MATH 135 Algebra, Solutions to Assignment 3

1: (a) Let $a_1 = 1$ and $a_{n+1} = 3a_n + 2$ for $n \geq 1$. Show that $a_n = 2 \cdot 3^{n-1} - 1$ for all $n \geq 1$.

Solution: We claim that $a_n = 2 \cdot 3^{n-1} - 1$ for all $n \geq 1$. When $n = 1$ we have $a_n = a_1 = 1$ and $2 \cdot 3^{n-1} - 1 = 2 \cdot 3^0 - 1 = 2 \cdot 1 - 1 = 1$ so the claim is true when $n = 1$. Let $k \geq 1$ and suppose the claim is true when $n = k$, that is suppose that $a_k = 2 \cdot 3^{k-1} - 1$. Then when $n = k+1$ we have

$$a_n = a_{k+1} = 3a_k + 2 = 3\big(2 \cdot 3^{k-1} - 1\big) + 2 = 2 \cdot 3^k - 3 + 2 = 2 \cdot 3^k - 1 = 2 \cdot 3^{n-1} - 1\,.$$

Thus the claim is true when $n = k+1$. By Mathematical Induction, $a_n = 2 \cdot 3^{n-1} - 1$ for all $n \geq 1$.

(b) Let $a_1 = 3$ and $a_{n+1} = 2a_n - 1$ for $n \geq 1$. Find a closed form formula for a_n.

Solution: Using the given recursion formula, we find that $a_1 = 3$, $a_2 = 5$, $a_3 = 9$, $a_4 = 17$ and $a_5 = 33$. Notice that $a_n = 2^n + 1$ for $n = 1, 2, 3, 4, 5$. We claim that $a_n = 2^n + 1$ for all $n \geq 1$. When $n = 1$ the claim is true. Let $k \geq 1$ and suppose that the claim is true when $n = k$, that is suppose that $a_k = 2^k + 1$. Then when $n = k+1$ we have

$$a_n = a_{k+1} = 2 \cdot a_k - 1 = 2(2^k + 1) - 1 = 2^{k+1} + 2 - 1 = 2^{k+1} + 1 = 2^n + 1\,.$$

Thus the claim is true when $n = k+1$. By Mathematical Induction, we have $a_n = 2^n + 1$ for all $n \geq 1$.

(c) Let $a_1 = 2$ and $a_{n+1} = \dfrac{5a_n - 4}{a_n}$ for $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 4$ for all $n \geq 1$.

Solution: We claim that $1 \leq a_n \leq a_{n+1} \leq 4$ for all $n \geq 1$. We have $a_1 = 2$ and the recursion formula gives $a_2 = \dfrac{5a_1 - 4}{a_1} = \dfrac{5 \cdot 2 - 4}{2} = 3$, and so we do have $1 \leq a_1 \leq a_2 \leq 4$ and so the claim is true when $n = 1$. Let $k \geq 1$ and suppose the claim is true when $n = k$, that is suppose that $1 \leq a_k \leq a_{k+1} \leq 4$. We have

$$\begin{aligned} 1 \leq a_k \leq a_{k+1} \leq 4 &\Longrightarrow 1 \geq \tfrac{1}{a_k} \geq \tfrac{1}{a_{k+1}} \geq \tfrac{1}{4} \Longrightarrow 4 \geq \tfrac{4}{a_k} \geq \tfrac{4}{a_{k+1}} \geq 1 \Longrightarrow -4 \leq -\tfrac{4}{a_k} \leq -\tfrac{4}{a_{k+1}} \leq -1 \\ &\Longrightarrow 1 \leq 5 - \tfrac{4}{a_k} \leq 5 - \tfrac{4}{a_{k+1}} \leq 4 \Longrightarrow 1 \leq \tfrac{5a_k - 4}{a_k} \leq \tfrac{5a_{k+1} - 4}{a_{k+1}} \leq 4 \Longrightarrow 1 \leq a_{k+1} \leq a_{k+2} \leq 4\,. \end{aligned}$$

Thus the claim is true when $n = k+1$. By Mathematical Induction, $1 \leq a_n \leq a_{n+1} \leq 4$ for all $n \geq 1$.

2: (a) Let $a_0 = 0$ and $a_1 = 1$ and for $n \geq 2$ let $a_n = a_{n-1} + 6a_{n-2}$. Show that $a_n = \frac{1}{5}\big(3^n - (-2)^n\big)$ for all $n \geq 0$.

