# ECE 206 Advanced Calculus 2, Solutions to the Midterm Test, Spring 2016

[10] **1:** (a) Let $S$ be the surface given parametrically by $(x,y,z)=\sigma(s,t)=\left(s-t^2\,,\,\frac{s}{t}\,,\,\sqrt{s\,t}\right)$. Find an implicit equation (in the form $ax+by+cz=d$) for the tangent plane to the surface $S$ at the point where $(s,t)=(4,1)$.

Solution: We have

$$\sigma_s\times\sigma_t=\left(1\,,\,\frac{1}{t}\,,\,\frac{t}{2\sqrt{st}}\right)\times\left(-2t\,,\,-\frac{s}{t^2}\,,\,\frac{s}{2\sqrt{st}}\right)$$
$$(\sigma_s\times\sigma_t)(4,1)=\left(1,1,\tfrac14\right)\times\left(-2,-4,1\right)=\left(2,-\tfrac32,-2\right)$$

Thus the tangent plane has normal vector $(4,-3,-4)$ and so its equation is of the form $4x-3y-4z=d$ $(*)$ for some constant $d$. When $(s,t)=(4,1)$ we have $(x,y,z)=\sigma(4,1)=(3,4,2)$, so we put $(x,y,z)=(3,4,2)$ into equation $(*)$ to get $d=-8$. Thus the tangent plane has equation

$$4x-3y-4z=-8.$$

(b) Let $C$ be the curve of intersection of the paraboloid $z=1-x^2-y^2$ with the plane $z=1-2x$. Find a parametric equation for the tangent line to $C$ at the point $(1,1,-1)$.

Solution 1: We find the intersection of the tangent planes to the two surfaces. The paraboloid is given by $g(x,y)=1$ where $g(x,y)=x^2+y^2+z$, and we have $\nabla g=(2x,2y,1)$ so the tangent plane to the paraboloid at $(1,1,-1)$ has normal vector $u=\nabla g(1,1,-1)=(2,2,1)$. On the other hand, the plane $2x+z=1$ has normal vector $v=(2,0,1)$. Thus the tangent line to $C$ has direction vector $u\times v=(2,2,1)\times(2,0,1)=(2,0,-4)$ and so it is given parametrically by

$$(x,y,z)=(1,1,-1)+t(2,0,-4).$$

Solution 2: We find a parametric equation for $C$. When $(x,y,z)$ lies on both the paraboloid and the plane we have

$$1-x^2-y^2=z=1-2x\Longrightarrow x^2-2x+y^2=0\Longrightarrow (x-1)^2+y^2=1.$$

This is the equation of the circle in the $xy$-plane of radius 1 centred at $(1,0)$, and we can parametrize the circle by $(x,y)=(1+\cos t\,,\,\sin t)$ for $0\le t\le 2\pi$. We also need $z=1-2x=1-2(1+\cos t)=-1-2\cos t)$, and so the curve $C$ can be parametrized by

$$(x,y,z)=\alpha(t)=\left(1+\cos t\,,\,\sin t\,,\,-1-2\cos t\right)\text{ for }0\le t\le 2\pi.$$

and then

$$\alpha'(t)=\left(-\sin t\,,\,\cos t\,,\,2\sin t\right).$$

Since $\alpha\left(\frac{\pi}{2}\right)=(1,1,-1)$ and $\alpha'\left(\frac{\pi}{2}\right)=(-1,0,2)$, the tangent line is given parametrically by

$$(x,y,z)=(1,1,-1)+t(-1,0,2).$$

[10] **2:** (a) Let $f(x,y,z)=\dfrac{xy^3}{x+z^2}$, $a=(1,2,1)$ and $u=\frac{1}{3}(2,1,-2)$. Find $D_uf(a)$.

