MATH 135 Algebra, Solutions to Assignment 1

1: (a) Find all real numbers x such that $\dfrac{\sqrt{x}}{1+x} = \dfrac{1}{3-\sqrt{x}}$.

Solution: By cross multiplying and then using the quadratic formula, we have

$$\frac{\sqrt{x}}{1+x} = \frac{1}{3-\sqrt{x}} \iff \sqrt{x}\,(3-\sqrt{x}) = 1+x \iff 3\sqrt{x} - x = 1+x \iff 2x - 3\sqrt{x} + 1 = 0$$
$$\iff \sqrt{x} = \frac{3 \pm \sqrt{9-8}}{4} \iff \sqrt{x} = 1 \text{ or } \tfrac{1}{2} \iff x = 1 \text{ or } \tfrac{1}{4}\,.$$

(b) Solve $y = x + \frac{1}{x}$ with $0 < x \le 1$ for x in terms of y.

Solution: Multiply both sides of the equation by x then use the quadratic formula to get

$$y = x + \tfrac{1}{x} \iff xy = x^2 + 1 \iff x^2 - xy + 1 = 0 \iff x = \frac{y \pm \sqrt{y^2-4}}{2}\,.$$

Note that $0 < x \le 1 \Longrightarrow \frac{1}{x} \ge 1 \ge x \Longrightarrow y = x + \frac{1}{x} \ge 2x \Longrightarrow x \le \frac{y}{2}$, so we must use the negative sign. Thus

$$x = \frac{y - \sqrt{y^2-4}}{2}\,.$$

2: (a) Find all ordered pairs of integers (x, y) such that $xy = 6 + 2x$.

Solution: We have $xy = 6 + 2x \iff xy - 2x = 6 \iff x(y-2) = 6$. So x must be a factor of 6, that is $x = \pm1, \pm2, \pm3, \pm6$, and we have $y = \frac{6}{x} + 2$. Thus

$$(x, y) = (-6, 1), (-3, 0), (-2, -1), (-1, -4), (1, 8), (2, 5), (3, 4) \text{ or } (6, 3)\,.$$

Note that these are the 6 points with integer coordinates which lie on the hyperbola $y = \frac{6}{x} + 2$.

(b) Find all ordered pairs of integers (x, y) such that $x^2 + y^2 = 4x + 2y$.

Solution: Complete the square to get

$$x^2+y^2 = 4x+2y \iff x^2-4x+y^2-2y = 0 \iff (x-2)^2-4+(y-1)^2-1 = 0 \iff (x-2)^2+(y-1)^2 = 5\,.$$

Since $(x-2)^2 = 5 - (y-1)^2 \le 5$ we must have $|x-2| \le \sqrt{5}$ so $(x-2) = 0, \pm1$, or ±2. Also, we have $(y-1)^2 = 5 - (x-2)^2$ so $y = 1 \pm \sqrt{5-(x-2)^2}$, and so when $(x-2) = 0$ we have $y = 1 \pm \sqrt{5}$ which is not an integer, when $(x-2) = \pm1$ we have $y = 1 \pm 2$, and and when $(x-2) = \pm2$ we have $y = 1 \pm 1$. Thus

$$(x, y) = (0, 0), (0, 2), (1, -1), (1, 3), (3, -1), (3, 3), (4, 0) \text{ or } (4, 2)\,.$$

These are the 8 points with integer coordinates which lie on the circle of radius $\sqrt{5}$ centered at $(2, 1)$.

3: (a) Determine whether there exists an integer x such that $x^2 + 20$ is a perfect square.

Solution: There does exist such an integer x, indeed when $x = 4$ we have $x^2 + 20 = 16 + 20 = 36 = 6^2$.

(b) Determine whether there exists an integer x such that $x^2 + 10$ is a perfect square.

Solution: We claim that there is no such value of x. Note that it suffices to consider only non-negative values of x since $(-x)^2 = x^2$. We list the first few values of $x^2 + 10$.

x	0	1	2	3	4
x^2+10	10	11	14	19	26

We see from the table that for $0 \le x \le 4$, $x^2 + 10$ is not a perfect square. Now suppose, for a contradiction, that $x \ge 5$ and $x^2 + 10$ is a perfect square, say $x^2 + 10 = y^2$ where $y \ge 0$. Then

$$y^2 = x^2 + 10 \Longrightarrow y^2 > x^2 \Longrightarrow y > x \Longrightarrow y \ge x + 1 \Longrightarrow y^2 \ge (x+1)^2 = x^2 + 2x + 1$$
$$\Longrightarrow y^2 \ge x^2 + 2 \cdot 5 + 1 = x^2 + 11 \Longrightarrow y^2 > x^2 + 10 \Longrightarrow y^2 \ne x^2 + 10$$

giving us the desired contradiction.