Solution: We claim that $a_n = \frac{1}{5}\big(3^n - (-2)^n\big)$ for all $n \geq 0$. When $n = 0$ we have $a_n = a_0 = 0$ and $\frac{1}{5}\big(3^n - (-2)^n\big) = \frac{1}{5}\big(3^0 - (-2)^0\big) = 0$, so the claim is true when $n = 0$. When $n = 1$ we have $a_n = a_1 = 1$ and $\frac{1}{5}\big(3^n - (-2)^n\big) = \frac{1}{5}\big(3 - (-2)\big) = 1$, so the claim is true when $n = 1$. Let $k \geq 2$ and suppose the claim is true for all $n < k$. In particular we suppose the claim is true when $n = k-1$ and when $n = k-2$, that is we suppose $a_{k-1} = \frac{1}{5}\big(3^{k-1} - (-2)^{k-1}\big)$ and $a_{k-2} = \frac{1}{5}\big(3^{k-2} - (-2)^{k-2}\big)$. Then when $n = k$ we have

$$\begin{aligned} a_n = a_k &= a_{k-1} + 6a_{k-2} \\ &= \tfrac{1}{5}\big(3^{k-1} - (-2)^{k-1}\big) + \tfrac{6}{5}\big(3^{k-2} - (-2)^{k-2}\big) \\ &= \big(\tfrac{1}{5} \cdot 3^{k-1} + \tfrac{6}{5} \cdot 3^{k-2}\big) - \big(\tfrac{1}{5}(-2)^{k-1} + \tfrac{6}{5}(-2)^{k-2}\big) \\ &= \big(\tfrac{3}{5} \cdot 3^{k-2} + \tfrac{6}{5} \cdot 3^{k-2}\big) - \big(-\tfrac{2}{5}(-2)^{k-2} + \tfrac{6}{5}(-2)^{k-2}\big) \\ &= \tfrac{9}{5} \cdot 3^{k-2} - \tfrac{4}{5}(-2)^{k-2} = \tfrac{1}{5} \cdot 3^k - \tfrac{1}{5}(-2)^k \\ &= \tfrac{1}{5}\big(3^k - (-2)^k\big) = \tfrac{1}{5}\big(3^n - (-2)^n\big)\,. \end{aligned}$$

Thus the claim is true when $n = k$. By Strong Mathematical Induction, the claim is true for all $n \geq 0$.

(b) Let $a_0 = 1$ and $a_1 = 1$ and for $n \geq 2$ let $a_n = 2a_{n-1} + a_{n-2}$. Show that $a_n = \frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big)$ for all $n \geq 0$.

Solution: We claim that $a_n = \frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big)$ for all $n \geq 0$. When $n = 0$ we have $a_n = a_0 = 1$ and $\frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big) = \frac{1}{2}\big((1+\sqrt{2})^0 + (1-\sqrt{2})^0\big) = \frac{1}{2}\big(1-1\big) = 0$, so the claim is true when $n = 0$. When $n = 1$ we have $a_n = a_1 = 1$ and $\frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big) = \frac{1}{2}\big((1+\sqrt{2})^1 + (1-\sqrt{2})^1\big) = \frac{1}{2}\big(1+\sqrt{2}+1-\sqrt{2}\big) = 1$, so the claim is true when $n = 1$. Let $k \geq 2$ and suppose the claim is true for all $n < k$. In particular, suppose the claim is true when $n = k-1$ and when $= k-2$, that is suppose $a_{k-1} = \frac{1}{2}\big((1+\sqrt{2})^{k-1} + (1-\sqrt{2})^{k-1}\big)$ and $a_{k-2} = \frac{1}{2}\big((1+\sqrt{2})^{k-2} + (1-\sqrt{2})^{k-2}\big)$. Then when $n = k$ we have

