MATH 135 Algebra, Solutions to Assignment 2

1: (a) Make a truth table for the statement $(P \leftrightarrow \neg R) \wedge (R \to Q)$.

Solution: Here is a truth table.

P	Q	R	$\neg R$	$P \leftrightarrow \neg R$	$R \to Q$	$(P \leftrightarrow \neg R) \wedge (R \to Q)$
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	T	F	T	F	T	F
F	F	T	F	T	F	F
F	F	F	T	F	T	F

(b) Determine whether $(P \wedge \neg Q) \vee (R \leftrightarrow P)$ is equivalent to $Q \leftrightarrow R$.

Solution: We make a truth table for $(P \wedge \neg Q) \vee (R \leftrightarrow P)$ and $Q \leftrightarrow R$.

P	Q	R	$\neg Q$	$P \wedge \neg Q$	$R \leftrightarrow P$	$(P \wedge \neg Q) \vee (R \leftrightarrow P)$	$Q \leftrightarrow R$
T	T	T	F	F	T	T	T
T	T	F	F	F	F	F	F
T	F	T	T	T	T	T	F
T	F	F	T	T	F	T	T
F	T	T	F	F	F	F	T
F	T	F	F	F	T	T	F
F	F	T	T	F	F	F	F
F	F	F	T	F	T	T	T

Since the final two columns are not identical, the two statements are not equivalent (for example, as seen on the third row, when P is true, Q is false and R is true, the statement $(P \wedge \neg Q) \vee (R \leftrightarrow P)$ is true but the statement $Q \leftrightarrow R$ is false).

(c) Determine whether $P \to (Q \to R)$ is equivalent to $(P \to Q) \to R$.

Solution: We make a truth table for the two given statements.

P	Q	R	$Q \to R$	$P \to (Q \to R)$	$P \to Q$	$(P \to Q) \to R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

Since the column for $P \to (Q \to R)$ is not identical to the column for $(P \to Q) \to R$, these two statements are not equivalent (for example, as seen on the 6^{th} row, when P is false, Q is true and R is false, the statement $P \to (Q \to R)$ is true but the statement $(P \to Q) \to R$ is false).

2: Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. List all of the elements in each of the following sets.

(a) $A = \big\{x \in S \,\big|\, x \text{ is even or } x \text{ is a multiple of } 3\big\}$

Solution: We have

$$\begin{aligned} A &= \big\{x \in S \big|\ x \text{ is even}\big\} \cup \big\{x \in S \big|\ x \text{ is a multiple of } 3\big\} \\ &= \{2, 4, 6, 8, 10\} \cup \{3, 6, 9\} = \{2, 3, 4, 6, 8, 9, 10\}\,. \end{aligned}$$

(b) $B = \big\{x \in S \,\big|\, \text{if } x \text{ is even then } x \text{ is a multiple of } 3\big\}$

Solution: Since $P \to Q$ is equivalent to $\neg P \vee Q$, the statement "if x is even then x is a multiple of 3" is equivalent to the statement "x is odd or x is a multiple of 3", so we have

$$\begin{aligned} B &= \big\{x \in S \big|\ x \text{ is odd or } x \text{ is a multiple of } 3\big\} \\ &= \big\{x \in S \big|\ x \text{ is odd}\big\} \cup \big\{x \in S \big|\ x \text{ is a multiple of } 3\big\} \\ &= \{1, 3, 5, 7, 9\} \cup \{3, 6, 9\} = \{1, 3, 5, 6, 7, 9\}\,. \end{aligned}$$

(c) $C = \big\{(x, y) \in S \times S \,\big|\, 3x + 2y = 20 \text{ and } (y < x^2 \text{ if and only if } xy < 8)\big\}$

Solution: Note that for $(x, y) \in S \times S$ we have $3x + 2y = 20$ if and only if $(x, y) = (2, 7), (4, 4)$ or $(6, 1)$. When $(x, y) = (2, 7)$, the statement "$y < x^2$" is false and the statement "$xy < 8$" is false, so the statement "$y < x^2$ if and only if $xy < 8$" is true. When $(x, y) = (4, 4)$, the statement "$y < x^2$" is true and the statement "$xy < 8$' is false, so the statement "$y < x^2$ if and only if $xy < 8$" is false. When $(x, y) = (6, 1)$, the statement "$y < x^2$" is true and the statement "$xy < 8$"is true, so the statement "$y < x^2$ if and only if $xy < 8$" is true. Thus $C = \big\{(2, 7), (6, 1)\big\}$.

3: Express each of the statements below symbolically, taking the universe of discourse to be **R** and using only symbols from the following list:

$$\neg\ ,\ \wedge\ ,\ \vee\ ,\ \to\ ,\ \leftrightarrow\ ,\ (\ ,\ )\ ,\ \forall\ ,\ \exists\ ,\ 0\ ,\ 1\ ,+\ ,\ \times\ ,\ =\ ,\ <\ ,\ \leq\ ,\ x\ ,\ y\ ,\ z$$

(a) x is equal to the minimum of y and z.

Solution: Here are three ways to express this statement symbolically.

$$\begin{gathered} (y \leq z \wedge x = y) \vee (z \leq y \wedge x = z) \\ (y \leq z \to x = y) \wedge (z \leq y \to x = z) \\ (x = y \vee x = z) \wedge (x \leq y \wedge x \leq z) \end{gathered}$$

Note that many slight variations of these expressions will be incorrect. For example, although the statement $(y \leq z \to x = y) \vee (z \leq y \to x = z)$ may look similar to the second of the above expressions, it has a completely different meaning. Indeed it is true whenever $y \neq z$ (can you see why?).

