MATH 135 Algebra, Solutions to Assignment 12

1: Express each of the following complex numbers in cartesian form.

(a) $4\,e^{i\,5\pi/3}$

Solution: We have $4\,e^{i\,5\pi/3} = 4\left(\cos\frac{5\pi}{3} + i\,\sin\frac{5\pi}{3}\right) = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = 2 - 2\sqrt{3}\,i$

(b) $(1+i\sqrt{3})^{10}$

Solution: We have

$$\begin{aligned}(1+i\sqrt{3})^{10} &= \left(2e^{i\,\pi/3}\right)^{10} = 2^{10}e^{i\,10\pi/3} = 2^{10}\left(\cos\tfrac{10\pi}{3} + i\sin\tfrac{10\pi}{3}\right)\\ &= 1024\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\,i\right) = -512 - 512\sqrt{3}\,i\,.\end{aligned}$$

(c) $5\,e^{i\,\theta}$, where $\theta = \tan^{-1} 2$

Solution: We have $5\,e^{i\,\theta} = 5\,(\cos\theta + i\sin\theta) = 5\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}\,i\right) = \sqrt{5} + 2\sqrt{5}\,i.$

2: Express each of the following complex numbers in the polar form $re^{i\,\theta}$.

(a) $-2+2i$

Solution: We have $-2+2i = 2\sqrt{2}\,e^{i\,3\pi/4}$.

(b) $\dfrac{(1-i)^2}{(1+i\sqrt{3})}$

Solution: We have $\dfrac{(1-i)^2}{(1+i\,\sqrt{3})} = \dfrac{\left(\sqrt{2}\,e^{-i\,\pi/4}\right)^2}{2\,e^{i\pi/3}} = \dfrac{2\,e^{-i\,\pi/2}}{2\,e^{i\,\pi/2}} = e^{-i\left(\frac{\pi}{2}+\frac{\pi}{3}\right)} = e^{-i\,5\pi/6}$.

(c) $-3-i$

Solution: We have $-3-i = \sqrt{10}\,e^{i\,\theta}$, where $\theta = \pi + \tan^{-1}\frac{1}{3}$.

3: Solve each of the following for $z \in \mathbf{C}$. Express your answers in the polar form $re^{i\,\theta}$.

(a) $z^3 + 8i = 0$

Solution: Write $z = r\,e^{i\,\theta}$ with $r > 0$ and $\theta \in [0, 2\pi)$. Then

$$\begin{aligned} z^3 + 8i = 0 &\iff \left(r\,e^{i\,\theta}\right)^3 = -8i \iff r^3 e^{i\,3\theta} = 8\,e^{i\,3\pi/2} \\ &\iff r^3 = 8 \text{ and } 3\theta = \tfrac{3\pi}{2} + 2\pi\,k \text{ for some } k \in \mathbf{Z} \\ &\iff r = 2 \text{ and } \theta = \tfrac{\pi}{2} + \tfrac{2\pi}{3}\,k \text{ where } k = 0, 1 \text{ or } 2\,. \end{aligned}$$

Thus the solutions are $z = 2\,e^{i\,\pi/2}$, $z = 2\,e^{i\,7\pi/6}$ and $z = 2\,e^{i\,11\pi/6}$.

(b) $z = \dfrac{z-i}{z+2+i}$

Solution: We have

$$\begin{aligned} z = \frac{z-i}{z+2+i} &\iff z^2 + (2+i)z = z - i \iff z^2 + (1+i)z + i = 0 \iff z = \frac{-(1+i) \pm \sqrt{(1+i)^2 - 4i}}{2} \\ &\iff z = \frac{-(1+i) \pm \sqrt{2i-4i}}{2} = \frac{-(1+i) \pm \sqrt{-2i}}{2} = \frac{-(1+i) \pm (1-i)}{2} \\ &\iff z = -1 \text{ or } -i\,. \end{aligned}$$

In polar coordinates, the solutions are $z = e^{i\,\pi}$ and $z = e^{i\,3\pi/2}$.