$$\begin{aligned}
a_n = a_k &= 2a_{k-1} + a_{k-2} \\
&= 2 \cdot \tfrac{1}{2}\big((1+\sqrt{2})^{k-1} + (1-\sqrt{2})^{k-1}\big) + \tfrac{1}{2}\big((1+\sqrt{2})^{k-2} + (1-\sqrt{2})^{k-2}\big) \\
&= (1+\sqrt{2})^{k-1} + \tfrac{1}{2}(1+\sqrt{2})^{k-2} + (1-\sqrt{2})^{k-1} + \tfrac{1}{2}(1-\sqrt{2})^{k-2} \\
&= (1+\sqrt{2})(1+\sqrt{2})^{k-2} + \tfrac{1}{2}(1+\sqrt{2})^{k-2} + (1-\sqrt{2})(1-\sqrt{2})^{k-2} + \tfrac{1}{2}(1-\sqrt{2})^{k-2} \\
&= \tfrac{1}{2}(3+2\sqrt{2})(1+\sqrt{2})^{k-2} + \tfrac{1}{2}(3-2\sqrt{2})(1-\sqrt{2})^{k-2} \\
&= \tfrac{1}{2}(1+\sqrt{2})^2(1+\sqrt{2})^{k-2} + \tfrac{1}{2}(1-\sqrt{2})^2(1-\sqrt{2})^{k-2} \\
&= \tfrac{1}{2}\big((1+\sqrt{2})^k + (1-\sqrt{2})^k\big) \\
&= \tfrac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big)\,,
\end{aligned}$$

Thus the claim is true when $n = k$. By Strong Mathematical Induction, the claim is true for all $n \geq 0$.

3: (a) Show that $\displaystyle\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all $n \geq 1$.

Solution: We claim that $\displaystyle\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all $n \geq 1$. When $n = 1$ we have $\displaystyle\sum_{i=1}^{n}(2i-1)^2 = 1^2 = 1$ and $\displaystyle\frac{n(2n-1)(2n+1)}{3} = \frac{1 \cdot 1 \cdot 3}{3} = 1$, so the claim is true when $n = 1$. Let $k \geq 1$ and suppose the claim is true when $n = k$, that is suppose that $\displaystyle\sum_{i=1}^{k}(2i-1)^2 = \frac{k(2k-1)(2k+1)}{3}$. Then when $n = k+1$ we have

$$\begin{aligned}
\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{k+1}(2i-1)^2 &= \left(\sum_{i=1}^{k}(2i-1)^2\right) + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 = (2k+1)\left(\frac{k(2k-1)}{3} + (2k+1)\right) \\
&= (2k+1)\left(\frac{k(2k-1) + 3(2k+1)}{3}\right) = \frac{(2k+1)(2k^2-k+6k+3)}{3} \\
&= \frac{(2k+1)(2k^2+5k+3)}{3} = \frac{(2k+1)(k+1)(2k+3)}{3} = \frac{(2n-1)(n)(2n+1)}{3}\,.
\end{aligned}$$

Thus the claim is true when $n = k+1$. By Mathematical Induction, the claim is true for all $n \geq 0$.

(b) Find a closed form formula for $\sum_{i=1}^{n}(-1)^i(2i-1)^2$ for $n \geq 1$.

Solution: We have

$$\begin{aligned}
\sum_{i=1}^{1}(-1)^i(2i-1)^2 &= -1^2 = -1 \\
\sum_{i=1}^{2}(-1)^i(2i-1)^2 &= -1^2+3^2 = -1+9 = 8 = 2\cdot 4 \\
\sum_{i=1}^{3}(-1)^i(2i-1)^2 &= -1^2+3^2-5^2 = -1+9-25 = -17 = 1-2\cdot 9 \\
\sum_{i=1}^{4}(-1)^i(2i-1)^2 &= -1^2+3^2-5^2+7^2 = -1+9-25+49 = 32 = 2\cdot 16 \\
\sum_{i=1}^{5}(-1)^i(2i-1)^2 &= -1^2+3^2-5^2+7^2-9^2 = -1+9-25+49-81 = -49 = 1-2\cdot 25 \\
\sum_{i=1}^{6}(-1)^i(2i-1)^2 &= -1^2+3^2-5^2+7^2-9^2+11^2 = -1+9-25+49-81+121 = 72 = 2\cdot 36\,.
\end{aligned}$$

It appears that for all $n \geq 1$, $\sum_{i=1}^{n}(-1)^i(2i-1)^2 = \begin{cases} 2n^2 & \text{when } n \text{ is even,} \\ 1-2n^2 & \text{when } n \text{ is odd.} \end{cases}$ In other words, it appears that $\sum_{i=1}^{2m}(-1)^i(2i-1)^2 = 2(2m)^2$ for all $m \geq 1$ and that $\sum_{i=1}^{2m-1}(-1)^i(2i-1)^2 = 1-2(2m-1)^2$ for all $m \geq 1$.

