cdot 15 + 3\ ,\quad 15 = 5\cdot 3 + 0$$

so we have $\gcd(18, 33) = 3$. Back-Substitution gives

$$1\ ,\quad -1\ ,\quad 2$$

so we have $18(2) + 33(-1) = 3$. Note that $\frac{600}{3} = 200$, and multiplying both sides of the equation by 200 gives $18(400) + 33(-200) = 600$, and so one solution is $(x, y) = (400, -200)$. Note that $\frac{18}{3} = 6$ and $\frac{33}{3} = 11$, so by the Linear Diophantine Equation Theorem, the general solution is

$$(x, y) = (400, -200) + k(-11, 6)\ ,\ \ k \in \mathbf{Z}\,.$$

Note that

$$x \geq 0 \Longrightarrow 400 - 11k \geq 0 \Longrightarrow 11k \leq 400 \Longrightarrow k \leq \left\lfloor \tfrac{400}{11} \right\rfloor = 36$$
$$y \geq 0 \Longrightarrow -200 + 6k \geq 0 \Longrightarrow 6k \geq 200 \Longrightarrow k \geq \left\lceil \tfrac{200}{6} \right\rceil = 34\,,$$

so we obtain non-negative solutions when $k = 34, 25, 26$. Thus there are three pairs (x, y) such that x 18-cents stamps and y 33-cent stamps are worth 6 dollars; namely

$$(x, y) = (26, 4), (15, 10), (4, 16)\,.$$

(b) A shopper spends \$19.81 to buy some apples which cost 35 cents each and some oranges which cost 56 cents each. What is the minimum number of pieces of fruit that the shopper could have bought.

Solution: Let x be the number of apples purchased and let y be the number of oranges purchased. The fruit is worth\$ 19.81 when we have

$$35x + 56y = 1981\,.$$

The Euclidean Algorithm gives

$$56 = 1\cdot 35 + 21\ ,\quad 35 = 1\cdot 21 + 14\ ,\quad 21 = 1\cdot 14 + 7\ ,\quad 14 = 2\cdot 7 + 0$$

so we have $\gcd(35, 56) = 7$. Back-Substitution gives

$$1\ ,\quad -1\ ,\quad 2\ ,\quad -3$$

so we have $35(-3) + 56(2) = 7$. Note that $\frac{1981}{7} = 283$ and multiplying both sides of the equation by 93 gives $35(-849) + 56(566) = 1981$, and so one solution is $(x, y) = (-849, 566)$. Note that $\frac{35}{7} = 5$ and $\frac{56}{7} = 8$, and so by the Linear Diophantine Equation Theorem, the general solution is

$$(x, y) = (-849, 566) + k(-8, 5)\ ,\ \ k \in \mathbf{Z}\,.$$

Note that

$$x \geq 0 \Longrightarrow -849 - 8k \geq 0 \Longrightarrow 8k \leq -849 \Longrightarrow k \leq \left\lfloor -\tfrac{849}{8} \right\rfloor = -107$$
$$y \geq 0 \Longrightarrow 566 + 5k \geq 0 \Longrightarrow 5k \geq -566 \Longrightarrow k \geq \left\lceil -\frac{566}{5} \right\rceil = -113\,,$$

so we obtain non-negative solutions when $-107 \leq k \leq 113$. We wish to choose the value of k which minimizes $x + y$ (the total number of pieces of fruit purchased). Note that

$$x + y = -849 - 8k + 566 + 5k = -283 - 3k\,,$$

so to minimize $x + y$ we must choose the maximum possible value of k, that is $k = -107$. When $k = -107$ we have $x + y = -283 - 3k = 38$. Thus the minimum number of pieces of fruit is 38.

4: We can solve a pair of linear diophantine equations in three variables by first eliminating one of the variables and solving the resulting equation in the remaining two variables.

(a) Show that there is no solution to the pair of diophantine equations

$$\begin{aligned} 2x + 7y + z &= 45 \\ 3x + 8y + 4z &= 21\,. \end{aligned}$$

Solution: Multiply the first equation by 4 and subtract the second equation to get $5x + 20y = 159$. Notice that $\gcd(5, 20) = 5$ and 5 does not divide 159, so there is no solution.

