MATH 135 Algebra, Solutions to Assignment 11

1: Express each of the following complex numbers in cartesian form.

(a) $(2+i)(3+2\,i)-(5+3\,i)$

Solution: We have $(2+i)(3+2i)-(5+3i)=(6+4i+3i-2)-(5+3i)=(4+7i)-(5+3i)=-1+4i$.

(b) $\dfrac{1}{(\sqrt{2}+i)(1+i\sqrt{2})}$

Solution: We have $\dfrac{1}{(\sqrt{2}+i)(1+i\sqrt{2})}=\dfrac{1}{\sqrt{2}+2i+i-\sqrt{2}}=\dfrac{1}{3i}=-\frac{1}{3}\,i$.

(c) $\dfrac{(1+2\,i)^3}{(3+i)^2}$

Solution: We have

$$\begin{aligned}\frac{(1+2\,i)^3}{(3+i)^2}&=\frac{1+6i+12i^2+8i^3}{9+6i+i^2}=\frac{1+6i-12-8i}{9+6i-1}=\frac{-11-2i}{8+6i}\\&=\frac{-11-2i}{2(4+3i)}\cdot\frac{4-3i}{4-3i}=\frac{-44+33i-8i-6}{2(16+9)}=\frac{-50+25i}{50}=-1+\tfrac{1}{2}\,i\,.\end{aligned}$$

2: Solve each of the following equations for $z\in\mathbf{C}$. Express your answers in cartesian form.

(a) $\dfrac{z+1}{z+i}=3+i$

Solution: We have

$$\begin{aligned}&\frac{z+1}{z+i}=3+i\iff(z+1)=(3+i)(z+i)\iff z+1=3z+3i+iz-1\iff 2-3i=(2+i)z\\&\iff z=\frac{2-3i}{2+i}=\frac{2-3i}{2+i}\cdot\frac{2-i}{2-i}=\frac{1-8i}{5}\iff z=\tfrac{1}{5}-\tfrac{8}{5}i\,.\end{aligned}$$

(b) $z^2=\overline{z}$

Solution: Write $z=x+iy$ with $x,y\in\mathbf{R}$. Then we have

$$z^2=\overline{z}\iff(x+iy)^2=x-iy\iff(x^2-y^2)+i(2xy)=x-iy\iff\big(x^2-y^2=x\text{ and }2xy=-y\big)\,.$$

To get $2xy=-y$ we need $y=0$ or $2x=-\frac{1}{2}$. When $y=0$, the equation $x^2-y^2=x$ gives $x^2=x$ so $x=0$ or 1. When $2x=-1$ so $x=-\frac{1}{2}$, the equation $x^2-y^2=x$ gives $\frac{1}{4}-y^2=-\frac{1}{2}$ so $y^2=\frac{3}{4}$, that is $y=\pm\frac{\sqrt{3}}{2}$. Thus $z=x+iy=0,\ 1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ or $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$.

(c) $z^2=4+3\,i$

Solution: Write $z=x+iy$ with $x,y\in\mathbf{R}$. Then we have

$$z^2=4+3i\iff(x+iy)^2=4+3i\iff(x^2-y^2)+i(2xy)=4+3i\iff\big(x^2-y^2=4\text{ and }2xy=3\big)\,.$$

From $2xy=3$ we get $y=\frac{3}{2x}$. Put this into the equation $x^2-y^2=4$ to get

$$x^2-\frac{9}{4x^2}=4\implies 4x^4-9=16x^2\implies 4x^4-16x^2-9=0\implies(2x^2-9)(2x^2+1)=0\implies\big(x^2=\tfrac{9}{2}\text{ or }x^2=-\tfrac{1}{2}\big)\,.$$

Since $x\in\mathbf{R}$ we must have $x^2=\frac{9}{2}$ so $x=\pm\frac{3}{\sqrt{2}}$. When $x=\frac{3}{\sqrt{2}}$ we have $y=\frac{3}{2x}=\frac{3}{2}\cdot\frac{\sqrt{2}}{3}=\frac{1}{\sqrt{2}}$, and when $x=-\frac{3}{\sqrt{2}}$ we have $y=\frac{3}{2x}=-\frac{1}{\sqrt{2}}$. Thus $z=\pm\left(\frac{3}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i\right)$.

3: Solve the following pairs of equations for $z, w \in \mathbf{C}$.

(a) $z + i\,w = 2$
$i\,z + 2w = 3$

Solution: From the first equation we have $z = 2 - iw$. Put this into the second equation to get

$$i(2 - iw) + 2w = 3 \Longrightarrow 2i + w + 2w = 3 \Longrightarrow 3w = 3 - 2i \Longrightarrow w = 1 - \tfrac{2}{3}\,i\,.$$

Then $z = 2 - iw = 2 - i - \frac{2}{3} = \frac{4}{3} - i$. Thus the solution is $(z, w) = \left(\frac{4}{3} - i, 1 - \frac{2}{3}\,i\right)$.