We claim first that $\sum_{i=1}^{2m}(-1)^i(2i-1)^2 = 2(2m)^2$ for all $m \geq 1$. We have seen that this claim is true when $m=1$ (and when $m=2,3$). Let $k \geq 1$ and suppose that the claim is true when $m=k$, that is suppose that $\sum_{i=1}^{2k}(-1)^i(2i-1)^2 = 2(2k)^2$. Then when $m=k+1$ we have

$$\begin{aligned}
\sum_{i=1}^{2m}(-1)^i(2i-1)^2 &= \sum_{i=1}^{2k+2}(-1)^i(2i-1)^2 \\
&= \left(\sum_{i=1}^{2k}(-1)^i(2i-1)^2\right) + (-1)^{2k+1}(4k+1)^2 + (-1)^{2k+2}(4k+3)^2 \\
&= 2(2k)^2 - (4k+1)^2 + (4k+3)^2 = 8k^2 - (16k^2+8k+1) + (16k^2+24k+8) \\
&= 8k^2+16k+8 = 8(k+1)^2 = 2(2m)^2\,.
\end{aligned}$$

Thus the claim is true when $m=k+1$. By Mathematical Induction, the claim is true for all $m \geq 1$. Finally, note that for all $m \geq 1$ we have $1-2(2m-1)^2 = 1-2(4m^2-4m+1) = -8m^2+8m-1$ and

$$\begin{aligned}
\sum_{i=1}^{2m-1}(-1)^i(2i-1)^2 &= \left(\sum_{i=1}^{2m}(-1)^i(2i-1)^2\right) - (-1)^{2m}(4m-1)^2 = 2(2m)^2 - (4m-1)^2 \\
&= 8m^2 - (16m^2-8m+1) = -8m^2+8m-1 = 1-2(2m-1)^2\,.
\end{aligned}$$

4: (a) Expand $(2x+5)^4$.

Solution: We have

$$\begin{aligned}(2x+5)^4 &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(5)^1 + \binom{4}{2}(2x)^2(5)^2 + \binom{4}{3}(2x)^1(5)^3 + \binom{4}{4}(5)^4 \\ &= 1\cdot 16\,x^4 + 4\cdot 8\cdot 5\,x^3 + 6\cdot 4\cdot 25\,x^2 + 4\cdot 2\cdot 125\,x^1 + 1\cdot 625 \\ &= 16x^4 + 160x^3 + 600x^2 + 1000x + 625\,.\end{aligned}$$

(b) Expand $\left(x - \frac{1}{2x}\right)^8$.

Solution: We have

$$\begin{aligned}\left(x-\tfrac{1}{2x}\right)^8 &= \binom{8}{0}(x)^8 + \binom{8}{1}(x)^7\left(-\tfrac{1}{2x}\right)^1 + \binom{8}{2}(x)^6\left(-\tfrac{1}{2x}\right)^2 + \binom{8}{3}(x)^5\left(-\tfrac{1}{2x}\right)^3 + \binom{8}{4}(x)^4\left(-\tfrac{1}{2x}\right)^4 \\ &\quad + \binom{8}{5}(x)^3\left(-\tfrac{1}{2x}\right)^5 + \binom{8}{6}(x)^2\left(-\tfrac{1}{2x}\right)^6 + \binom{8}{7}(x)^1\left(-\tfrac{1}{2x}\right)^7 + \binom{8}{8} \\ &= 1\cdot x^8 - 8\cdot x^7\cdot\tfrac{1}{2x} + 28\cdot x^6\cdot\tfrac{1}{4x^2} - 56\cdot x^5\cdot\tfrac{1}{8x^3} + 70\cdot x^4\cdot\tfrac{1}{16x^4} \\ &\quad - 56\cdot x^6\cdot\tfrac{1}{32x^5} + 28\cdot x^2\cdot\tfrac{1}{64x^6} - 8\cdot x\cdot\tfrac{1}{128x^7} + \tfrac{1}{256x^8} \\ &= x^8 - 4x^6 + 7x^4 - 7x^2 + \tfrac{35}{8} - \tfrac{7}{4x^2} + \tfrac{7}{16x^4} - \tfrac{1}{16x^6} + \tfrac{1}{256x^8}\,.\end{aligned}$$

(c) Find the term involving x^8 in the expansion of $\left(\frac{x^3}{6} - \frac{12}{x^2}\right)^{11}$.