(b) Find all solutions to the pair of diophantine equations

$$\begin{aligned} 20x + 12y + 15z &= 85 \qquad (1) \\ 18x + 20y + 8z &= 110 \qquad (2) \end{aligned}$$

Solution: To eliminate z, multiply (2) by 15 and subtract 8 times (1). This gives

$$110x + 204y = 970 \qquad (3)$$

The Euclidean Algorithm gives

$$205 = 1 \cdot 110 + 94\,, \quad 110 = 1 \cdot 94 + 16\,, \quad 94 = 5 \cdot 16 + 14\,, \quad 16 = 1 \cdot 14 + 2\,, \quad 14 = 7 \cdot 2 + 0$$

so we have $\gcd(110, 204) = 2$. Back-Substitution gives

$$1\,, \quad -1\,, \quad 6\,, \quad -7\,, \quad 13$$

so we have $110(13) + 204(-7) = 2$. Note that $\frac{970}{2} = 485$, and multiplying both sides of the previous equation by 485 gives $110(6305) + 204(-3395) = 970$, and so one solution is $(x, y) = (6305, -3395)$. Note that $\frac{110}{2} = 55$ and $\frac{204}{2} = 102$, and so by the Linear Diophantine Equation Theorem, the general solution to equation (3) is

$$(x, y) = (6305, -3395) + k(-102, 55)\ ,\ k \in \mathbf{Z}$$

Notice that taking $k = 62$ gives the solution $(x, y) = (-19, 15)$, so the general solution to (3) is also given by

$$(x, y) = (-19, 15) + k(-102, 55)\ ,\ k \in \mathbf{Z}\,.$$

Put $x = -19 - 102k$ and $y = 15 + 55k$ into (1) to get

$$20(-19 - 102k) + 12(15 + 55k) + 15z = 85$$

that is

$$-1380k + 15z = 285 \qquad (4)$$

We don't need to use the Euclidean Algorithm with Back-Substitution to solve this diophantine equation beacause $15 \big| 1380$. By inspection, one solution is $(k, z) = (0, 19)$, and since $\frac{1380}{15} = 92$, the general solution is

$$(k, z) = (0, 19) + l(1, 92)\ ,\ l \in \mathbf{Z}\,.$$

The complete solution to the pair of equations (1) and (2) is given by

$$\begin{aligned} x &= -19 - 102k = -19 - 102l \\ y &= 15 + 55k = 15 + 55l \\ z &= 19 + 92l \end{aligned}$$

or equivalently

$$(x, y, z) = (-19, 15, 19) + l(-102, 55, 92)\ ,\ l \in \mathbf{Z}\,.$$

5: Let a, b and c be non-zero integers. The **greatest common divisor** $d = \gcd(a, b, c)$ is defined to be the largest positive integer d such that $d\big|a$, $d\big|b$ and $d\big|c$.

(a) Show that $\gcd(a, b, c) = \gcd\big(\gcd(a, b), c\big)$.

Solution: Let $d = \gcd(a, b, c)$, $e = \gcd(a, b)$ and $f = \gcd(e, c)$. Since d is a common divisor of a and b, we have $d\big|e$ by Proposition 2.29. Thus d is a common divisor of e and c, so (since f is the greatest common divisor of e and c) we must have $d \leq f$. On the other hand, since $f\big|e$ and $e\big|a$ we have $f\big|a$, and since $f\big|e$ and $e\big|b$ we have $f\big|b$. Thus f is a common divisor of a and b, and f also divides c, so (since d is the greatest common divisor of a, b and c), we must have $f \leq d$.

(b) Show that for any integer e, the linear diophantine $ax + by + cz = e$ has a solution if and only if $\gcd(a, b, c)\big|e$.