4: (a) The first 15 odd prime numbers are $3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$ and 53. Determine which of these 15 prime numbers are equal to the sum of two squares, then make a conjecture about which odd prime numbers are equal to the sum of two squares.

Solution: Note that each of the primes 5, 13, 17, 29, 37, 41 and 53 is a sum of two squares (indeed $5 = 1+4$, $13 = 4+9$, $17 = 1+16$, $29 = 4+25$, $37 = 1+36$, $41 = 16+25$ and $53 = 4+49$). By trying all possibilities, we can verify that none of the remaining primes 3, 7, 11, 19, 23, 31, 43 and 47 is a sum of two squares (for example, 47 is not a sum of two squares since $47-0 = 47$, $47-1 = 46$, $47-4 = 43$, $47-9 = 38$, $47-16 = 31$, $47-25 = 22$ and $47-36 = 11$, none of which are perfect squares).

Notice that each of the primes 5, 13, 17, 29, 37, 41 and 53 is equal to 1 more than a multiple of 4, while each of the remaining primes 3, 7, 11, 19, 23, 31, 43 and 47 is equal to 1 less than a multiple of 4. We conjecture that an odd prime is a sum of two squares if and only if it is equal to 1 more than a multiple of 4. (In fact, this conjecture is true, but we shall not prove it in this course).

(b) Find all integers n with $2 \leq n \leq 20$ with the property that n is a factor of $2^{n-1} - 1$, then make a conjecture about which integers $n \geq 2$ have the property that n is a factor of $2^{n-1} - 1$.

Solution: With the help of a calculator, we find that the values of n with $2 \leq n \leq 20$ such that n is a factor of $2^{n-1} - 1$ are 3, 5, 7, 11, 13, 17, and 19. We conjecture that an integer $n \geq 2$ is a factor of $2^{n-1} - 1$ if and only if n is an odd prime. (In fact, this conjecture is false, as we shall show in Assignment 9).

5: (a) The first six terms of sequences $\{a_n\}$ and $\{b_n\}$ are listed below:

n	1	2	3	4	5	6
a_n	1	2	5	12	29	70
b_n	1	3	7	17	41	99

(i) Find a rule that governs how the terms a_{n+1} and b_{n+1} are obtained from the terms a_n and b_n, and use this rule to find the terms a_7, b_7, a_8 and b_8.

Solution: It appears that $a_{n+1} = a_n + b_n$ and that $b_{n+1} = a_n + a_{n+1} = 2a_n + b_n$. Using these rules we find that $a_7 = 169$, $b_7 = 239$, $a_8 = 408$ and $b_8 = 577$.

(ii) Use a calculator to find the ratio ${b_n}^2/{a_n}^2$ for $1 \leq n \leq 6$ and make a conjecture about the limit $\lim_{n\to\infty} (b_n/a_n)$.

Solution: With a calculator, we find the following approximate values for ${b_n}^2/{a_n}^2$.

n	1	2	3	4	5	6
${b_n}^2/{a_n}^2$	1.0000	2.2500	1.9600	2.0069	1.9988	2.0002

It appears that $\lim_{n\to\infty} {b_n}^2/{a_n}^2 = 2$, so we conjecture that $\lim_{n\to\infty} (b_n/a_n) = \sqrt{2}$. (In fact, this conjecture is true).

(b) For a real number x, let $\lfloor x \rfloor$ denote the largest integer which is less than or equal to x. For each positive integer n, let $a_n = \lfloor n\sqrt{2} \rfloor$ and let b_n be the n^{th} positive integer which does not occur in the sequence $\{a_n\}$. The first 20 terms of the sequence $\{a_n\}$ and the first 8 terms of the sequence $\{b_n\}$ are listed below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a_n	1	2	4	5	7	8	9	11	12	14	15	16	18	19	21	22	24	25	26	28
b_n	3	6	10	13	17	20	23	27												

Make a conjecture about a rule which allows us to determine the terms a_n and b_n without using a calculator and without using the value of $\sqrt{2}$.

Solution: It appears that for all $n \geq 1$ we have $b_n = a_n + 2n$, so we conjecture that the terms a_n and b_n can be obtained using the following rule: we start with $a_1 = 1$ and $b_1 = 3$, and once we have found $a_1, \cdots a_{n-1}$ and $b_1, \cdots, b_{n-1}$, we let a_n be the n^{th} positive integer which does not occur in the sequence $b_1, \cdots, b_{n-1}$, and then we set $b_n = a_n + 2n$. (This conjecture is true, and it is also true that $b_n = \lfloor n(2+\sqrt{2}) \rfloor$ for all n).