(b) $i\,z + (1+i)w = -3 + i$
$(2+i)z + (3-2i)w = 4i$

Solution: Multiply the first equation by $-i$ to get $z + (-i+1)w = 3i + 1$ so we have $z = (1+3i) - (1-i)w$. Put this into the second equation to get

$$\begin{aligned}(2+i)\big((1+3i) - (1-i)w\big) + (3-2i)w = 4i &\Longrightarrow (-1+7i) - (3-i)w + (3-2i)w = 4i \\ &\Longrightarrow -iw = 1 - 3i \Longrightarrow w = 3 + i\,.\end{aligned}$$

Then $z = (1+3i) - (1-i)w = (1+3i) - (1-i)(3+i) = (1+3i) - (4-2i) = -3 + 5i$. Thus the solution is $(z, w) = \big(-3 + 5i, 3 + i\big)$.

4: Draw a picture of each of the following subsets of the plane.

(a) $\left\{z \in \mathbf{C} \middle| 1 < |z-1| \leq \sqrt{5}\right\}$

Solution: We have $1 < |z-1| < \sqrt{5}$ when the distance between z and 1 is greater than 1, so z lies outside the unit circle centered at 1, and is less than or equal to 5, so z lies inside or on the circle of radius $\sqrt{5}$ centered at 1.

(b) $\left\{z \in \mathbf{C} \middle| |z - 2\,i| = |z - 4|\right\}$

Solution: We have $|z - 2i| = |z - 4|$ when z is equidistant from the points 2i and 4, that is when z lies on the perpendicular bisector of the line segment from 2i to 4.

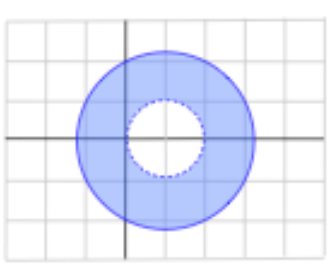
(c) $\left\{z \in \mathbf{C} \middle| |z| + |z - 4| = 8\right\}$

Solution: We have $|z| + |z-4| = 8$ when the sum of the distance from 0 to z with the distance from z to 4 is equal to 8. This happens when z is on the ellipse, with focii at 0 and 4, which passes through the points -2, $\pm 3i$, $4 \pm 3i$, and 6.

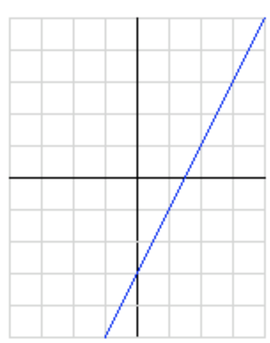
Alternatively, writing $z = x + iy$ with $x, y \in \mathbf{R}$ we can see that this set is the above ellipse as follows.

$$\begin{aligned}|z| + |z-4| = 8 &\Longleftrightarrow \sqrt{x^2+y^2} + \sqrt{(x-4)^2 + y^2} = 8 \Longleftrightarrow \sqrt{x^2 - 8x + 16 + y^2} = 8 - \sqrt{x^2+y^2} \\ &\Longleftrightarrow x^2 - 8x + 16 + y^2 = 64 - 16\sqrt{x^2+y^2} + x^2 + y^2 \Longleftrightarrow 16\sqrt{x^2+y^2} = 8x + 48 \\ &\Longleftrightarrow 2\sqrt{x^2+y^2} = x + 6 \Longleftrightarrow 4(x^2+y^2) = x^2 + 12x + 36 \Longleftrightarrow 3x^2 - 12x + 4y^2 = 36 \\ &\Longleftrightarrow 3(x-2)^2 + 4y^2 = 48 \Longleftrightarrow \frac{(x-2)^2}{16} + \frac{y^2}{12} = 1\,.\end{aligned}$$

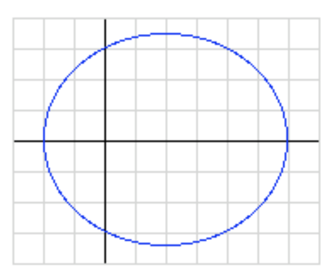
(a)



(b)



(c)



5: For real numbers x and y, we define $e^{x+iy} = e^x \cos y + i\, e^x \sin y$. For a complex number z, we define $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$.

(a) Show that for all $z, w \in \mathbf{C}$ we have $e^{z+w} = e^z e^w$.