Solution: Suppose first that $ax + by + cz = e$ has a solution, say $as + bt + cu = e$, and let $d = \gcd(a, b, c)$. Since $d\big|a$, $d\big|b$ and $d\big|c$, we can choose k, l and m so that $a = dk$, $b = dl$ and $c = dm$. Then

$$as + bt + cu = e \implies dks + dlt + dmu = e \implies d(ks + lt + mu) = e$$

and so $d\big|e$. Conversely, suppose that $d\big|e$ where again we let $d = \gcd(a, b, c)$. Using the Euclidean Algorithm with Back-Substitution, we can choose integers s and t such that $as + bt = \gcd(a, b)$. Also, since we have $d = \gcd\big(\gcd(a, b), c\big)$ by part (a), and $d\big|e$ so we have $\gcd\big(\gcd(a, b), c\big)\Big|e$, we can choose integers u and v so that $\gcd(a, b)u + cv = e$ by the Linear Diophantine Equation Theorem. Since $as + bt = \gcd(a, b)$ and $\gcd(a, b)u + cv = e$, we have $asu + btu + cv = e$, so the diophantine equation $ax + by + cz = e$ does indeed have a solution.

(c) Find all solutions to the linear diophantine equation $42x + 70y + 105z = 63$.

Solution: By the Linear Diophantine Equation Theorem, for any fixed value of z, in order for the diophantine equation $42x+70y = 63-107z$ to have a solution (x, y), we must have $\gcd(42, 70)\big|(63-105z)$. The Euclidean Algorithm gives

$$70 = 1 \cdot 42 + 28\ ,\quad 42 = 1 \cdot 28 + 14\ ,\quad 28 = 2 \cdot 14 + 0$$

so we have $\gcd(42, 70) = 14$. To have a solution, we need to have $14\big|(63 - 105z)$, that is we need to have $63 - 105z = 14k$ for some $k \in \mathbf{Z}$. Let us solve the diophantine equation $14k + 105z = 63$. The Euclidean Algorithm gives $105 = 14{\cdot}7+7$ and $14 = 7{\cdot}2+0$, so we have $\gcd(14, 105) = 7$. Back-Substitution immediately shows that $14(-7) + 105(1) = 7$, and we multiply both sides by 9 to get $14(-63) + 105(9) = 63$. Thus one solution to the diophantine equation $14k + 105z = 63$ is given by $(k, z) = (-63, 9)$. Note that $\frac{14}{7} = 2$ and $\frac{105}{7} = 15$, so the general solution is $(k, z) = (-63, 9) + l(-15, 2)\ ,\ l \in \mathbf{Z}$. Taking $l = -4$ gives $(k, z) = (-3, 1)$, so we can also say that the general solution to the diophantine equation $14k + 105z = 63$ is

$$(k, z) = (-3, 1) + l(-15, 2)\ ,\ l \in \mathbf{Z}\,.$$

Thus the original diophantine equation $42x + 70y + 105z = 63$ has a solution when $z = 1 + 2l$ for some $l \in \mathbf{Z}$. Now fix $z = 1 + 2l$. The original diophantine equation becomes $42x + 70y + 105(1 + 2l) = 63$, or equivalently

$$42x + 70y = -42 - 210l\,.$$

Let us solve this. We applied the Euclidean Algorithm earlier to show that $\gcd(42, 70) = 14$, and now Back-Substitution gives the sequence $1, -1, 2$, so we have $42(2) + 70(-1) = 14$. Note that $\frac{-42-210l}{14} = (-3 - 15l)$, so we multiply both sides by $(-3 - 15l)$ to get $42(-6 - 30l) + 70(3 + 15l) = -42 - 210l$. Thus one solution is $(x, y) = (-6 - 30l, 3 + 15l)$. Note that $\frac{42}{14} = 3$ and $\frac{70}{14} = 5$, so the general solution is

$$(x, y) = (-6 - 30l, 3 + 15l) + m(-5, 3)\ ,\ m \in \mathbf{Z}\,.$$

Since we also have $z = 1 + 2l$, the general solution to the original diophantine equation is

$$(x, y, z) = (-6, 3, 1) + l(-30, 15, 2) + m(-5, 3, 0)\ ,\ l, m \in \mathbf{Z}\,.$$

We remark that there are many equivalent ways to express this result, for example we could also write

$$(x, y, z) = (-1, 0, 1) + l(0, 3, -2) + m(5, 0, -2)\ ,\ l, m \in \mathbf{Z}\,.$$