Solution: The i^{th} term in the expansion is

$$\binom{11}{i}\left(\frac{x^3}{6}\right)^{11-i}\left(-\tfrac{12}{x^2}\right)^i = \binom{11}{i}\left(\tfrac{1}{6}\right)^{11-i}(-12)^i x^{3(11-i)-2i} = (-1)^i\binom{11}{i}\tfrac{12^i}{6^{11-i}}\,x^{33-5i}\,.$$

The term involving x^8 occurs when $33-5i=8$, that is when $i=5$. The 5^{th} is

$$(-1)^5\binom{11}{5}\tfrac{12^5}{6^6}\,x^8 = -\tfrac{11\cdot10\cdot9\cdot8\cdot7}{5\cdot4\cdot3\cdot2\cdot1}\,\tfrac{2^5\cdot6^5}{6^6}\,x^8 = -11\cdot7\cdot2^5\,x^8 = -2464\,x^8\,.$$

5: (a) Evaluate $\displaystyle\sum_{i=0}^{n}\binom{n}{i}\frac{1}{2^i}$.

Solution: By the Binomial Theorem, we have $\displaystyle\sum_{i=0}^{n}\binom{n}{i}\tfrac{1}{2^i} = \left(1+\tfrac{1}{2}\right)^n = \left(\tfrac{3}{2}\right)^n$.

(b) Evaluate $\displaystyle\sum_{i=0}^{n}\binom{2n}{2i}\frac{1}{2^i}$.

Solution: By the Binomial Theorem, we have

$$\binom{2n}{0} + \binom{2n}{1}\tfrac{1}{\sqrt{2}} + \binom{2n}{2}\tfrac{1}{\sqrt{2}^2} + \binom{2n}{3}\tfrac{1}{\sqrt{2}^3} + \binom{2n}{4}\tfrac{1}{\sqrt{2}^4} + \cdots + \binom{2n}{2n}\tfrac{1}{\sqrt{2}^{2n}} = \left(1+\tfrac{1}{\sqrt{2}}\right)^{2n}$$

and

$$\binom{2n}{0} - \binom{2n}{1}\tfrac{1}{\sqrt{2}} + \binom{2n}{2}\tfrac{1}{\sqrt{2}^2} - \binom{2n}{3}\tfrac{1}{\sqrt{2}^3} + \binom{2n}{4}\tfrac{1}{\sqrt{2}^4} - \cdots + \binom{2n}{2n}\tfrac{1}{\sqrt{2}^{2n}} = \left(1-\tfrac{1}{\sqrt{2}}\right)^{2n}.$$

Adding these gives

$$2\left(\binom{2n}{0} + \binom{2n}{2}\tfrac{1}{\sqrt{2}^2} + \binom{2n}{4}\tfrac{1}{\sqrt{2}^4} + \cdots + \binom{2n}{2n}\tfrac{1}{\sqrt{2}^{2n}}\right) = \left(1+\tfrac{1}{\sqrt{2}}\right)^{2n} + \left(1-\tfrac{1}{\sqrt{2}}\right)^{2n}.$$

Thus $\displaystyle\sum_{i=0}^{n}\binom{2n}{2i}\tfrac{1}{2^i} = \tfrac{1}{2}\left(\left(1+\tfrac{1}{\sqrt{2}}\right)^{2n} + \left(1-\tfrac{1}{\sqrt{2}}\right)^{2n}\right)$.

(c) Evaluate $\displaystyle\sum_{i=0}^{n} \binom{n+i}{i} \frac{1}{2^i}$.