Solution: We have

$$\begin{aligned}\nabla f &= \left(\frac{y^3z^2}{(x+z^2)^2}\,,\,\frac{3xy^2}{x+z^2}\,,\,\frac{-2xy^3z}{(x+z^2)^2}\right)\\ \nabla f(a) &= \nabla f(1,2,1)=(2,6,-4)\\ D_uf(a) &= \nabla f(a)\cdot u=\tfrac{1}{3}\,(2,6,-4)\cdot(2,1,-2)=6.\end{aligned}$$

(b) Let $(u,v)=f(x,y)=\left(xe^{-xy}\,,\,1+x^2+x\sin y\right)$, let $z=g(u,v)=\sqrt{u^2+2v^2}$, and let $h(x,y)=g\big(f(x,y)\big)$. Find $\nabla h(1,0)$.

Solution: We have

$$Dh=Dg\,Df=\begin{pmatrix} z_u & z_v\end{pmatrix}\begin{pmatrix} u_x & u_y\\ v_x & v_y\end{pmatrix}=\begin{pmatrix}\dfrac{u}{\sqrt{u^2+2v^2}} & \dfrac{2v}{\sqrt{u^2+2v^2}}\end{pmatrix}\begin{pmatrix} e^{-xy}-xye^{-xy} & -x^2e^{-xy}\\ 2x+\sin y & x\cos y\end{pmatrix}$$

When $(x,y)=(1,0)$ we have $(u,v)=f(1,0)=(1,2)$ and so

$$Dh(1,0)=Dg(1,2)Df(1,0)=\begin{pmatrix}\frac{1}{3} & \frac{4}{3}\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}=(3,1).$$

Thus $\nabla h(1,0)=(3,1)$.

(c) Let $F(x,y,z)=\left(z^2+\dfrac{1}{y}\,,\,z-\dfrac{x}{y^2}\,,\,y+2(x+1)z\right)$. Find $g$ such that $F=\nabla g$.

Solution: To get $\frac{\partial g}{\partial x}=z^2+\frac{1}{y}$ we need $g=xz^2+\frac{x}{y}+\phi(y,z)$. To get $\frac{\partial g}{\partial y}=z-\frac{x}{y^2}$ we need $-\frac{x}{y^2}+\frac{\partial\phi}{\partial y}=z-\frac{x}{y^2}$ hence $\frac{\partial\phi}{\partial y}=z$ and hence $\phi=yz+\psi(z)$. Thus we need $g=xz^2+\frac{x}{y}+\phi=xz^2+\frac{x}{y}+yz+\psi(z)$. Finally, to gey $\frac{\partial g}{\partial z}=y+2xz+2z$ we need $2xz+y+\psi'(z)=y+2xz+2z$ hence $\psi'=2z$ and we can take $\psi=z^2$. Thus we can take

$$g(x,y,z)=xz^2+\frac{x}{y}+yz+z^2.$$

[10] **3:** (a) Find the mass of the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,2,2)$ with density (mass per unit volume) given by $\rho(x,y,z)=2z-xy$.

Solution: The top view of the tetrahedron is the triangle $T$ in the $xy$-plane with vertices at $(0,0)$, $(1,0)$ and $(0,2)$ which is given by $T=\big\{(x,y)\big|0\le x\le 1\,,\,0\le y\le 2-2x\big\}$. The top face of the tetrahedron is the plane through $(0,0,0)$, $(1,0,0)$ and $(0,2,2)$, which is the plane $y=z$, so the given tetrahedron is the set

$$D=\Big\{(x,y,z)\Big|0\le x\le 1\,,\,0\le y\le 2-2x\,,\,0\le z\le y\Big\}.$$

The mass is

$$\begin{aligned}M&=\iiint_D \rho(x,y,z)\,dV=\int_{x=0}^1\int_{y=0}^{2-2x}\int_{z=0}^{y}2z-xy\;dz\,dy\,dx=\int_{x=0}^1\int_{y=0}^{2-2x}y^2-xy^2\;dy\,dx\\&=\int_{x=0}^1\tfrac13(1-x)(2-2x)^3\;dx=\int_{x=0}^1\tfrac83(1-x)^4\,dx=\int_{u=0}^1\tfrac83\,u^4\,du=\tfrac{8}{15},\end{aligned}$$

where we made use of the substitution $u=x-1$.