(c) $z^4 = 8\overline{z}$

Solution: Write $z = r\,e^{i\,\theta}$ where $r > 0$ and $\theta \in [0, 2\pi)$. Then

$$z^4 = 8\overline{z} \iff r^4 e^{i\,4\theta} = 8r\,e^{-i\theta} \iff \left(r = 0 \text{ or } r^3\,e^{i\,5\theta} = 8\right) \iff \left(r = 0 \text{ or } (r = 2 \text{ and } 5\theta = 2\pi\,k,\ k \in \mathbf{Z})\right).$$

Thus the solutions are $z = 0$, $z = 2$, $z = 2\,e^{i\,2\pi/5}$, $z = 2\,e^{i\,4\pi/5}$, $z = 2\,e^{i\,6\pi/5}$ and $z = 2\,e^{i\,8\pi/5}$.

4: For each of the following polynomials $f(x)$, first solve $f(z)=0$ for $z\in\mathbf{C}$, and then factor $f(x)$ over the real numbers. (You may find it useful to read section 9.2 in the text book).

(a) $f(x)=x^6+7x^3-8$

Solution: For $z\in\mathbf{C}$ we have

$$\begin{aligned} f(z)=0 &\iff z^6+7z^3-8=o \iff (z^3+8)(z^3-1)=0 \iff z^3=1 \text{ or } z^3=-8=8\,e^{i\,\pi} \\ &\iff z=1\,,\ e^{i\,2\pi/3}\,,\ e^{i\,4\pi/3}\,,\ 2\,e^{i\pi/3}\,,\ 2\,e^{i\,\pi}\ ,\text{ or } 2e^{i\,5\pi/3} \\ &\iff z=1\,,\ \left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\,i\right)\,,\ \left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\,i\right)\,,\ \left(1+\sqrt{3}i\right)\,,\ -2\,,\ \left(1-\sqrt{3}\,i\right)\,. \end{aligned}$$

It follows that $f(x)$ factors over the reals as $f(x)=(x-1)(x+2)(x^2+x+1)(x^2-2x+4)$.

(b) $f(x)=x^6+1$

Solution: For $z\in\mathbf{C}$ we have

$$\begin{aligned} f(z)=0 &\iff z^6+1=0 \iff z^6=-1=e^{i\,\pi} \\ &\iff z=e^{i\,\pi/6}\,,\ e^{i\,\pi/2}\,,\ e^{i\,5\pi/6}\,,\ e^{i\,7\pi/6}\,,\ e^{i\,3\pi/2} \text{ or } e^{i\,11\pi/6} \\ &\iff z=\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}\,i\right)\,,\ i\,,\ \left(-\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}\,i\right)\,,\ \left(-\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}\,i\right)\,,\ -i\,,\ \left(\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}\,i\right)\,. \end{aligned}$$

It follows that $f(x)$ factors over the reals as $f(x)=(x^2+1)(x^2-\sqrt{3}\,x+1)(x^2+\sqrt{3}\,x+1)$.

(c) $f(x)=x^4+4x^3+6x^2+4x+5$

Solution: For $z\in\mathbf{C}$ we have

$$\begin{aligned} f(z)=0 &\iff z^4+4x^3+6x^2+4x+5=0 \iff (z+1)^4+4=0 \iff (z+1)^4=-4=4\,e^{i\,\pi} \\ &\iff z+1=\sqrt{2}\,e^{i\,\pi/4}\,,\ \sqrt{2}\,e^{i\,3\pi/4}\,,\ \sqrt{2}\,e^{i\,5\pi/4} \text{ or } \sqrt{2}\,e^{i\,7\pi/4} \\ &\iff z+1=(1+i)\,,\ (-1+i)\,,\ (-1-i) \text{ or } (1-i) \\ &\iff z=i\,,\ (-2+i)\,,\ (-2-i) \text{ or } -i\,. \end{aligned}$$

It follows that $f(x)$ factors over the reals as $f(x)=(x^2+1)(x^2+4x+5)$.

5: An equation of the form $ax^3+bx^2+cx+d=0$, where $a,b,c,d\in\mathbf{C}$ with $a\neq 0$ can be solved as follows. First, divide by a to obtain an equation of the form $x^3+Bx^2+Cx+D=0$. Next, make the substitution $x=z-\frac{B}{3}$. This will convert the equation to the form $z^3+pz+q=0$. Thirdly, make the substitution $z=w-\frac{p}{3w}$. This will convert the equation to the form $w^3+r\,w^{-3}+q=0$. Finally, multiply through by w^3 to obtain $w^6+q\,w^3+r=0$, which we can solve for w^3 using the Quadratic Formula. (You may find section 9.6 useful).