Solution: We have

$$\begin{aligned}
\sum_{i=0}^{0} \tbinom{n+i}{i} \tfrac{1}{2^i} &= \tbinom{0}{0} = 1 \\
\sum_{i=0}^{1} \tbinom{n+i}{i} \tfrac{1}{2^i} &= \tbinom{1}{0} + \tbinom{2}{1}\tfrac{1}{2} = 1 + \tfrac{2}{2} = 2 \\
\sum_{i=0}^{2} \tbinom{n+i}{i} \tfrac{1}{2^i} &= \tbinom{2}{0} + \tbinom{3}{1}\tfrac{1}{2} + \tbinom{4}{2}\tfrac{1}{4} = 1 + \tfrac{3}{2} + \tfrac{6}{4} = 4 \\
\sum_{i=0}^{3} \tbinom{n+i}{i} \tfrac{1}{2^i} &= \tbinom{3}{0} + \tbinom{4}{1}\tfrac{1}{2} + \tbinom{5}{2}\tfrac{1}{4} + \tbinom{6}{3}\tfrac{1}{8} = 1 + \tfrac{4}{2} + \tfrac{10}{4} + \tfrac{20}{8} = 8\,.
\end{aligned}$$

We claim that $\displaystyle\sum_{i=0}^{n} \tbinom{n+i}{i} \tfrac{1}{2^i} = 2^n$ for all $n \geq 0$. When $n = 0$ (and also when $n = 1$, 2 and 3) we have seen that the claim is true. Let $k \geq 0$ and suppose that the claim is true when $n = k$, that is suppose $\displaystyle\sum_{i=0}^{k} \binom{k+i}{i} \frac{1}{2^i} = 2^k$. Let $n = k+1$ and write $S = \displaystyle\sum_{i=0}^{n} \tbinom{n+i}{i} \tfrac{1}{2^i} = \sum_{i=0}^{k+1} \binom{k+1+i}{i} \frac{1}{2^i}$. Then we have

$$\begin{aligned}
S &= \binom{k+1}{0} + \binom{k+2}{1}\frac{1}{2} + \binom{k+3}{2}\frac{1}{2^2} + \binom{k+4}{3}\frac{1}{2^3} + \cdots + \binom{2k+1}{k}\frac{1}{2^k} + \binom{2k+2}{k+1}\frac{1}{2^{k+1}} \\
&= 1 + \left(\binom{k+1}{0} + \binom{k+1}{1}\right)\frac{1}{2} + \left(\binom{k+2}{1} + \binom{k+2}{2}\right)\frac{1}{2^2} + \left(\binom{k+3}{2} + \binom{k+3}{3}\right)\frac{1}{2^2} \\
&\quad + \cdots + \left(\binom{2k}{k-1} + \binom{2k}{k}\right)\frac{1}{2^k} + \left(\binom{2k+1}{k} + \binom{2k+1}{k+1}\right)\frac{1}{2^{k+1}} \\
&= \left(\binom{k+1}{0}\frac{1}{2} + \binom{k+2}{1}\frac{1}{2^2} + \binom{k+3}{2}\frac{1}{2^3} + \cdots + \binom{2k}{k-1}\frac{1}{2^k} + \binom{2k+1}{k}\frac{1}{2^{k+1}}\right) \\
&\quad + \left(1 + \binom{k+1}{1}\frac{1}{2} + \binom{k+2}{2}\frac{1}{2^2} + \binom{k+3}{3}\frac{1}{2^3} + \cdots + \binom{2k}{k}\frac{1}{2^k} + \binom{2k+1}{k+1}\frac{1}{2^{k+1}}\right) \\
&= \left(\frac{1}{2}S - \binom{2k+2}{k+1}\frac{1}{2^{k+2}}\right) + \left(\sum_{i=0}^{k}\binom{k+i}{i}\frac{1}{2^i} + \binom{2k+1}{k+1}\frac{1}{2^{k+1}}\right).
\end{aligned}$$

Subtract $\frac{1}{2}S$ from each side to get

$$\frac{1}{2}S = \sum_{i=0}^{k}\binom{k+i}{i}\frac{1}{2^i} + \binom{2k+1}{k+1}\frac{1}{2^{k+1}} - \binom{2k+2}{k+1}\frac{1}{2^{k+2}}\,.$$

Notice that

$$\binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{(2k+2)(2k+1)!}{(k+1)k!(k+1)!} = \frac{2(2k+1)!}{k!(k+1)!} = 2\binom{2k+1}{k+1}$$

and so we have $\frac{1}{2}S = \displaystyle\sum_{i=0}^{k}\binom{k+i}{i}\frac{1}{2^i} = 2^k$, that is $S = 2^{k+1} = 2^n$. Thus the claim holds when $n = k+1$, and so by Mathematical Induction, the claim holds for all $n \geq 0$.