(b) Find the total charge on the cone $S=\Big\{(x,y,z)\Big|z=\sqrt{x^2+y^2}\,,\,z\le 1\Big\}$ with charge density (charge per unit area) given by $\rho(x,y,z)=x^2z$.

Solution: The cone $S$ can be parametrized by

$$(x,y,z)=\sigma(r,\theta)=\big(r\cos\theta\,,\,r\sin\theta\,,\,r\big)\ \text{for}\ 0\le r\le 1\,,\,0\le\theta\le 2\pi.$$

We have

$$\begin{aligned}\rho\big(\sigma(r,\theta)\big)&=(r\cos\theta)^2\cdot r=r^3\cos^2\theta,\\ \sigma_r\times\sigma_\theta&=\big(\cos\theta\,,\,\sin\theta\,,\,1\big)\times\big(-r\sin\theta\,,\,r\cos\theta\,,\,0\big)\\&=\big(-r\cos\theta\,,\,-r\sin\theta\,,\,r\big)\ \text{and}\\ \big|\sigma_r\times\sigma_\theta\big|&=\sqrt{r^2+r^2}=\sqrt2\,r,\end{aligned}$$

and so the charge is

$$\begin{aligned}Q&=\iint_S\rho(x,y,z)dA=\int_{r=0}^1\int_{\theta=0}^{2\pi}\rho\big(\sigma(r,\theta)\big)\big|\sigma_r\times\sigma_\theta\big|\,d\theta\,dr\\&=\int_{r=0}^1\int_{\theta=0}^{2\pi}r^3\cos^2\theta\cdot\sqrt2\,r\;d\theta\,dr=\int_{r=0}^1\sqrt2\,\pi\,r^4\;dr=\tfrac{\sqrt2\,\pi}{5}.\end{aligned}$$

[10] **4:** (a) Find the work done by the force $F(x,y,z) = \left(\dfrac{z}{1+x}\,,\,\sqrt{y}\,,\,2x\right)$ acting on a small object moving along the curve $C$ given by $(x,y,z) = \alpha(t) = \big(t\,,\,1+2t^2\,,\,1+t^3\big)$ for $0 \le t \le 2$.

Solution: We have

$$F\big(\alpha(t)\big) = \left(\frac{1+t^3}{1+t}\,,\,\sqrt{1+2t^2}\,,\,2t\right) = \left(1-t+t^2\,,\,\sqrt{1+2t^2}\,,\,2t\right) \text{ and}$$
$$\alpha'(t) = \big(1, 4t, 3t^2\big),$$

and so the work is

$$W = \int_C F \cdot T\,dL = \int_{t=0}^{2} F\big(\alpha(t)\big) \cdot \alpha'(t)\,dt = \int_{t=0}^{2} 1 - t + t^2 + 4t\sqrt{1+2t^2} + 6t^3\,dt$$
$$= \Big[t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 + \tfrac{2}{3}(1+2t^2)^{3/2} + \tfrac{3}{2}t^4\Big]_{t=0}^{2} = 2 - 2 + \tfrac{8}{3} + 18 - \tfrac{2}{3} + 24 = 44.$$

(b) Find the flux of the vector field $F = \big(xy^2\,,\,yz\,,\,x^2z\big)$ outwards across the boundary surface $S = \partial D$ of the cylinder $D = \Big\{(x,y,z)\Big|x^2+y^2 \le 1\,,\,0 \le z \le 1\Big\}$.