Solution: Write $z = x+iy$ and $w = u+iv$ with $x,y,u,v \in \mathbf{R}$. Then

$$\begin{aligned} e^z e^w &= \big(e^x \cos y + i\, e^x \sin y\big)\big(e^u \cos v + i\, e^u \sin v\big) \\ &= \big(e^x e^u \cos y \cos v - e^x e^u \sin y \sin v\big) + i\,\big(e^x e^u \cos y \sin v + e^x e^u \sin y \cos v\big) \\ &= e^{x+u}\big(\cos y \cos v - \sin y \sin v\big) + i\, e^{x+u}\big(\sin y \cos v + \cos y \sin v\big) \\ &= e^{x+u}\big(\cos(y+v) + i \sin(y+v)\big) \\ &= e^{(x+u)+i(y+v)} = e^{z+w}\,. \end{aligned}$$

(b) Show that for all $z, w \in \mathbf{C}$ we have $\sin(z+w) = \sin z \cos w + \cos z \sin w$.

Solution: We have

$$\begin{aligned} \sin z \cos w + \cos z \sin w &= \frac{e^{iz}-e^{-iz}}{2i} \cdot \frac{e^{iw}+e^{-iw}}{2} + \frac{e^{iz}+e^{-iz}}{2} \cdot \frac{e^{iw}-e^{-iw}}{2i} \\ &= \frac{e^{iz}e^{iw} + e^{iz}e^{-iw} - e^{-iz}e^{iw} - e^{-iz}e^{-iw} + e^{iz}e^{iw} - e^{iz}e^{-iw} + e^{-iz}e^{iw} - e^{-iz}e^{-iw}}{4i} \\ &= \frac{2e^{iz}e^{iw} - 2e^{-iz}e^{-iw}}{4i} = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z+w)\,. \end{aligned}$$

Note that we used the formula from part (a) on the last line above.

(c) Solve $e^z = 1 + i\sqrt{3}$ for $z \in \mathbf{C}$.

Solution: First we claim that for $r, s > 0$ and $\theta, \phi \in \mathbf{R}$ we have

$$re^{i\,\theta} = se^{i\,\phi} \iff \big(r = s \text{ and } \theta = \phi + 2\pi\, k \text{ for some } k \in \mathbf{Z}\big)$$

Indeed, if $re^{i\,\theta} = se^{i\,\phi}$ then we must have $r = \left|re^{i\,\theta}\right| = \left|se^{i\,\phi}\right| = s$, and when $r = s$ we have

$$\begin{aligned} re^{i\,\theta} = se^{i\,\phi} &\iff e^{i\,\theta} = e^{i\,\phi} \iff \cos\theta + i \sin\theta = \cos\phi + i\sin\phi \\ &\iff \big(\cos\theta = \cos\phi \text{ and } \sin\theta = \sin\phi\big) \iff \theta = \phi + 2\pi\, k \text{ for some } k \in \mathbf{Z}\,. \end{aligned}$$

It follows that for $x, y \in \mathbf{R}$ we have

$$e^{x+iy} = re^{i\,\theta} \iff e^x e^i y = re^{i\theta} \iff \big(x = \ln r \text{ and } y = \theta + 2\pi\, k \text{ for some } k \in \mathbf{Z}\,.\big)$$

In particular, writing $z = x+iy$ with $x, y \in \mathbf{R}$, we have

$$e^z = 1 + i\sqrt{3} \iff e^{x+iy} = 2e^{i\pi/3} \iff \big(x = \ln 2 \text{ and } y = \tfrac{\pi}{3} + 2\pi\, k \text{ for some } k \in \mathbf{Z}\big).$$

Thus the solution is $z = \ln 2 + i\left(\frac{\pi}{3} + 2\pi\, k\right)$ where $k \in \mathbf{Z}$.

(d) Solve $\sin z = i$ for $z \in \mathbf{C}$.

Solution: Write $z = x+iy$. Then we have

$$\begin{aligned} \sin z = i &\iff \frac{e^{iz}-e^{-iz}}{2i} = i \iff e^{iz} - e^{-iz} = -2 \iff e^{i\,2z} + 2e^{iz} - 1 = 0 \iff \big(e^{iz}+1\big)^2 = 2 \\ &\iff e^{iz} + 1 = \pm\sqrt{2} \iff e^{iz} = -1 \pm \sqrt{2} \iff e^{i(x+iy)} = -1 \pm \sqrt{2} \iff e^{-y+ix} = -1 \pm \sqrt{2}\,. \end{aligned}$$

We have $e^{-y+ix} = \sqrt{2} - 1 = (\sqrt{2}-1)e^{i\,0} \iff -y = \ln(\sqrt{2}-1)$ and $x = 2\pi\, k$ for some $k \in \mathbf{Z}$ and we have $e^{-y+ix} = -(\sqrt{2}+1) = (\sqrt{2}+1)e^{i\,\pi} \iff -y = \ln(\sqrt{2}+1)$ and $x = \pi + 2\pi\, k$ for some $k \in \mathbf{Z}$. Thus the solution is $z = 2\pi\, k - i\, \ln(\sqrt{2}-1)$ or $z = (\pi + 2\pi\, k) - i\, \ln(\sqrt{2}+1)$ for some $k \in \mathbf{Z}$.