(b) There is no smallest positive real number.

Solution: Here are two ways to express this statement symbolically.

$$\begin{gathered} \neg\exists x \big(0 < x \wedge \forall y\,(0 < y \to x \leq y)\big) \\ \forall x \big(0 < x \to \exists y\,(0 < y \wedge y < x)\big) \end{gathered}$$

(c) Every real number has a unique cubed root.

Solution: Here are two ways to express this symbolically.

$$\begin{gathered} \forall x\, \exists y \big(y \times y \times y = x\ \ \wedge\ \ \forall z\,(z \times z \times z = x \to z = y)\big) \\ \forall x\, \exists y \big(y \times y \times y = x\big)\ \wedge\ \forall y \forall z \big((y \times y \times y = z \times z \times z) \to y = z\big) \end{gathered}$$

4: For each of the following statements, determine whether it is true when the universe of discourse is $\mathbf{Z}$ and whether it is true when the universe of discourse is $\mathbf{R}$.

(a) $\forall x\, \exists y\, (x = y \times y \ \vee\ y \times y + x = 0)$ We can also write this statement as $\forall x\, \exists y\ x = \pm y^2$. This is false in $\mathbf{Z}$, for example when $x = 2$ there is no integer y for which $x = \pm y^2$. On the other hand, it is true in $\mathbf{R}$. Indeed given $x \in \mathbf{R}$ we can choose $y = \sqrt{|x|}$ and then we have $x = \pm y^2$.

(b) $\forall x\, \exists y\, \big(y < x \ \wedge\ \forall z\, (z < x \to z \le y)\big)$

Solution: This is true in $\mathbf{Z}$. Indeed given $x \in \mathbf{Z}$ we can choose $y = x - 1$. Then we have $y < x$, and for all $z \in \mathbf{Z}$, if $z < x$ then we have $z \le x - 1$, that is $z \le y$. On the other hand, the given statement is false in $\mathbf{R}$. Indeed given $x \in \mathbf{R}$ and given $y \in \mathbf{R}$ with $y < x$, we can choose $z \in \mathbf{R}$ with $y < z < x$ (for example we can choose $z = \frac{x+y}{2}$).

(c) $\forall x\, \forall y\, \big((0 < x \ \wedge\ x + y < x \times y) \to 0 < y\big)$

Solution: This is true in $\mathbf{Z}$ and we give a proof. Let $x, y \in \mathbf{Z}$. Suppose $0 < x$ and $x + y < xy$. Since $x + y < xy$ we have $xy - y > x$, that is $y(x-1) > x$. Note that $x \ne 1$ since if we had $x = 1$ then $y(x-1) > x$ would give $0 > 1$, which is false. Since $0 < x$ and $x \ne 1$ we know that $x > 1$, and so $y(x-1) > x$ implies that $y > \frac{x}{x-1} = 1 + \frac{1}{x-1} > 1$. Thus $0 < y$, as required. On the other hand, the given statement is false in $\mathbf{R}$. For example when $x = \frac{1}{2}$ and $y = -2$, we have $0 < x$ and $x + y < xy$, but we do not have $0 < y$.

5: Prove each of the following statements, where the universe of discourse is $\mathbf{R}$.

(a) $\forall x\, \big(\, |x^2 - 4x| \le x \ \leftrightarrow\ \big(x = 0 \ \vee\ (3 \le x \ \wedge\ x \le 5)\big)\big)$

Solution: Let $x \in \mathbf{R}$. Then

$$\begin{aligned} |x^2 - 4x| \le x &\iff -x \le x^2 - 4x \text{ and } x^2 - 4x \le x \\ &\iff x^2 - 3x \ge 0 \text{ and } x^2 - 5x \le 0 \\ &\iff x(x-3) \ge 0 \text{ and } x(x-5) \le 0 \\ &\iff (x \le 0 \text{ or } x \ge 3) \text{ and } (0 \le x \le 5) \\ &\iff (x \le 0 \text{ and } 0 \le x \le 5) \text{ or } (x \ge 3 \text{ and } 0 \le x \le 5) \\ &\iff x = 0 \text{ or } 3 \le x \le 5\,. \end{aligned}$$

(b) $\forall x\, \forall y\, \big(y^2 = x^3 + x^2 \ \leftrightarrow\ \exists z\, (x = z^2 - 1 \ \wedge\ y = z^3 - z)\big)$

Solution: Let $x, y \in \mathbf{R}$. Suppose that $y^2 = x^3 + x^2$. If $x = 0$ then we must have $y = 0$ and we can choose $z = 1$ to get $x = z^2 - 1$ and $y = z^3 - z$. If $x \ne 0$ then we can choose $z = \frac{y}{x}$ to get $z^2 - 1 = \frac{y^2}{x^2} - 1 = \frac{x^3 + x^2}{x^2} - 1 = x$ and $z^3 - z = z(z^2 - 1) = \frac{y}{x} \cdot x = y$. Conversely, suppose that $x = z^2 - 1$ and $y = z^3 - z^2$. Then we have $y^2 = (z^3 - z)^2 = \big(z(z^2 - 1)\big)^2 = z^2(z^2 - 1)^2$ and we have $x^3 + x^2 = x^2(x+1) = (z^2 - 1)z^2$, and so $y^2 = x^3 + x^2$.