Solution 1: By the Divergence Theorem, the flux is

$$\Phi = \iint_S F \cdot N\,dA = \iiint_D \nabla \cdot F\,dV = \iiint_D x^2 + z + y^2\,dV$$
$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{1} (r^2+z)r\,dz\,d\theta\,dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^3 + \tfrac{1}{2}r\,d\theta\,dr$$
$$= \int_{r=0}^{1} 2\pi r^3 + \pi r\,dr = \tfrac{\pi}{2} + \tfrac{\pi}{2} = \pi.$$

Solution 2: The surface $S$ consists of three parts: the top disc $S_{\text{top}} = \big\{(x,y,z)\big|z=1, x^2+y^2 \le 1\big\}$, the bottom disc $S_{\text{bot}} = \big\{(x,y,z)\big|z=0, x^2+y^2 \le 1\big\}$, and the vertical side $S_{\text{side}} = \big\{(x,y,z)\big|0 \le z \le 1\,,\,x^2+y^2 = 1\big\}$. We calculate the flux across each of these surfaces. The flux across $S_{\text{top}}$ is calculated by setting $z=1$ in $F$ and taking $N = (0,0,1)$ to get

$$\Phi_{\text{top}} = \iint_{S_{\text{top}}} F \cdot N\,dA = \iint_{x^2+y^2\le 1} \big(xy^2, y, x^2\big) \cdot (0,0,1)\,dA = \iint_{x^2+y^2 \le 1} x^2\,dA$$
$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} (r\cos\theta)^2\,r\,d\theta\,dr = \int_{r=0}^{1} \pi\,r^3\,dr = \tfrac{\pi}{4}.$$

The flux across $S_{\text{bot}}$ is calculated by setting $z=0$ in $F$ and taking $N = (0,0,-1)$ to get

$$\Phi_{\text{bot}} = \iint_{S_{\text{bot}}} F \cdot N\,dA = \iint_{x^2+y^2\le 1} (xy^2, 0, 0) \cdot (0,0,-1)\,dA = \iint_{x^2+y^2\le 1} 0\,dA = 0.$$

To find the flux across $S_{\text{side}}$ we parametrize the surface $S_{\text{side}}$ by

$$(x,y,z) = \sigma(\theta, z) = \big(\cos\theta, \sin\theta, z\big) \quad \text{for } 0 \le \theta \le 2\pi\,,\,0 \le z \le 1.$$

We then have

$$F\big(\sigma(\theta,z)\big) = \Big(\cos\theta\sin^2\theta\,,\,z\sin\theta\,,\,z\cos^2\theta\Big) \text{ and}$$
$$\sigma_\theta \times \sigma_z = (-\sin\theta, \cos\theta, 0) \times (0,0,1) = (\cos\theta, \sin\theta, 0).$$

We note that $\sigma_\theta \times \sigma_z$ points outwards (as desired) so the flux across $S_{\text{side}}$ is

$$\Phi_{\text{side}} = \iint_{S_{\text{side}}} F \cdot N\,dA = \int_{\theta=0}^{2\pi}\int_{z=0}^{1} F\big(\sigma(\theta,z)\big) \cdot (\sigma_\theta \times \sigma_z)\,dz\,d\theta$$
$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{1} \cos^2\theta\sin^2\theta + z\sin^2\theta\,dz\,d\theta = \int_{\theta=0}^{2\pi} \cos^2\theta\sin^2\theta + \tfrac{1}{2}\sin^2\theta d\theta$$
$$= \int_{\theta=0}^{2\pi} \tfrac{1}{4}\sin^2 2\theta + \tfrac{1}{2}\sin^2\theta\,d\theta = \tfrac{\pi}{4} + \tfrac{\pi}{2} = \tfrac{3\pi}{4}.$$

Thus the total flux across $S$ is

$$\Phi_{\text{tot}} = \Phi_{\text{top}} + \Phi_{\text{bot}} + \Phi_{\text{side}} = \tfrac{\pi}{4} + 0 + \tfrac{3\pi}{4} = \pi.$$