(a) Solve $x^3-3x+1=0$ for $x\in\mathbf{R}$.

Solution: Write $x=w+w^{-1}$. The equation becomes

$$\begin{aligned}(w+w^{-1})^3-3(w+w^{-1})+1=0\\ w^3+3w+3w^{-1}+w^{-3}-3w-3w^{-1}+1=0\\ w^3+w^{-3}+1=0\end{aligned}$$

Multiply by w^3 to get $w^6+1+w^3=0$, that is $w^6+w^3+1=0$. Solve for w^3 using the Quadratic Formula to get $w^3=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\,i$. When $w^3=-\frac{1}{2}+\frac{\sqrt{3}}{2}\,i=e^{i\,2\pi/3}$, we have $w=e^{i\,2\pi/9}$, $e^{i\,8\pi/9}$ or $e^{i\,14\pi/9}$. Notice that in all cases we have $|w|=1$ so that $w^{-1}=\overline{w}$ and hence $x=w+w^{-1}=w+\overline{w}=2\text{Re}\,w$. Thus we have $x=2\cos\frac{2\pi}{9}$, $2\cos\frac{8\pi}{9}$ or $x=2\cos\frac{14\pi}{9}$. Alternatively, in degrees we have

$$x=2\cos(40^\circ)\,,\ 2\cos(160^\circ)\ ,\text{ or }2\cos(280^\circ)\,.$$

Note that we do not need to consider the case that $w^3=-\frac{1}{2}-\frac{\sqrt{3}}{2}\,i$ since we have found all three real roots.

(b) Solve $x^3+3x^2-3x-7=0$ for $x\in\mathbf{R}$.

Solution: Write $x=z-1$. Then the equation becomes

$$\begin{aligned}(z-1)^3+3(z-1)^2-3(z-1)-z=0\\ z^3-3z^2+3z-1+3z^2-6z+3-3z+3-7=0\\ z^3-6z-2=0\end{aligned}$$

Write $z=w+2w^{-1}$. Then the equation becomes

$$\begin{aligned}(w+2w^{-1})^3-6(w+2w^{-1})-2=0\\ w^3+6w+12w^{-1}+8w^{-3}-6w-12w^{-1}-2=0\\ w^3+8w^{-3}-2=0\end{aligned}$$

Multiply by w^3 to get $w^6-2w^3+8=0$. Use the Quadratic Formula to get $w^3=\frac{2\pm\sqrt{4-32}}{2}=1\pm\sqrt{7}\,i$. When $w^3=1+\sqrt{7}\,i=2\sqrt{2}\,e^{i\,\theta}$, where $\theta=\tan^{-1}\sqrt{7}$, we have $w=\sqrt{2}\,e^{i\,\theta/3}$, $\sqrt{2}\,e^{i\,(\theta+2\pi)/3}$, or $\sqrt{2}\,e^{i\,(\theta+4\pi)/3}$. Notice that in all cases, w is of the form $w=\sqrt{2}\,e^{i\,\phi}$, so we have $z=w+2w^{-1}=\sqrt{2}e^{i\,\phi}+\sqrt{2}\,e^{-i\,\phi}=2\sqrt{2}\text{Re}w$. Thus $z=2\sqrt{2}\cos\left(\frac{\theta}{3}\right)$, $2\sqrt{2}\cos\left(\frac{\theta+2\pi}{3}\right)$, or $2\sqrt{2}\cos\left(\frac{\theta+4\pi}{3}\right)$. Finally, since $x=z-1$ we have

$$x=2\sqrt{2}\cos\left(\tfrac{\theta}{3}\right)-1\,,\ 2\sqrt{2}\cos\left(\tfrac{\theta+2\pi}{3}\right)-1\ ,\text{ or }2\sqrt{2}\cos\left(\tfrac{\theta+4\pi}{3}\right)-1\,,$$

where $\theta=\tan^{-1}\sqrt{7}$.